Winter term 2025/26

Prof. Dr. Stefan Schwede
Dr. Tobias Lenz

Exercises for **Algebraic Topology I**
Sheet 8

Definition. Let X be a simplicial set and let $n \geq 0$. An n-simplex $x \in X_n$ is called *degenerate* if there exists a non-injective map $f\colon [n] \to [m]$ in Δ and a $y \in X_m$ with $x = f^*y$. We call x *non-degenerate* if it is not degenerate.

Exercise 1 (*10 points*)**.**

1. Show that any map in Δ admits a unique factorization as a surjective monotone map followed by an injective one.
2. Conclude that a simplex $x \in X_n$ is degenerate if and only if there exists a surjection $f\colon [n] \to [m]$ in Δ for some $m < n$ and a $y \in X_m$ such that $x = f^*y$.
3. Let $x \in X_n$ be arbitrary. Show that there exists a *unique* pair of a monotone surjection $f\colon [n] \to [m]$ and a non-degenerate $y \in X_m$ such that $x = f^*y$.

   Hint. For existence pick a pair of a surjection $f\colon [n] \to [m]$ and an (a priori possibly degenerate) $y \in X_m$ with $x = f^*y$ where m is minimal. For uniqueness use that any surjective monotone map has a monotone section.

Exercise 2 (*8 points*)**.** Let X be a simplicial set. Show that any point in the geometric realization $|X| = \big(\coprod_{n\geq 0} X_n \times \nabla^n\big)/\!\sim$ admits a representative $(x, \sigma) \in X_n \times \nabla^n$ where x is non-degenerate and $\sigma \in (\nabla^n)^\circ = \{(s_0, \dots, s_n) \in (0,1]^{1+n} : s_0 = + \cdots + s_n = 1\}$.

Remark. We will show later in class that this representative is in fact unique.

Exercise 3 (*22 points*)**.** Let $m, n \geq 0$.

1. Show that a k-simplex $(f\colon [k] \to [m], g\colon [k] \to [n])$ of $\Delta^m \times \Delta^n$ is non-degenerate if and only if the map of sets $(f, g)\colon [k] \to [m] \times [n]$ is injective.
2. Show that the continuous map $(|\mathrm{pr}_1|, |\mathrm{pr}_2|)\colon |\Delta^m \times \Delta^n| \to |\Delta^m| \times |\Delta^n|$ is bijective.

   Hint. For surjectivity, use induction to more generally construct for any $x = (x_0, \dots, x_m) \in [0,1]^{1+m}$ and $y \in [0,1]^{1+n}$ with $\sum_{i=0}^m x_i = \sum_{i=0}^n y_i$ monotone maps $f\colon [k] \to [m]$, $g\colon [k] \to [n]$ and a $z \in [0,1]^{1+k}$ such that $x = f_*z$ and $y = g_*z$.

   For injectivity, use the previous exercise and a similar inductive argument.
3. Show that $|\Delta^m \times \Delta^n|$ is quasi-compact and use this to deduce that the continuous bijection from the previous subtask is in fact a homeomorphism.

Remark. On the next sheet, we will build on this exercise to show that geometric realization 'almost' preserves finite products.

Please hand in your solutions by **December 15, 2pm** via eCampus.