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Exercises for **Algebraic Topology I**
Sheet 9

Definition. A simplicial set is called *finite* if it has only finitely many non-degenerate simplices.

Exercise 1 (*14 points*). 1. Let K be any simplicial set. Show that $K \times -: \mathbf{SSet} \to \mathbf{SSet}$ is a left adjoint.

2. Show that for any simplicial set K and any $n \geq 0$, the map $(|\mathrm{pr}_1|, |\mathrm{pr}_2|): |K \times \Delta^n| \to |K| \times |\Delta^n|$ is a homeomorphism.

3. Show that for any two simplicial sets K, L such that at least one of them is *finite*, the natural map $(|\mathrm{pr}_1|, |\mathrm{pr}_2|): |K \times L| \to |K| \times |L|$ is a homeomorphism.

   Remark. The natural map $|K \times L| \to |K| \times |L|$ is not a homeomorphism in full generality, related to the fact that the product topology on a product of two CW-complexes does not always coincide with the weak topology with respect to the natural cell structure.

Exercise 2 (*13 points*). Throughout, let G be a finite (discrete) group.

1. Let X be a Hausdorff space with a free G-action, and let $p: X \to X/G$ denote the projection. Show that $\mathrm{Sing}(p): \mathrm{Sing}(X) \to \mathrm{Sing}(X/G)$ descends to an isomorphism $\mathrm{Sing}(X)/G \cong \mathrm{Sing}(X/G)$.

2. Let R be a ring such that the order $|G|$ of G is invertible in R, and let C be a cochain complex of R-modules, equipped with a G-action through R-linear chain maps. Show that the inclusion $C^G \hookrightarrow C$ of the levelwise G-fixed points induces an isomorphism of R-modules $H^*(C^G) \cong (H^*C)^G$.

3. Conclude: if X is a Hausdorff space with a free G-action and if R is a ring such that $|G|$ is invertible in R, then the quotient map $X \to X/G$ induces an isomorphism $H^*(X/G; R) \cong H^*(X; R)^G$.

Definition. Let G, H be groups and let $\phi: H \to \mathrm{Aut}(G)$ be a homorphism into the group of automorphisms of G. The *semidirect product* $G \rtimes_\phi H$ is defined as the set $G \times H$, with group multiplication given by $(g, h) \cdot (g', h') := (g\phi(h)(g'), hh')$.

Exercise 3 (*13 points*). Let $n \geq 1$, and let $\phi: C_2 \to \mathrm{Aut}(C_n)$ be the homomorphism sending the generator of C_2 to the automorphism $C_n \to C_n, x \mapsto x^{-1}$. The *dihedral group* is defined as $D_{2n} := C_n \rtimes_\phi C_2$.

1. We let the generator of C_2 act on the first factor of $S^\infty \times S^\infty$ via complex conjugation and on the second one via multiplication by -1. Moreover, we let the generator of C_n act on the first factor via multiplication by $\exp(2\pi i/n)$ and on the second factor via the identity.

   Show that this extends (necessarily uniquely) to an action of D_{2n} on $S^\infty \times S^\infty$, and that this action is free. Conclude that $BD_{2n} := (S^\infty \times S^\infty)/D_{2n}$ is an Eilenberg–MacLane space of type $K(D_{2n}, 1)$.

2. Assume now that $n = p$ is an odd prime. Use the previous exercise to construct an isomorphism $H^*(BD_{2p}; \mathbb{F}_p) \cong H^*(S^\infty/C_p; \mathbb{F}_p)^{C_2}$, where the C_2-action on the target is via the complex conjugation on S^∞.

3. In class, we described explicit cohomology classes $x \in H^1(S^\infty/C_p; \mathbb{F}_p)$ and $y \in H^2(S^\infty/C_p; \mathbb{F}_p)$ such that $H^*(S^\infty/C_p; \mathbb{F}_p) = \mathbb{F}_p[x, y]/(x^2)$.

   Show that $H^*(S^\infty/C_p; \mathbb{F}_p)^{C_2} \subset H^*(S^\infty/C_p; \mathbb{F}_p)$ is generated as a ring by the classes xy and y^2.

Please hand in your solutions by **January 12, 2pm** via eCampus.