Winter term 2025/26

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Exercises for **Algebraic Topology I**
Sheet 5

Exercise 1 (*10 points*)**.** 1. Let $n \geq 1$ and let X be a CW-complex of dimension $\leq n$. Construct a natural bijection $H^n(X;\mathbb{Z}) \cong [X, S^n]$.

2. Describe representatives of all homotopy classes of maps $S^1 \times S^1 \to S^2$ and $\mathbb{R}\mathrm{P}^2 \to S^2$.

Definition. Let B be a topological space. A *line bundle* over B consists of a continuous map $\zeta\colon E \to X$, together with for every $x \in B$ the structure of a 1-dimensional complex vector space on the fiber $E_x \coloneqq \zeta^{-1}(x)$ satisfying the following condition: For every $x \in B$ there exists an open neighborhood U together with a homeomorphism $\phi\colon U \times \mathbb{C} \to \zeta^{-1}(U)$ fitting into a commutative diagram

$$\begin{array}{ccc} U \times \mathbb{C} & \xrightarrow[\cong]{\phi} & \zeta^{-1}(U) \\ & {\scriptstyle \mathrm{pr}} \searrow \quad \swarrow {\scriptstyle \zeta} & \\ & U & \end{array}$$

and such that the induced map $\phi_x\colon \mathbb{C} \to E_x$ on fibers is $\mathbb{C}$-linear for every $x \in U$. Whenever such a ϕ exists for a given U, we say that the bundle is *trivial* over U.

By abuse of terminology, one often refers to E as a line bundle, leaving the map ζ and the vector space structures implicit. A *morphism* of line bundles $\zeta_i\colon E_i \to B$ $(i = 1, 2)$ over B is a continuous map $f\colon E_1 \to E_2$ such that $\zeta_2 f = \zeta_1$ and such that the restriction of f to each fiber is complex linear.

Construction. Let $\zeta\colon E \to B$ be a line bundle, and let $f\colon A \to B$ be any continuous map. We define $f^*E \subset E \times A$ as the subspace of all those (e, a) where $\zeta(e) = f(a)$. This comes with a map $f^*\zeta\colon f^*E \to A$ given by the projection to the second factor; one easily checks (and you may use without proof) that this is again a line bundle when we equip each fiber $(f^*E)_a = E_{f(a)} \times \{a\}$ with the obvious vector space structure.

Exercise 2 (*20 points + 5 bonus points*)**.** Consider the polynomial ring $\mathbb{C}[X]$; for every $n \geq 0$, we topologize the subspace spanned by $1, \dots, X^n$ in the usual way as $\mathbb{C}^{1+n}$, and we give $\mathbb{C}[X]$ the weak topology with respect to these subspaces. We define $\mathbb{C}\mathrm{P}^\infty$ as the set of 1-dimensional complex linear subspaces of $\mathbb{C}[X]$, topologized as a quotient of $\mathbb{C}[X] \smallsetminus \{0\}$. Recall from class that $\mathbb{C}\mathrm{P}^\infty$ is a $K(\mathbb{Z}, 2)$.

1. We let E_{taut} be the subspace of $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}[X]$ of all those pairs (ℓ, p) such that $p \in \ell$. Show that the projection defines a line bundle $\zeta_{\mathrm{taut}}\colon E_{\mathrm{taut}} \to \mathbb{C}\mathrm{P}^\infty$.

2. Let B be a finite CW-complex, and let $\zeta\colon E \to B$ be any line bundle. Show that there exists a finite-dimensional vector space V and a closed embedding $E \to B \times V$ over B such that each of the maps $E_x \to V$ on fibers is complex linear.

   Hint. Cover B by open sets over which E is trivial and consider an appropriate partition of unity.

3. We write $\mathrm{Pic}(B)$ for the collection of isomorphism classes of line bundles over the finite CW-complex B. Show that

$$\begin{aligned} \hom_{\mathbf{Top}}(B, \mathbb{C}\mathrm{P}^\infty) &\longrightarrow \mathrm{Pic}(B) \\ (f\colon B \to \mathbb{C}\mathrm{P}^\infty) &\longmapsto f^*\zeta_{\mathrm{taut}} \end{aligned} \tag{$*$}$$

   is surjective.

please turn over

Please hand in your solutions by **November 24, 2pm** via eCampus.

4. Show that the usual multiplication on $\mathbb{C}[X]$ descends to a commutative monoid structure on $\mathbb{C}\mathrm{P}^\infty$.

$*$5. (*5 bonus points*) Given two elements of $\mathrm{Pic}(B)$ with representatives $\zeta_i \colon E_i \to B$ we define their *tensor product* $[\zeta_1] \otimes [\zeta_2]$ as follows: we pick closed embeddings $\kappa_i \colon E_i \hookrightarrow B \times V_i$ as in the second subtask, and we define $E_1 \otimes E_2$ as the subspace of $B \times (V_1 \otimes V_2)$ whose fiber over $x \in B$ is $\kappa_1(E_1)_x \otimes \kappa_2(E_2)_x$.

Prove that the above is well-defined, gives $\mathrm{Pic}(B)$ the structure of a commutative group, and that the map $(*)$ is a monoid homomorphism.

Hint. Use surjectivity of $(*)$ to avoid checking the axioms of a commutative monoid by hand.

6. Show that $(*)$ descends to a well-defined bijection $[B, \mathbb{C}\mathrm{P}^\infty] \xrightarrow{\cong} \mathrm{Pic}(B)$.

Hint. Use the group and monoid structures constructed above.

Combining the above with the representability of singular cohomology proven in class, we altogether obtain a natural isomorphism $H^2(B;\mathbb{Z}) \cong \mathrm{Pic}(B)$ of abelian groups for any finite CW-complex B.

Exercise 3 (*10 points*)**.** Let

$$1 \to B \xrightarrow{i} E \xrightarrow{p} A \to 1$$

be a short exact sequence of groups, where A and B are abelian and i maps B into the center of the not necessarily abelian group E; such a short exact sequence is called a *central extension* (of A by B).

If X is a simplicial set (e.g. the singular complex of a topological space), and $f \colon X_1 \to A$ is a 1-cocycle in $C^*(X;A)$, then we pick an arbitrary lift $\bar{f} \colon X_1 \to E$ and we define

$$(\delta f)(x) = \bar{f}(d_0^* x) \cdot \bar{f}(d_1^* x)^{-1} \cdot \bar{f}(d_2^* x) \in E$$

for all $x \in X_2$.

Show that δf factors (necessarily uniquely) through the homomorphism $i \colon B \to E$, that the resulting $X_2 \to B$ is a 2-cocycle, and that this construction defines a homomorphism $\delta \colon H^1(X;A) \to H^2(X;B)$ independent of any choices and natural in the simplicial set X. (In particular, we therefore obtain a cohomology operation $H^1(-;A) \to H^2(-;B)$ by precomposing with the singular complex functor.)