Winter term 2025/26

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Exercises for **Algebraic Topology I**
Sheet 6

Exercise 1 (*10 points*)**.** Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ denote the quaternion group. The central subgroup $\{\pm 1\} \subset Q$ is isomorphic to $\mathbb{Z}/2$ (in a unique way), and every element of the quotient $Q/\{\pm 1\}$ has order ≤ 2, so the quotient is abstractly isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. We therefore obtain a central extension

$$1 \to \mathbb{Z}/2 \to Q \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1,$$

to which we can associate a cohomology operation $\delta\colon H^1(-;\mathbb{Z}/2 \times \mathbb{Z}/2) \to H^2(-;\mathbb{Z}/2)$ via Exercise 3 from the previous sheet.

Show that for a suitable choice of the isomorphism $Q/\{\pm 1\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, this operation is given by

$$\delta(x) = (p_1)_*(x) \cup (p_2)_*(x)$$

where $(p_i)_*$ $(i = 1, 2)$ denotes the coefficient homomorphism associated to the projection $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2$ to the i-th factor.

Exercise 2 (*13 points*)**.** Let $n \geq 1$. Show that there is no finite CW-complex which is a $K(\mathbb{Z}/2, n)$.

Exercise 3 (*17 points + 5 bonus points*)**.**

1. Let X be a topological space. We define the map

$$\mathrm{Sq}\colon \bigoplus_{n\geq 0} H^n(X;\mathbb{F}_2) \to \bigoplus_{n\geq 0} H^n(X;\mathbb{F}_2)$$

as the unique group homomorphism given on the n-th summand by $x \mapsto \sum_{i=0}^{\infty} \mathrm{Sq}^i(x) = \sum_{i=0}^{n} \mathrm{Sq}^i(x)$. Prove that Sq is a ring homomorphism.

2. Conclude that $\mathrm{Sq}^k(x^n) = \binom{n}{k} x^{n+k}$ for any $x \in H^1(X;\mathbb{F}_2)$.

3. Show that $\binom{n}{k}$ is odd if and only if the following holds: for every $\ell \geq 1$, whenever the ℓ-th bit in the binary expansion of k is non-zero, so is the ℓ-th bit of the binary expansion of n (counted from the right in both cases).

4. The previous subtasks together completely describe the Steenrod operations on $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2) \cong \mathbb{F}_2[x]$. Similarly describe the Steenrod operations on $H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{F}_2)$.

∗5. (*5 bonus points*) Determine the maximal r such that there exist a sequence $k_1, \ldots, k_r$ of positive integers for which

$$\mathrm{Sq}^{k_r} \circ \cdots \circ \mathrm{Sq}^{k_1}\colon H^*(\mathbb{R}\mathrm{P}^{42};\mathbb{F}_2) \to H^{*+k_1+\cdots+k_r}(\mathbb{R}\mathrm{P}^{42};\mathbb{F}_2)$$

is nonzero.

Please hand in your solutions by **December 1, 2pm** via eCampus.