Winter term 2025/26

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Exercises for **Algebraic Topology I**
Sheet 7

Definition. Let $\mathcal{C}, \mathcal{D}$ be categories. An *adjunction* between $\mathcal{C}$ and $\mathcal{D}$ is a quadruple (F, G, η, ϵ) where $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are functors and $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ and $\epsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ are natural transformations satisfying the two *triangle identities*:

1. For every $X \in \mathcal{C}$, the composite $FX \xrightarrow{F\eta_X} FGFX \xrightarrow{\epsilon_{FX}} FX$ is the identity.
2. For every $Y \in \mathcal{D}$, the composite $GY \xrightarrow{\eta_{GY}} GFGY \xrightarrow{G\epsilon_Y} GY$ is the identity.

Put differently, the following two diagrams of natural transformations commute:

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow^{=} & \downarrow{\scriptstyle \epsilon F} \\ & & F \end{array} \qquad\qquad \begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow^{=} & \downarrow{\scriptstyle G\epsilon} \\ & & G. \end{array}$$

In this case we say that F is a *left adjoint*, that F is *left adjoint to* G, or that F *admits a right adjoint*. Dually, we say that G is a right adjoint or that it admits a left adjoint (namely, F). We moreover call η the *unit* of the adjunction and ϵ the *counit*.

Exercise 1 (*14 points*)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors.

1. Let $\eta\colon \mathrm{id} \to GF$ be a natural transformation. Show that there exists at most one natural transformation $\epsilon\colon FG \to \mathrm{id}$ such that (F, G, η, ϵ) is an adjunction, and such an ϵ exists if and only if the composite
$$\hom_{\mathcal{D}}(FX, Y) \xrightarrow{G} \hom_{\mathcal{C}}(GFX, GY) \xrightarrow{-\circ\eta} \hom_{\mathcal{C}}(X, GY)$$
is bijective for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.

In this case we say that η *is the unit of an adjunction* (between F and G). Dually, one shows (and you may use without proof from now on) that for any given $\epsilon\colon FG \to \mathrm{id}$ there is at most one $\eta\colon \mathrm{id} \to GF$ such that (F, G, η, ϵ) is an adjunction, and such an η exists if and only if
$$\hom_{\mathcal{C}}(X, GY) \xrightarrow{F} \hom_{\mathcal{D}}(FX, FGY) \xrightarrow{\epsilon\circ-} \hom_{\mathcal{D}}(FX, Y)$$
is bijective; we then say that ϵ *is the counit of an adjunction*.

2. Let $G'\colon \mathcal{D} \to \mathcal{C}$ and assume we are given two adjunctions (F, G, η, ϵ) and $(F, G', \eta', \epsilon')$. Show that there exists a unique natural transformation $\phi\colon G \to G'$ making the following triangle commute:
$$\begin{array}{ccccc} FG & & \xrightarrow{F\phi} & & FG' \\ & {\scriptstyle \epsilon}\searrow & & \swarrow{\scriptstyle \epsilon'} & \\ & & \mathrm{id}. & & \end{array}$$
Moreover, prove that ϕ is a natural isomorphism. Put more informally: if a right adjoint of a given F exists, it is unique up to unique isomorphism.

3. Prove that any left adjoint $F\colon \mathcal{C} \to \mathcal{D}$ preserves all colimits that exist in $\mathcal{C}$, i.e. for any category I (not necessarily small), any functor $X_\bullet\colon I \to \mathcal{C}$, and any colimit $(\zeta_i\colon X_i \to X)_{i\in I}$, the cocone $(F\zeta_i\colon FX_i \to FX)_{i\in I}$ is a colimit of $FX_\bullet$.

A dual argument then shows (and you may use from now on) that right adjoints preserve limits.

please turn over

Please hand in your solutions by **December 8, 2pm** via eCampus.

Construction. For a simplicial set X, we define its *geometric realization* $|X|$ as the quotient of the topological disjoint union $\coprod_{n\geq 0} X_n \times \nabla^n$ by the equivalence relation generated by $(f^*x, \sigma) \sim (x, f_*\sigma)$ for every morphism $f\colon [m] \to [n]$ in Δ, $x \in X_n$, and $\sigma \in \nabla^m = \{(s_0, \dots, s_m) \in [0,1]^{1+m} : s_0 + \cdots + s_m = 1\}$.

Exercise 2 (*14 points*)**.** 1. Let $f\colon X \to Y$ be a morphism of simplicial sets (i.e., a natural transformation). Show that the map $\coprod_{n\geq 0} f_n \times \mathrm{id}_{\nabla^n}$ descends to a continuous map $|f|\colon |X| \to |Y|$ and that this way geometric realization becomes a functor $|\cdot|\colon \mathbf{SSet} \to \mathbf{Top}$ from simplicial sets to topological spaces.

2. Show that the map $\nabla^n \to |\Delta^n|, \sigma \mapsto [\mathrm{id}_{[n]}, \sigma]$ is a homeomorphism for any $n \geq 0$.

3. Let Z be any topological space. Show that the map

$$\begin{aligned} \tilde{\epsilon}_Z\colon \coprod_{n\geq 0} \hom_{\mathbf{Top}}(\nabla^n, Z) \times \nabla^n &\longrightarrow X \\ (f, \sigma) &\longmapsto f(\sigma) \end{aligned}$$

descends to a continuous map $\epsilon_Z\colon |\mathrm{Sing}(Z)| \to Z$ from the geometric realization of the singular complex of Z to Z itself.

4. Show that the maps ϵ_Z define a natural transformation, and that this is the counit of an adjunction between geometric realization and singular complex.

Exercise 3 (*12 points*)**.** Write **hTop** for the category of topological spaces and homotopy classes of maps, and let $\mathbf{hCW} \subset \mathbf{hTop}$ denote the full subcategory spanned by the CW-complexes. Show that the inclusion $\mathbf{hCW} \hookrightarrow \mathbf{hTop}$ admits a right adjoint, but does not admit a left adjoint.

∗ **Exercise 4** (*10 bonus points*)**.** Let I be any small category. We write $\mathrm{PSh}(I) \coloneqq \mathrm{Fun}(I^{\mathrm{op}}, \mathbf{Set})$ and call it the *category of presheaves* on I. It comes with a functor $y\colon I \to \mathrm{PSh}(I), i \mapsto \hom(-, i)$ (with functoriality via composition), and the Yoneda lemma shows that this functor is fully faithful.

1. Let $X \in \mathrm{PSh}(I)$ be any presheaf, and write $I \downarrow X$ for the category whose objects are pairs (i, x) where $i \in I$ and $x \in X(i)$, and with morphisms $(i, x) \to (j, y)$ given by those $f\colon i \to j$ in I such that $X(f)(y) = x$. This category comes with a 'forgetful' functor $\mathrm{fgt}\colon I \downarrow X \to I$ sending an object (i, x) to i and a morphism f to f.

   Exhibit X as a colimit of the composite

$$I \downarrow X \xrightarrow{\mathrm{fgt}} I \xrightarrow{y} \mathrm{PSh}(I).$$

   Hint. Use the full strength of the Yoneda lemma.

2. Show: if $\mathcal{C}$ is any cocomplete category, and $f\colon I \to \mathcal{C}$ is any functor, then there exists a left adjoint functor $F\colon \mathrm{PSh}(I) \to \mathcal{C}$ together with a natural isomorphism $\tau\colon Fy \cong f$.

   Hint. The previous subtask tells you how to define F on objects, and the Yoneda lemma tells you how a right adjoint must look like, if it exists.

3. Let $\mathcal{C}$ be cocomplete. Show that restriction along the Yoneda embedding $y\colon I \to \mathrm{PSh}(I)$ defines an equivalence $\mathrm{Fun}^{\mathrm{L}}(\mathrm{PSh}(I), \mathcal{C}) \to \mathrm{Fun}(I, \mathcal{C})$, where the left hand side denotes the full subcategory of $\mathrm{Fun}(\mathrm{PSh}(I), \mathcal{C})$ spanned by the left adjoint functors.

Specializing the above to $I = \Delta$, we see that constructing a left adjoint functor from $\mathbf{SSet} = \mathrm{PSh}(\Delta)$ to a cocomplete $\mathcal{C}$ is essentially equivalent to giving a functor $\Delta \to \mathcal{C}$ (a so-called *cosimplicial object*). Exercise 2 then shows that for the standard cosimplicial object $\nabla^\bullet$ this yields the adjunction between geometric realization and singular complex.