LECTURE NOTES ON EQUIVARIANT STABLE HOMOTOPY THEORY

We expand on the existing notes on equivariant stable homotopy theory, by Stefan Schwede, by including a section about the unstable version of equivariant homotopy theory (pre-spectra). Like in the notes, we fix the following notation: Given a Lie group G, we denote that $H \subseteq G$ is a closed subgroup by writing $H \leq G$. We also denote by ρ_G the *regular representation* $\mathbb{R}[G]$ with obvious G-action. Finally, denote with $\mathcal{O}_G$ the category with objects the quotients G/H by closed subgroups and G-equivariant maps.

1. Unstable Equivariant Homotopy Theory

Let G be a finite (discrete) group. Denote with G-Top the category of G-spaces and G-equivariant maps. Let Δ^1 be with the trivial groups action, then a homotopy $f \simeq g$ between two maps $X \to Y$ is a G-equivariant map

$$H : X \times \Delta^1 \to Y$$

such that $f = H \circ i_0$ and $g = H \circ i_1$. A morphism $f : X \to Y$ is a *G-homotopy equivalence* if there is $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are G-homotopic to the identity.

Remark 1.1. Being a G-equivariant homotopy equivalence of spaces is not enough to be a G-homotopy equivalence. For example, the unique map $EG \to *$ is G-equivariant and EG is contractible but there is no G-equivariant map $* \to EG$, since such a map corresponds to a G-fixed point $\in EG$ and G acts freely.

The forgetful functor $U : G\text{-Top} \to \text{Top}$ has a left adjoint $G \times -$ and also a right adjoint $\text{map}(G, -)$, with G acting on $f \in \text{map}(G, X)$ as

$$(g \cdot f)(h) = f(hg)$$

In particular, U preserves both limits and colimits, meaning that *limits and colimits of G-spaces are created on the underlying spaces.* The forgetful functor is a special case of the following definition: Given $\alpha : K \to G$ a (continuous) map of groups, denote with $\alpha^* : G\text{-Top} \to K\text{-Top}$ the functor giving to a G-space X the K-action

$$k \cdot x = \alpha(k) \cdot x$$

If $\alpha = i$, the inclusion of a subgroup $K \leq G$, we might also denote i^* by res^G_K. Define then the following two functors

$$G \times_K - : K\text{-Top} \to G\text{-Top}, \qquad \text{map}^K(G, -) : K\text{-Top} \to G\text{-Top}$$

Where $G \times_K -$ takes a K-space X and returns $G \times_K X = (G \times X)/\sim$, where $\sim$ identifies $(\alpha(k)g, x)$ with $(g, k \cdot x)$ (we can think of $G \times_K X$ as the tensor product of the two K-spaces G, X). On the other hand, $\text{map}^K(G, -)$ maps X into the space of K-equivariant maps $G \to X$.

Theorem 1.1. *$G \times_K -$ is left adjoint to α^*, which is left adjoint to* $\text{map}^K(G, -)$.

Given a G-map $f : X \to Y$ and $H \leq G$, there is a (continuous) restriction $f^H : X^H \to Y^H$, where X^H are the points $\in X$ fixed by all elements in H (with the subspace topology). The slogan of equivariant homotopy theory is that we reduce to the non-equivariant case by looking at the fixed points for all (closed) subgroups of G.

Definition 1.1. A map $f : X \to Y$ is a *G-weak equivalence* if f^H is a weak homotopy equivalence (of spaces), for all $H \leq G$.

Definition 1.2. A *relative G-CW complex* is a pair (X, A) of G-spaces admitting a filtration by closed G-subspaces $A = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X$ such that

- $X = \bigcup_{n \geq 0} X_n$ and X has the weak topology coming from the filtration (i.e. $D \subseteq X$ is open if $D_n = X_n \cap D$ is open, for all n).
- For all $n \geq 0$, there is a G-set I and pushout square in G-Top

$$\begin{array}{ccc} I \times \partial D^n & \longrightarrow & X_{n-1} \\ {\scriptstyle\subseteq}\downarrow & & \downarrow{\scriptstyle\subseteq} \\ I \times D^n & \xrightarrow{c} & X_n \end{array}$$

where each Δ^n has the trivial G-action. For each $i \in I$, the induced map $c_i : D^n \to I \times D^n \xrightarrow{c} X_n$ is called the *i-th characteristic map*. The images of the interior of D^n under any characteristic map are called *open cells*.

Theorem 1.2 (Illman)**.** *Let G be a finite group. Then every smooth, compact G-manifold admits a G-equivariant triangulation (in particular, it's a G-CW complex).*

Remark 1.2. Let $s\mathcal{S}et$ be the category of simplicial sets, then we can form the category G-$s\mathcal{S}et$ of G-simplicial sets. Since geometric realization commutes with finite products, it induces a functor G-$s\mathcal{S}et \to G$-Top with image inside G-CW complexes. The skeleta filtration of $|X|$ is given by $X_n = |\mathrm{sk}_n X|$. Like in non-equivariant homotopy theory, the adjunction between the simplicial nerve and geometric realization induces an equivalence between the ∞-categorical localization of G-spaces at G-weak equivalences and G-simplicial sets and maps $f : X \to Y$ such that $|f| : |X| \to |Y|$ is a G-weak equivalence.

Recall the following result from non-equivariant unstable homotopy theory

Theorem 1.3. *Given a continuous map $f : X \to Y$ of spaces and $n \in \mathbb{N} \cup \{\infty\}$, the following are equivalent:*

1. *For all $0 \leq k < n$, the map $f_* : \pi_k(X) \to \pi_k(Y)$ is bijective and surjective for $k = n$.*
2. *For all $k \leq n$ and diagrams like the following*

$$\begin{array}{ccc} \partial D^k & \xrightarrow{\alpha} & X \\ \downarrow & \overset{\lambda}{\nearrow} & \downarrow{\scriptstyle f} \\ D^n & \xrightarrow[\beta]{} & Y \end{array}$$

a lifting λ exists such that the upper triangle commutes (on the nose) and the lower triangle commutes up to homotopy (relative to ∂D^k).

A map f satisfying any of the equivalent conditions above is called *n-connected* and a ∞-connected morphism is simply a weak homotopy equivalence.

Theorem 1.4. *A relative G-CW complex (X, A) has the G-homotopy extension property, i.e. for every G-map $f : X \to Y$ and G-homotopy $H : A \times \Delta^1 \to Y$ between $f|_A = H(-, 0)$ and $g = H(-, 1)$, there is a homotopy $\bar{H} : X \times \Delta^1 \to Y$ such that $\bar{H}(-, 0) = f$ and $\bar{H}|_{A \times \Delta^1} = H$.*

Proof. Let $\mathcal{C}$ be the class of maps in G-Top having the G-HEP. This class is closed under disjoint union, pushout (clear using that $- \times \Delta^1$ preserves colimits) and transfinite compositions. To show transfinite compositions, assume we're working with subspace inclusions $i_n : X_n \subseteq X_{n+1}$, starting from $A = X_0$, a G-homotopy $H_0 : A \times \Delta^1 \to Y$ and $f : X \to Y$ (with $X = \bigcup_{n \geq 0} X_n$). Let $f_n = f|_{X_n}$ and assume we defined

$$H_n : X_n \times \Delta^1 \to Y, \qquad H_n(-, 0) = f_n$$

(for $n \geq 0$) extending H_{n-1} (for $n \geq 1$). Apply the G-HEP to the inclusion $X_n \subseteq X_{n+1}$ to get a G-homotopy H_{n+1} extending H_n and such that $H_{n+1}(-, 0) = f_{n+1}$. Finally, define $H : X \times \Delta^1 \to Y$ as

$$(x, t) \mapsto H_n(x, t), \text{ if } x \in X_n$$

This is continuous and well defined and satisfies the condition $H(-, 0) = f$ and $H|_{A \times \Delta^1} = H_0$.

Finally, because of the above closure property of $\mathcal{C}$, we only need to show the theorem for inclusions $i : G/H \times \partial D^n \subseteq G/H \times D^n$ to conclude that $\mathcal{C}$ contains inclusions $A \subseteq X$ of relative G-CW complexes. But then a G-equivariant map $\beta : G/H \times D^n \to Y$ and G-homotopy $H : G/H \times \partial D^n \times \Delta^1 \to Y$ correspond to unique maps (of spaces) $D^n \to Y^H$ and $\partial D^n \times \Delta^1 \to Y^H$, so we can apply the (non-equivariant) HEP to get an homotopy $\bar{H} : D^n \times \Delta^1 \to Y^H$ which corresponds to a G-equivariant homotopy $G/H \times D^n \times \Delta^1 \to Y$ satisfying the desired conditions (easy check). $\square$

Consider now a function $\underline{n}$ defined on the set of (closed) subgroups of G into $\mathbb{N} \cup \{\infty\}$ such that

$$\underline{n}(H) = \underline{n}({}^gH), \text{ for all } H \leq G, g \in G$$

where ${}^gH = gHg^{-1}$. We also write $\underline{\infty}$ for the constant map $= \infty$ on all subgroups.

Remark 1.3. As an example, consider X a G-CW complex. Define the *dimenion function* $\underline{dim}(X)$ by mapping H into the dimension (as a normal CW complex) of X^H. Then, multiplication by g induces a (cellular) homeomorphism between H-fixed points and ${}^gH$-fixed points, hence these have the same dimension.

Definition 1.3. Consider a function $\underline{n}$, then a map $f : X \to Y$ is called $\underline{n}$*-connected* if $f^H : X^H \to Y^H$ is $\underline{n}(H)$-connected.

Consider a relative G-CW complex (X, A) and a map $\phi : A \to Y$, we denote by $[X, Y; \phi]_G$ the set of G-homotopy classes relative to A of maps $X \to Y$ (for $A = \emptyset$, we simply write $[X, Y]_G$). Now, we want to establish a connection between G-homotopy equivalences and G-weak equivalences in the context of G-CW complexes. The goal is to prove a Whitehead's Theorem type of result in the equivariant world and is based on the following technical result:

Theorem 1.5. *Let (X, A) be a relative G-CW complex, $\underline{n}$ a conjugation-invariant map. Let $f : Y \to Z$ be a G-map that is $\underline{n}$-connected and $\phi : A \to Y$ a G-map. Then, the induced map*

$$f_* : [X, Y; \phi]_G \to [X, Z; f\phi]_G$$

is surjective if $\underline{dim}(X, A) \leq \underline{n}$ and bijective if $\underline{dim}(X, A) < \underline{n}$.

Proof. We begin by proving surjectivity when (X, A) is a finite relative CW-complex (so we obtain X by attaching cells to A in finite many dimensions). We start induction with $dim(X, A) = -1$, so $X = A$ and there's nothing to prove. In general, let $dim(X, A) = n + 1$ and $X' = X_n$, then (X', A) is a relative CW complex of dimension $\leq n$ and X is obtained from X' by attaching equivariant n-cells.

Consider a class $[v] \in [X, Z; f\phi]_G$ with representative v. By applying the inductive hypothesis, we conclude that there is $s : X' \to Y$ such that fs is G-homotopic (relative A) to $v' = v|_{X'}$. By applying the G-HEP to the pair (X, X'), we conclude that v is homotopic (rel. A) to $\bar{v}$ such that $\bar{v}|_{X'} = fs$ (and so, up to changing v, we might assume $v|_{X'} = fs$). Let then X be obtained from X' by attaching a cell along a G-map $G/H \times \partial D^n \to X'$ and consider the following diagram

$$\begin{array}{ccccc} G/H \times \partial D^n & \longrightarrow & X' & \xrightarrow{s} & Y \\ \downarrow & & \downarrow & & \downarrow f \\ G/H \times D^n & \longrightarrow & X & \xrightarrow{v} & Z \end{array}$$

This diagram is equivalent to the diagram (of spaces)

$$\begin{array}{ccccc} \partial D^n & \longrightarrow & (X')^H & \longrightarrow & Y^H \\ \downarrow & & \downarrow & & \downarrow f^H \\ D^n & \longrightarrow & X^H & \longrightarrow & Z^H \end{array}$$

At this point we apply the hypothesis on the dimension. Namely, since X has a n-dimensional H-equivariant cell, then $n \leq \underline{dim}(X, A)(H) \leq \underline{n}(H)$ and we can apply Theorem 1.3 to conclude that the outer square in the above diagram has a lift $\lambda : D^n \to Y^H$, which induces a map $\lambda' : G/H \times D^n \to Y$ that can be glued to s to form a morphism $s' : X \to Y$, since the diagram

$$\begin{array}{ccc} G/H \times \partial D^n & \longrightarrow & X' \\ \downarrow & & \downarrow s \\ G/H \times D^n & \xrightarrow{\lambda} & Y \end{array}$$

commutes (by the properties required on the lift λ) and such that $fs' \simeq v$. The homotopy $fs' \simeq v$ is defined as the identity on X', while on $G/H \times D^n$ is defined as the extension of the homotopy $H : D^n \times \Delta^1 \to Y^H$ between $f^H\lambda$ and $v^H|_{D^n}$ (again, in the conditions on the lift). These two homotopies glue together since H is the identity on ∂D^n.

In general, consider a filtration $A = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X$ and $v : X \to Z$ such that $v|_A = f\phi$. Then we construct a sequence of maps $\phi = \psi_{-1}, \psi_0, \psi_1, \cdots$ such that $\psi_n : X_n \to Y$ is an extension of ψ_{n-1} and

(1) $f\psi_n$ is G-homotopic (rel. X_{n-1}) to $v_n = v|_{X_n}$.

Clearly $n = -1$ is already defined. Take $n \geq -1$ such that ψ_n is defined, then using the G-HEP for the pair (X, X_n), we can homotopy v to a map $\bar{v} : X \to Z$ such that $\bar{v}|_{X_n} = f\psi_n$, so that up to exchanging v with $\bar{v}$, we can assume that v_{n+1} represents a class in $[X_{n+1}, Z; f\psi_n]$. Since $\underline{dim}(X_{n+1}, X_n) \leq \underline{dim}(X, A) \leq n$, we can apply surjectivity for finite G-CW complexes and conclude that there is $\psi_{n+1} : X_{n+1} \to Y$ such that $\psi_{n+1}|_{X_n} = \psi_n$ and $f\psi_{n+1}$ is homotopic (rel X_n) to v_{n+1}. All these maps ψ_n glue into a map $\psi : X \to Y$ and by (1) all the G-homotopies $f\psi_n \simeq v_n$ can also be glued into an homotopy $f\psi \simeq v$.

Finally, we show injectivity as a consequence of surjectivity. Consider $s, s' : X \to Y$ and a G-homotopy H (rel. A) between fs and fs'. Consider then the relative G-CW complex

$$(X \times \Delta^1, X \times \partial\Delta^1 \cup A \times \Delta^1)$$

and define the map $\xi : X \times \partial\Delta^1 \cup A \times \Delta^1 \to Y$ as $s \cup s'$ on $X \times \partial\Delta^1$ and constant equal to ϕ on $A \times \Delta^1$, then the homotopy H represents an element of $[X \times \Delta^1, Z; f\xi]_G$. Then we can apply surjectivity to f_* since

$$\underline{dim}(X \times \Delta^1, X \times \partial\Delta^1 \cup A \times \Delta^1) = \underline{dim}(X, A) + 1 \leq \underline{n}$$

which implies the existence of a G-homotopy $\bar{H} : X \times \Delta^1 \to Y$ extending ξ, that is a G-homotopy (rel. A) between s, s', concluding injectivity. □

An immediate corollary of the previous theorem is the following (notice that f is $\underline{\infty}$-connected map if and only if f is a G-weak equivalence).

Corollary 1.1. *Let (X, A) be a relative G-CW complex and $f : Y \to Z$ a G-weak equivalence. The map $f_* : [X, Y; \phi]_G \to [X, Z; f\phi]_G$ is bijective, for any $\phi : A \to Y$.*

Corollary 1.2. *Let $f : X \to Y$ be a G-weak equivalence between G-spaces admitting a G-CW structure, then f is a G-homotopy equivalence.*

Proof. The proof goes as the one for Whitehead's theorem for non-equivariant spaces. Since $f_* : [Y, X]_G \to [Y, Y]_G$ is bijective, there is a G-map $g : Y \to X$ such that fg is G-homotopic to id. Then $f_*[\mathrm{id}] = [f] = [fgf] = f_*[gf]$, so we can apply injectivity to conclude gf is G-homotopic to the identity, so g is a G-homotopy inverse to f. On the other hand, clearly if f is a G-homotopy equivalence, then it is also a G-weak equivalence. □

2. Orbit Category And Fixed Points Data

A $\mathcal{O}_G$*-space* is a functor (enriched over Top) $\mathcal{O}_G^{op} \to$ Top. Denote the functor category of $\mathcal{O}_G$-spaces as $\mathcal{O}_G$-Top and notice that, since $\mathcal{O}_G$ is small and Top is cofibrantly generated, the projective model structure exists on $\mathcal{O}_G$-Top, which is defined by declaring

1. A morphism is a weak equivalence if it is point-wise a weak equivalence.
2. A morphism is a fibration if it is point-wise a Serre fibration.
3. A morphism is a cofibration if it has the LLP with respect to all acyclic fibrations.

Remark 2.1. $\mathcal{O}_G$-Top is also cofibrantly generated, with cofibrations being the smallest class containing $L \times \partial D^n \hookrightarrow L \times D^n$ (for all n and representable functors L) and closed under isomorphisms, pushouts, disjoint unions and transfinite compositions.

Let X be a G-space, then define $\bar{X} : \mathcal{O}_G \to \mathrm{Top}$ as $G/H \mapsto X^H$. This induces a functor $\phi : G\text{-Top} \to \mathcal{O}_G\text{-Top}$ and it's immediate to see that a G-weak equivalence is sent to a weak equivalence of $\mathcal{O}_G$-spaces.

Theorem 2.1 (Elmendorff)**.** *The functor ϕ induces an equivalence between the ∞-categorical localizations at the respective weak equivalences.*

Proof. We put on G-Top the Quillen model structure making ϕ into a right Quillen functor. On the other hand, we define a functor $\lambda : \mathcal{O}_G\text{-Top} \to G\text{-Top}$ sending $F \mapsto F(G)$ (that is, λ is induced by the inclusion $\mathbf{B}G \hookrightarrow \mathcal{O}_G$). The G-action is given by the functoriality of F and maps $R_g : h \mapsto hg$. Let $\pi : G \to G/H$ be the projection, then $\pi R_g = \pi$, hence $\pi^* : F(G/H) \to F(G)$ lands inside $F(G)^H$, which induces a morphism $\eta_F : F \to \phi\lambda(F)$. Conversely, for every G-space X, we have $\lambda\phi(X) = X$ (in particular, ϕ is fully faithful).

The essential image of ϕ are the functors F such that the map $F(G/H) \to F(G)^H$ is a homeomorphism. Then we claim that *for every cofibrant $\mathcal{O}_G$-object F, the transformation η_F is an isomorphism.* Again we define $\mathcal{C}$ the class of $\mathcal{O}_G$-spaces F for which η_F is an isomorphism, this is closed under isomorphisms, disjoint unions, pushouts and transfinite compositions (along object-wise closed embeddings). This reduces the proof of the claim to check to check it for $F = L \times S$, for any L representable and S a space (with trivial G-action), but then let $L \simeq \mathcal{O}_G(-, G/K)$, then

$$\begin{aligned} F(G)^H &= \mathcal{O}_G(G, G/K)^H \times S \\ &\simeq (G/K)^H \times S \\ &\simeq \mathcal{O}_G(G/H, G/K) \times S \\ &= F(G/H) \end{aligned}$$

proving the claim. ☐

In general, the category $\mathcal{O}_G$-Top is useful when we want to define a G-space based on its behaviour on fixed points. For example, a *family of subgroups* of G is a set $\mathcal{F}$ of subgroups that is closed under conjugation and subgroups.

Definition 2.1. Given a family of subgroups $\mathcal{F}$, a *universal G-space* for $\mathcal{F}$ is a cofibrant G-space $E\mathcal{F}$ such that $(E\mathcal{F})^H = \emptyset$ if $H \notin \mathcal{F}$ and $(E\mathcal{F})^H \simeq *$ if $H \in \mathcal{F}$.

Theorem 2.2. *For every family $\mathcal{F}$, there is a universal G-space $E\mathcal{F}$, unique up to G-homotopy equivalence. Moreover, let X be a G-CW complex, if X is such that $G_x = \{g \in G | gx = x\} \in \mathcal{F}$, for all x, then $[X, E\mathcal{F}]_G \simeq *$, otherwise $[X, E\mathcal{F}]_G = \emptyset$.*

Proof. A said above, we first define $E \in \mathcal{O}_G$-Top as the $\mathcal{O}_G$-space with $E(G/H) = \emptyset$, if $H \notin \mathcal{F}$, and $= *$, if $H \in \mathcal{F}$. Then we define E' as a cofibrant replacement for E and $E\mathcal{F} := \lambda(E')$. Then EG is cofibrant, since λ is a left Quillen functor and, for all $H \leq G$

$$(E\mathcal{F})^H = \phi\lambda(E')(G/H) \simeq E'(G/H) \simeq E(G/H)$$

where the second equivalence is an isomorphism, coming from E' being cofibrant and the claim in the proof of the previous theorem, the third equivalence is by definition of cofibrant replacement.

Consider X as in the statement of the theorem. If $G_x \notin \mathcal{F}$, then there cannot be a map $X \to E\mathcal{F}$, since x would land inside $(E\mathcal{F})^H = \emptyset$. On the other hand, since ϕ induces an equivalence of homotopy categories and X and $E\mathcal{F}$ are fibrant-cofibrant objects, then

$$[X, E\mathcal{F}]_G \simeq \mathrm{Ho}(G\text{-Top})(X, E\mathcal{F}) \simeq \mathrm{Ho}(\mathcal{O}_G\text{-Top})(\phi(X), \phi(E\mathcal{F})) \simeq [\phi(X), \phi(E\mathcal{F})] \simeq *$$

The last equality comes from the fact that pointwise $\phi(E\mathcal{F})$ is equivalent to $*$.

In particular, we can apply the last point characterizing maps into $E\mathcal{F}$ to conclude that it is unique up to a G-homotopy equivalence. ☐

Remark 2.2. If $\mathcal{F} = \{\{e\}\}$, then $E\mathcal{F} \simeq EG$, since G acts freely on EG.

Remark 2.3. Consider a G-representation V (hence a inner product space V with a orthogonal G-action). Let $S(V)$ be the unit sphere in V (which is a G-CW complex by Theorem 1.2). Define

$$S(\infty V) = \bigcup_{n\geq 0} S(nV)$$

where $S(nV) = S(V^n)$. Finally, define $\mathcal{F}_V = \{H \leq G | V^H \neq 0\}$, then we claim $S(\infty V)$ *is a* $E\mathcal{F}_V$*-space*, which we prove as follows

$$\begin{array}{rcl} S(\infty V)^H & \simeq & \bigcup_{n\geq 0} S(nV)^H \\ & = & \bigcup_{n\geq 0} S(nV^H) \\ & \simeq & \begin{cases} \emptyset & H \notin \mathcal{F}_V \\ S^\infty & H \in \mathcal{F}_V \end{cases} \end{array}$$

The first isomorphism comes from $(-)^H$ being representable by a finite set and the final isomorphism $\simeq S^\infty$ comes from nV^H being a linear subspace.