THE BLAKERS-MASSEY THEOREM

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Definition 1. An *excisive triad* (Y, Y_1, Y_2) consists of a topological space Y and two subspaces Y_1, Y_2 of Y such that

$$Y \;=\; \mathring{Y}_1 \cup \mathring{Y}_2 \;,$$

i.e. the open cores of Y_1 and Y_2 cover Y.

The name 'excisive triad' comes from the connection to excision in singular homology. We write $Y_0 = Y_1 \cap Y_2$ for the intersection of the subspaces in an excisive triad (Y, Y_1, Y_2). Then excision in homology shows that for every coefficient group A and all $n \geq 0$ the inclusion induces an isomorphism

$$H_n(Y_2, Y_0; A) \;\longrightarrow\; H_n(Y, Y_1; A) \;.$$

Indeed, if we set $C = Y - Y_1$, then the excisiveness is equivalent to $\bar{C} \subseteq \mathring{Y}_2$, and moreover $Y_2 - A = Y_2 \cap Y_1 = Y_0$.

The excision property is a key ingredient in the identification of cellular and singular homology for CW-complexes, and hence one of the main reasons why homology can often be effectively calculated. Among other things, excision produces the long exact *Mayer-Vietoris sequence* of an excisive triad:

$$\cdots \;\longrightarrow\; H_n(Y_0, A) \;\longrightarrow\; H_n(Y_1, A) \oplus H_n(Y_2, A) \;\longrightarrow\; H_n(Y, A) \;\xrightarrow{\partial}\; H_{n-1}(Y_0, A) \;\longrightarrow\; \cdots$$

All maps except for the connecting homomorphism are induced by the inclusions, with a sign thrown in at one point.

Homotopy groups, on the other hand, do *not* satisfy excision, which is why in general they are much harder to calculate than homology groups. The Blakers-Massey theorem states that homotopy groups do satisfy excision *in range of dimensions* that is roughly the sum of the connectivities of the pairs (Y, Y_1) and (Y, Y_2). So the Blakers-Massey theorem often goes under the name *homotopy excision theorem*. Here is the precise statement:

Theorem 2 (Blakers-Massey theorem)**.** *Let (Y, Y_1, Y_2) be an excisive triad such that $Y_0 = Y_1 \cap Y_2$ is non-empty. Let $p, q \geq 1$ be such that*

$$\pi_i(Y_1, Y_0, y_0) \;=\; 0 \quad \text{for } 0 < i < p, \text{ and}$$
$$\pi_i(Y_2, Y_0, y_0) \;=\; 0 \quad \text{for } 0 < i < q$$

for every basepoint $y_0 \in Y_0$. Then for every $y_0 \in Y_0$ the map

$$\pi_i(Y_2, Y_0, y_0) \;\longrightarrow\; \pi_i(Y, Y_1, y_0)$$

induced by the inclusions is bijective for $1 \leq i < p + q - 2$ and surjective for $i = p + q - 2$.

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This theorem was first proved under slightly weaker hypothesis in the paper

A. L. Blakers, W. S. Massey, *The homotopy groups of a triad. II.* Ann. of Math. (2) **55** (1952), 192–201.

Blakers and Massey use simplicial approximation and general position arguments. The proof given in Section 4.2 of Hatcher's book *Algebraic Topology* is based on the same techniques.

A completely different proof is explained in Section 6.4 of tom Dieck's book with the same name *Algebraic Topology*; tom Dieck attributes the proof to Puppe. This proof is by elementary homotopy theory techniques, and is possibly the most direct argument.

We give yet another proof that is essentially an expansion of Lemma 1 in

M. Mather, *Hurewicz theorems for pairs and squares.* Math. Scand. 32 (1973), 269–272.

This proof, however, only works if the connectivities satisfy the stronger hypothesis $p, q \geq 3$. The advantage of this proof is that is uses previously established tools (mainly the Hurewicz theorem and homotopy fibers) to reduce the theorem to one main ingredient, namely an estimate of the connectivity of the inclusion $B \vee C \longrightarrow B \times C$ of a wedge into a product, in terms of the connectivities of B and C, see Proposition 8.

1. Homotopy invariance for excisive triads

We recall that a continuous map $f : A \longrightarrow B$ is *n-connected*, for $n \geq 0$, if the following equivalent conditions hold:

- for every basepoint in $a \in A$, the map $f_* : \pi_i(A, a) \longrightarrow \pi_i(B, f(a))$ is bijective for all $0 \leq i < n$ and surjective for $i = n$;
- for every basepoint $b \in B$, the homotopy fiber $\mathrm{hofib}_b(f)$ is $(n-1)$-connected;
- for all $0 \leq i \leq n$ and every commutative diagram

$$\begin{array}{ccc} \partial D^i & \xrightarrow{\Psi} & A \\ {\scriptstyle \mathrm{incl}}\downarrow & \overset{\lambda}{\nearrow} & \downarrow{\scriptstyle f} \\ D^i & \xrightarrow[\varphi]{} & B \end{array}$$

there is a map $\lambda : D^n \longrightarrow A$ such that $\lambda|_{\partial D^n} = \Psi$ and $f\lambda$ is homotopic, relative ∂D^n, to φ.

The following theorem is a powerful homotopy invariance property for excisive triads:

Theorem 3. *Let (Y, Y_1, Y_2) and (Z, Z_1, Z_2) be excisive pairs and $f : Y \longrightarrow Z$ a continuous map with $f(Y_1) \subseteq Z_1$ and $f(Y_2) \subseteq Z_2$. Suppose that for some $n \geq 0$*

$$\begin{aligned} f|_{Y_1} &: Y_1 \longrightarrow Z_1 \quad \text{is } n\text{-connected,} \\ f|_{Y_2} &: Y_2 \longrightarrow Z_2 \quad \text{is } n\text{-connected, and} \\ f|_{Y_1 \cap Y_2} &: Y_1 \cap Y_2 \longrightarrow Z_1 \cap Z_2 \quad \text{is } (n-1)\text{-connected.} \end{aligned}$$

Then f is n-connected.

The case $n = \infty$ of Theorem 3 is worth making explicit. Indeed, a continuous map is a weak homotopy equivalence if and only if it is n-connected for all $n \geq 0$, which gives the following corollary.

Corollary 4. *Let (Y, Y_1, Y_2) and (Z, Z_1, Z_2) be excisive pairs and $f : Y \longrightarrow Z$ a continuous map with $f(Y_1) \subseteq Z_1$ and $f(Y_2) \subseteq Z_2$. Suppose that the three restrictions*

$$f|_{Y_1} : Y_1 \longrightarrow Z_1 \ , \quad f|_{Y_2} : Y_2 \longrightarrow Z_2 \quad \text{and} \quad f|_{Y_1 \cap Y_2} : Y_1 \cap Y_2 \longrightarrow Z_1 \cap Z_2$$

are weak homotopy equivalences. Then f is a weak homotopy equivalence.

A proof of Theorem 3 can be found as Theorem 6.7.9 of tom Dieck's *Algebraic Topology.* Corollary 4 is also proved as Lemma 16.24 of

B. Gray, *Homotopy theory. An introduction to algebraic topology.*
Pure and Applied Mathematics, Vol. 64. Academic Press, 1975. xiii+368 pp.

In fact, Gray's and tom Dieck's arguments are essentially the same, and Gray's proof also works to prove the stronger version Theorem 3. Since Theorem 3 is well documented in the literature, we do not reproduce the proof here.

Here is another interesting corollary of Theorem 3.

Corollary 5. *Let (Y, Y_1, Y_2) be an excisive triad. If the pair (Y_2, Y_0) is n-connected for some $n \geq 0$, then so is the pair (Y, Y_1).*

Proof. The triple (Y_1, Y_1, Y_0) is an excisive triad for tautological reasons, and the inclusion $Y_1 \longrightarrow Y$ satisfies the hypotheses of Theorem 3 for (Y, Y_1, Y_2) as the target triad. So Theorem 3 specializes to the conclusion that the pair (Y, Y_1) is n-connected. □

2. Connectivity of wedge to product

As a warm up we spell out a special case of the Blakers-Massey theorem which we then prove directly. This special case will then enter into the proof of the general case.

We let B and C be topological spaces with basepoints $b_0 \in B$ and $c_0 \in C$. We define the *long wedge* by

$$B \tilde{\vee} C \;=\; B \cup_{b_0} [0,1] \cup_{c_0} C \;=\; (B \amalg [0,1] \amalg C)/\sim$$

where the equivalence relation identifies $b_0 \in B$ with $0 \in [0,1]$ and it identifies $c_0 \in C$ with $1 \in [0,1]$. We cover the long wedge by the open subspaces

$$Y_1 \;=\; B \cup_{b_0} [0,1) \qquad \text{and} \qquad Y_2 \;=\; (0,1] \cup_{c_0} C \ .$$

The intersection $Y_0 = Y_1 \cap Y_2 = (0,1)$ is an open interval, hence contractible. For every point $y_0 \in Y_0$ the relative homotopy groups $\pi_i(Y_1, Y_0, y_0)$ are thus isomorphic to the absolute homotopy groups $\pi_i(Y_1, y_0)$. Moreover, the half-open interval in Y_1 can be contracted to 0, which shows that the inclusion $B \longrightarrow Y_1$ is a homotopy equivalence. So altogether

$$\pi_i(Y_1, Y_0, y_0) \;\cong\; \pi_i(Y_1, y_0) \;\cong\; \pi_i(B, b_0) \ .$$

Similarly,

$$\pi_i(Y_2, Y_0, y_0) \;\cong\; \pi_i(C, c_0) \ .$$

Now we assume that B is $(p-1)$-connected and C is $(q-1)$-connected, for $p, q \geq 1$. Then the hypothesis of the Blakers-Massey theorem are satisfied. The conclusion is equivalent to the claim that the map

$$\pi_i([0,1] \cup_{c_0} C, 0) \longrightarrow \pi_i(B \tilde{\vee} C, B, b_0)$$

induced by the inclusion is bijective for $1 \leq i < p+q-2$ and surjective for $i = p+q-2$.

The main reason for using the long wedge instead of the actual wedge (one-point union) is the following homotopy invariance property.

Proposition 6. *Let $f : B \longrightarrow B'$ and $g : C \longrightarrow C'$ be weak equivalences of topological spaces. Then the map*

$$f \tilde{\vee} g \ : \ B \tilde{\vee} C \longrightarrow B' \tilde{\vee} C'$$

is a weak equivalence.

Proof. We apply Corollary 4 to the standard open cover of the long wedge $Y = B \tilde{\vee} C$, namely the open subsets

$$Y_1 \ = \ B \cup_{b_0} [0,1) \qquad \text{and} \qquad Y_2 \ = \ (0,1] \cup_{c_0} C \ ,$$

and we use the analogous open cover for $Z = B' \tilde{\vee} C'$.

In the commutative square

$$\begin{array}{ccc} B & \xrightarrow{\quad f \quad} & B' \\ {\scriptstyle \mathrm{incl}} \downarrow & & \downarrow {\scriptstyle \mathrm{incl}} \\ Y_1 = B \cup_{b_0} [0,1) & \xrightarrow[f \cup [0,1)]{} & B' \cup_{b_0} [0,1) = Z_1 \end{array}$$

the two vertical inclusions are homotopy equivalences, and f is a weak equivalence by hypothesis. So the restriction of $f \tilde{\vee} g$ to a map $Y_1 \longrightarrow Z_1$ is a weak equivalence, and similarly for the restriction to a map $Y_2 \longrightarrow Z_2$. On the other hand, $Y_1 \cap Y_2$ and $Z_1 \cap Z_2$ are homeomorphic to the open interval $(0,1)$, and the restriction of $f \tilde{\vee} g$ to a map $Y_1 \cap Y_2 \longrightarrow Z_1 \cap Z_2$ is even a homeomorphism. So $f \tilde{\vee} g$ satisfies the hypotheses of Corollary 4. □

The next result shows that if the basepoint b_0 'sits nicely' in B, then the long wedge is weakly homotopy equivalent to the wedge. Here we offer two interpretations of 'sitting nicely inside', namely by the homotopy extension property, or as a neighborhood deformation retract. All points in a CW-complex satisfy both of these 'niceness' criteria, so for CW-complexes, long wedges and wedges are weakly equivalent.

Proposition 7. (i) *Suppose that the basepoint b_0 of B is* non-degenerate*, i.e., the inclusion $\{b_0\} \longrightarrow B$ has the homotopy extension property. Then for every based space (C, c_0) the map*

$$c \ : \ B \tilde{\vee} C \longrightarrow B \vee C$$

that collapses the interval to the basepoint of $B \vee C$ is a homotopy equivalence.

(ii) *Suppose that the set $\{b_0\}$ is closed in B and a neighborhood deformation retract. Then for every based space (C, c_0) the map*

$$c \ : \ B \tilde{\vee} C \longrightarrow B \vee C$$

that collapses the interval to the basepoint of $B \vee C$ is a weak homotopy equivalence.

Proof. (i) We construct a continuous map in the other direction. Since the basepoint inclusion into B has the HEP, we can choose a homotopy

$$H \ : \ B \times [0,1] \ \longrightarrow \ B\tilde{\vee}C$$

that starts with the inclusion of B into $B\tilde{\vee}C$ and such that

$$H(b_0,-) \ : \ [0,1] \ \longrightarrow \ B\tilde{\vee}C$$

is the inclusion of the interval. The map

$$H(-,1) \ : \ B \ \longrightarrow \ B\tilde{\vee}C$$

then sends b_0 to c_0, so the map

$$s \ : \ H(-,1) \cup \mathrm{incl} \ : \ B \vee C \ \longrightarrow \ B\tilde{\vee}C$$

is well-defined at the gluing point and continuous.

We claim that s is homotopy inverse to the collapse map c, but we leave that as an exercise.

(ii) We let U be a neighborhood of b_0 inside B that can be contracted into b_0. We use the excisive cover of $Y = B\tilde{\vee}C$ by the subsets

$$Y_1 \ = \ B \qquad \text{and} \qquad Y_2 \ = \ U\tilde{\vee}C \ ,$$

and we use the excisive cover of $Z = B \vee C$ by the subsets

$$Z_1 \ = \ B \qquad \text{and} \qquad Z_2 \ = \ U \vee C \ .$$

Then c restricts to homeomorphisms $c|_{Y_1} : Y_1 \cong Z_1$ and $c|_{Y_1 \cap Y_2} : Y_1 \cap Y_2 \cong Z_1 \cap Z_2$. On the other hand, the inclusion of C into Z_2 is a homotopy equivalence since U can be contracted into $\{b_0\}$. The inclusion of C into Y_2 is also a homotopy equivalence since $U\tilde{\vee}C$ can be contracted into $[0,1] \cup_{c_0} C$, which can then be contracted further into C. So the restriction $c|_{Y_2} : Y_2 \longrightarrow Z_2$ is a homotopy equivalence, and Corollary 4 gives the desired conclusion. □

We will now show this special case of the Blakers-Massey theorem directly. The main substance of the special case is the following connectivity estimate for a canonical map

$$j \ : \ B\tilde{\vee}C \ \longrightarrow \ B \times C$$

from a long wedge to a product. The map j is defined as the union of the maps

$$(-,c_0) \ : \ B \ \longrightarrow \ B \times C \ , \quad (b_0,-) \ : \ C \ \longrightarrow \ B \times C$$

and the constant map sending $[0,1]$ to (b_0,c_0).

Proposition 8. *Let B be a $(p-1)$-connected space and C a $(q-1)$-connected space, for some $p,q \geq 1$. Then for every point $b_0 \in B$ and every point $c_0 \in C$ the map*

$$j \ : \ B\tilde{\vee}C \ \longrightarrow \ B \times C$$

is $(p+q-1)$-connected.

Proof. Since B is $(p-1)$-connected we can choose a CW-approximation $f : \bar{B} \longrightarrow B$ with a single 0-cell and all other cells of dimension at least p. We can also arrange things so that f takes the 0-cell to b_0. Similarly, there is a CW-approximation $g : \bar{C} \longrightarrow C$ with a single 0-cell mapping to c_0 and all other cells of dimension at least q. We base B and C at their 0-cells.

We give the product $\bar{B} \times \bar{C}$ the compactly generated topology. In the commutative square

$$\begin{array}{ccc} \bar{B}\tilde{\vee}\bar{C} & \xrightarrow{j} & \bar{B} \times \bar{C} \\ {\scriptstyle f\tilde{\vee}g}\downarrow & & \downarrow{\scriptstyle f\times g} \\ B\tilde{\vee}C & \xrightarrow[j]{} & B \times C \end{array}$$

the left vertical map is a weak equivalence by Proposition 6. The right vertical map is a weak equivalence because homotopy groups commute with products and the change-of-topology map on $\bar{B} \times \bar{C}$ is a weak equivalence.

So it suffices to show that the upper horizontal map is $(p+q-1)$-connected. The top map j factors as the composite

$$\bar{B}\tilde{\vee}\bar{C} \ \xrightarrow{c}\ \bar{B} \vee \bar{C} \ \longrightarrow\ B \times C \ ,$$

where the first map collapses the interval $[0,1]$ to the wedge point; this map is a homotopy equivalence by Proposition 7 since all 0-cells in a CW-complex are non-degenerate.

The CW-structures on $\bar{B}$ and $\bar{C}$ induce a CW-structure on the product $\bar{B} \times \bar{C}$ (in the compactly generated topology) with filtration

$$(\bar{B} \times \bar{C})_k \ = \ \bigcup_{i+j=k} \bar{B}_i \times \bar{C}_j \ .$$

All cells of $\bar{B} \times \bar{C}$ that do not belong to $\bar{B} \vee \bar{C}$ then have dimension at least $p+q$. So the inclusion $\bar{B} \vee \bar{C} \longrightarrow \bar{B} \times \bar{C}$ is $(p+q-1)$-connected by cellular approximation. □

Now we are ready to prove the special case of the Blakers-Massey theorem.

Proposition 9. *Let B be a $(p-1)$-connected space and C a $(q-1)$-connected space, for some $p, q \geq 1$. Then for all basepoints $b_0 \in B$ and $c_0 \in C$, the map*

$$\pi_i([0,1] \cup_{c_0} C, 0) \ \longrightarrow\ \pi_i(B\tilde{\vee}C, B, b_0)$$

induced by the inclusion is bijective for $1 \leq i \leq p+q-2$.

Proof. We know from Proposition 8 that the map $j : B\tilde{\vee}C \longrightarrow B \times C$ is $(p+q-1)$-connected. By comparing the long exact homotopy group sequences of the pairs $(B\tilde{\vee}C, B)$ and $(B \times C, B \times \{c_0\})$ we see that the induced map

$$j_* \ : \ \pi_i(B\tilde{\vee}C, B, b_0) \ \longrightarrow\ \pi_i(B \times C, B \times \{c_0\}, (b_0, c_0))$$

on relative homotopy groups is bijective for $i \leq p+q-2$. So it suffices to show that the composite

$$\pi_i(C, c_0) \ \longrightarrow\ \pi_i(B\tilde{\vee}C, B, b_0) \ \xrightarrow{j_*}\ \pi_i(B \times C, B \times \{c_0\}, (b_0, c_0)) \tag{10}$$

is bijective for all $i \geq 1$. The projection of $B \times C$ to the second factor induces retraction

$$\pi_i(B \times C, B \times \{c_0\}, (b_0, c_0)) \longrightarrow \pi_i(C, c_0) ,$$

so the composite (10) is injective.

For surjectivity we consider any continuous map

$$(\beta_B, \beta_C) \; : \; D^i \longrightarrow B \times C$$

that represents an element of the target of (10). The map must take S^{i-1} into $B \times \{c_0\}$ and the basepoint $z \in S^{i-1}$ to (b_0, c_0). The first component β_B is then a continuous map $D^i \longrightarrow B$ with no restriction except that it must send the basepoint to b_0. Since we can contract the disc onto the basepoint z_0, the first component β_B is homotopic to the constant map at b_0. Moreover, combining such a homotopy with the constant homotopy in the second factor shows that the class of (β_B, β_C) is in the image of $\pi_i(C, c_0)$. So the composite (10) is also surjective. □

3. Proof of the Blakers-Massey theorem for $p, q \geq 3$

In this section we prove the Blakers-Massey theorem in the case where the connectivities satisfy the more restrictive hypotheses $p, q \geq 3$.

Proof of Theorem 2 for $p, q \geq 3$. We let (Y, Y_1, Y_2) be an excisive triad such that (Y_1, Y_0) is $(p-1)$-connected and (Y_2, Y_0) is $(q-1)$-connected. In particular, both pairs (Y_1, Y_0) and (Y_2, Y_0) are 2-connected.

We start with another **special case:** we suppose that the space Y is contractible, the space Y_1 is $(q-2)$-connected and the space Y_2 is $(p-2)$-connected. All the hypotheses together imply that $Y_0 = Y_1 \cap Y_2$ is simply connected.

Claim 1: if Y is contractible, then the conclusion of the Blakers-Massey theorem is equivalent to the statement that the diagonal map

$$\Delta \; : \; Y_0 \longrightarrow Y_1 \times Y_2 \; , \quad z \longmapsto (z, z)$$

is $(p+q-3)$-connected.

The proof of Claim 1 uses the commutative diagram, for $i \geq 1$,

$$\begin{array}{ccc}
\pi_i(Y_2, Y_0, y_0) & \longrightarrow & \pi_i(Y, Y_1, y_0) \\
\cong \downarrow & & \cong \downarrow \partial \\
\pi_{i-1}(\mathrm{hofib}_{y_0}(Y_0 \longrightarrow Y_2), *) & \longrightarrow & \pi_{i-1}(Y_1, y_0)
\end{array}$$

in which the vertical maps are bijective and the lower horizontal map is induced by the projection map

$$\psi \; : \; \mathrm{hofib}_{y_0}(Y_0 \longrightarrow Y_2) \longrightarrow Y_1 \; , \quad (\omega, z) \longmapsto z \; .$$

So the conclusion of the Blakers-Massey theorem is equivalent to the assertion that the map ψ is $(p+q-3)$-connected. Equivalently, the homotopy fiber of ψ is $(p+q-4)$-connected.

A point in the homotopy fiber of ψ over y_0 consists of

- a path $\lambda : [0,1] \longrightarrow Y_1$ starting at y_0
- an element (ω, z) in $\mathrm{hofib}_{y_0}(Y_0 \longrightarrow Y_2)$ such that $z = \lambda(1)$.

This entire data is determined by the pair $(\lambda,\omega):[0,1]\longrightarrow Y_1\times Y_2$, subject to the condition that $(\lambda,\omega)(0)=(y_0,y_0)$ and $(\lambda,\omega)(1)\in\Delta(Y_0)$. In other words, the iterated homotopy fiber is homeomorphic to the homotopy fiber of the diagonal map $\Delta:Y_0\longrightarrow Y_1\times Y_2$:

$$\begin{array}{ccc}\mathrm{hofib}_{y_0}(\psi:\mathrm{hofib}_{y_0}(Y_0\longrightarrow Y_2)\longrightarrow Y_1) & \cong & \mathrm{hofib}_{(y_0,y_0)}(\Delta:Y_0\longrightarrow Y_1\times Y_2)\\ ((\omega,z),\lambda) & \longmapsto & ((\omega,\lambda,(z,z))\ .\end{array}$$

So the conclusion of the Blakers-Massey theorem is equivalent to the assertion that the homotopy fiber of Δ is $(p+q-4)$-connected. Equivalently, we have to show that $\Delta:Y_0\longrightarrow Y_1\times Y_2$ is $(p+q-3)$-connected. This proves Claim 1.

By our hypothesis that Y_0,Y_1 and Y_2 are simply connected, so it suffices to show – by the Hurewicz theorem – that map

$$\Delta_*\ :\ \tilde H_n(Y_0)\ \longrightarrow\ \tilde H_n(Y_1\times Y_2)$$

induced by the diagonal map on reduced integral homology is an isomorphism for $0\le n\le p+q-3$. Here, and in the rest of the proof, all homology groups are taken with integral coefficients, which we omit from the notation.

Since Y is contractible, all its reduced homology groups are trivial by the Hurewicz theorem. So the Mayer-Vietoris sequence of the excisive triad degenerates and shows that the map

$$(i^1_*,i^2_*)\ :\ \tilde H_n(Y_0)\ \longrightarrow\ \tilde H_n(Y_1)\oplus\tilde H_n(Y_2) \tag{11}$$

is an isomorphism, where $i^1:Y_0\longrightarrow Y_1$ and $i^2:Y_0\longrightarrow Y_2$ are the inclusions.

Now we let $y_0\in Y_0$ be any point. We form the long wedge $Y_1\tilde\vee Y_2$ with y_0 in the role of both basepoints. The standard covering of $Y_1\tilde\vee Y_2$ by

$$Y_1\cup_{y_0}[0,1)\qquad\text{and}\qquad(0,1]\cup_{y_0}Y_2$$

gives rise to a Mayer-Vietoris sequence. Since the intersection of these two open is the open interval $(0,1)$, hence contractible, this Mayer-Vietoris sequence degenerates into an isomorphism

$$\tilde H_n(Y_1)\oplus\tilde H_n(Y_2)\ \longrightarrow\tilde H_n(Y_1\tilde\vee Y_2)$$

induced by the inclusions of Y_1 and Y_2 into $Y_1\tilde\vee Y_2$.

Since Y_1 is $(q-2)$-connected and Y_2 is $(p-2)$-connected, the map $j:Y_1\tilde\vee Y_2\longrightarrow Y_1\times Y_2$ is $(p+q-3)$-connected by Proposition 8. So j induces isomorphisms in homology up to dimension $p+q-4$ and an epimorphism in H_{p+q-3}. But slightly more is true: the composite

$$\tilde H_n(Y_1)\oplus\tilde H_n(Y_2)\ \xrightarrow{\cong}\tilde H_n(Y_1\tilde\vee Y_2)\ \xrightarrow{j_*}\ \tilde H_n(Y_1\times Y_2)\ \xrightarrow{(p^1_*,p^2_*)}\ \tilde H_n(Y_1)\times\tilde H_n(Y_2) \tag{12}$$

is an isomorphism, where p^1,p^2 denote the projections. So j_* is split injective in all dimensions, and hence j in fact induces isomorphisms in homology up to dimension $p+q-3$.

Combining this with the isomorphism (11) shows that he map

$$\tilde H_n(Y_0)\ \longrightarrow\ \tilde H_n(Y_1\times Y_2)\ ,\quad x\ \longmapsto\ \alpha_*(x)+\beta_*(x) \tag{13}$$

is an isomorphism for all $n \leq p+q-3$, where

$$\alpha \ : \ Y_0 \longrightarrow Y_1 \times Y_2 \ , \quad \alpha(z) \ = \ (z, y_0)$$
$$\beta \ : \ Y_0 \longrightarrow Y_1 \times Y_2 \ , \quad \beta(z) \ = \ (y_0, z) \ .$$

Claim 2: the map (13) coincides with the map induced by the diagonal $\Delta : Y_0 \longrightarrow Y_1 \times Y_2$ in the relevant dimensions $n \leq p+q-3$. Hence the diagonal Δ induces an isomorphism in reduced integral homology up to dimension $p+q-3$.

To prove Claim 2 we use again that the composite (12) is an isomorphism in all dimensions. Since j_* is an isomorphism up to dimension $p+q-3$, the last map (p^1_*, p^2_*) is an isomorphism in those dimensions as well. So we can test the desired relation $\Delta_* = \alpha_* + \beta_*$ after applying the isomorphism. We have

$$p^1_* \circ (\alpha_* + \beta_*) \ = \ (p^1 \circ \alpha)_* + (p^1 \circ \beta) \ = \ \text{incl}_* \ = \ p^1_* \circ \Delta_* \ : \ \tilde{H}_n(Y_0) \longrightarrow \tilde{H}_n(Y_1)$$

because $p^1 \circ \alpha$ is the identity and $p^1 \circ \beta$ is constant. Similarly, $p^2_* \circ (\alpha_* + \beta_*) = p^2 \circ \Delta_*$, so this shows Claim 2 and also completes the proof of the special case. □

Proof of BM-theorem, general case. Now we reduce the **general case** to the special case by taking suitable homotopy fibers. We let $y_0 \in Y_0$ be any basepoint and consider the space

$$P \ = \ \{\omega \in Y^{[0,1]} \mid \omega(0) = y_0\}$$

of all paths in Y that start at y_0. The space P is contractible to the constant path at y_0 and the endpoint map

$$\text{ev}_1 \ : \ P \longrightarrow Y \ , \quad \omega \longmapsto \omega(1)$$

is a Serre fibration. We also consider the subspaces

$$P_1 \ = \ \text{ev}_1^{-1}(Y_1) \ = \ \{\omega \in P \mid \omega(1) \in Y_1\} \ = \ \text{hofib}_{y_0}(Y_1 \longrightarrow Y)$$
$$P_2 \ = \ \text{ev}_1^{-1}(Y_2) \ = \ \{\omega \in P \mid \omega(1) \in Y_2\} \ = \ \text{hofib}_{y_0}(Y_2 \longrightarrow Y) \quad \text{and}$$
$$P_0 \ = \ \text{ev}_1^{-1}(Y_0) \ = \ \{\omega \in P \mid \omega(1) \in Y_0\} \ = \ \text{hofib}_{y_0}(Y_0 \longrightarrow Y) \ = \ P_1 \cap P_2 \ .$$

The triple (P, P_1, P_2) is another excisive triad. Indeed, if $\mathring{Y}_1$ is the open core of Y_1 in Y, then $\text{ev}_1^{-1}(\mathring{Y}_1)$ is open in P and contained in P_1, so $\text{ev}_1^{-1}(\mathring{Y}_1) \subset \mathring{P}_1$. Hence

$$P \ = \ \text{ev}_1^{-1}(\mathring{Y}_1 \cup \mathring{Y}_2) \ = \ \text{ev}_1^{-1}(\mathring{Y}_1) \cup \text{ev}_1^{-1}(\mathring{Y}_2) \ \subset \ \mathring{P}_1 \cup \mathring{P}_2 \ .$$

We claim that moreover P_1 is $(q-2)$-connected and P_2 is $(p-2)$-connected. Indeed,

$$\pi_i(P_1, c) \ = \ \pi_i(\text{hofib}_{y_0}(Y_1 \longrightarrow Y)) \ \cong \ \pi_{i+1}(Y, Y_1, y_0) \ .$$

Since the pair (Y_2, Y_0) is $(q-1)$-connected, so is the pair (Y, Y_1), by Corollary 5. So the group $\pi_i(P_1, c)$ vanishes for $i+1 \leq q-1$ i.e., $i \leq q-2$. A similar argument shows that P_2 is $(p-2)$-connected.

Since the map $\text{ev}_1 : P_1 \longrightarrow Y_1$ is a Serre fibration and $P_0 = \text{ev}_1^{-1}(Y_0)$, the projection induces a bijection

$$\pi_i(P_1, P_0, \omega) \ \cong \ \pi_i(Y_1, Y_0, \omega(1))$$

for all $i \geq 1$ and all $\omega \in P_0$. Similarly, the projections induce bijections

$$\pi_i(P_2, P_0, \omega) \cong \pi_i(Y_2, Y_0, \omega(1)) \quad \text{and}$$
$$\pi_i(P, P_1, \omega) \cong \pi_i(Y, Y_1, \omega(1))$$

for all $i \geq 1$. So by the special case of the Blakers-Massey theorem proved above, we can conclude that for every $p_0 \in P_0$ the inclusion induces a bijection

$$\pi_i(P_2, P_0, \omega) \longrightarrow \pi_i(P, P_1, \omega)$$

for $1 \leq i < p+q-2$ and a surjection for $i = p+q-2$. In the commutative square

$$\begin{array}{ccc} \pi_i(P_2, P_0, \omega) & \longrightarrow & \pi_i(P, P_1, \omega) \\ \cong \downarrow & & \downarrow \cong \\ \pi_i(Y_2, Y_0, \omega(1)) & \longrightarrow & \pi_i(Y, Y_1, \omega(1)) \end{array}$$

the vertical maps are bijective. So the lower horizontal map is bijective for $1 \leq i < p+q-2$ and surjective for $i = p+q-2$. This concludes the proof. □

4. Freudenthal's suspension theorem

One important application of the Blakers-Massey theorem is the Freudenthal suspension theorem.

Definition 14. The *suspension* of a topological space X is the space

$$\Sigma X = X \times [0,1]/\sim$$

where the equivalence relation collapses $X \times \{0\}$ to a point and $X \times \{1\}$ to another point.

Example 15. The suspension of a sphere is a sphere of the next dimension. Indeed, if S^{n-1} denotes the unit sphere in $\mathbb{R}^n$, then the map

$$\Sigma S^{n-1} \longrightarrow S^n, \quad [x,t] \longmapsto (\cos(t\pi),\ sin(t\pi) \cdot x)$$

is a homeomorphism.

We agree to take the 'south pole' as the basepoint of ΣX, i.e., the class of the collapsed $X \times \{0\}$. Moreover, we take $(1, 0, \ldots, 0)$ as the basepoint of S^{n-1}. With these conventions, the homeomorphism of Example 15 preserves the basepoints.

Suspension is clearly functorial for continuous maps. Then suspension takes homotopic maps to maps that are based homotopic. So we can define the *suspension homomorphism*

$$\Sigma : \pi_{n-1}(X, *) \longrightarrow \pi_n(\Sigma X, *)$$

by sending the homotopy class of a based map $f : S^{n-1} \longrightarrow X$ to the class of the composite

$$S^n \cong \Sigma S^{n-1} \xrightarrow{\Sigma f} \Sigma X .$$

We omit the verification that this map is indeed a group homomorphism when $n \geq 2$.

Theorem 16. *Let X be an $(n-1)$-connected space based space, for $n \geq 1$. Then the suspension homomorphism*

$$\Sigma \; : \; \pi_k(X, x_0) \longrightarrow \pi_{k+1}(\Sigma X, *)$$

is an isomorphism for $1 \leq k \leq 2n-2$ and an epimorphism for $k = 2n-1$.

Freudenthal proved this result in the special case where X is a sphere. We will deduce the suspension theorem from the Blakers-Massey theorem. Historically, Freudenthal's theorem predates the Blakers-Massey theorem by quite a bit, and it was published in the paper

H. Freudenthal, *Über die Klassen der Sphärenabbildungen I. Große Dimensionen.* Compositio Math. 5 (1938), 299–314.

Proof of Theorem 16. The suspension ΣX comes with an excisive cover by the open subsets

$$Y_1 \;=\; \Sigma X - \{S\} \qquad \text{and} \qquad Y_2 \;=\; \Sigma X - \{N\} \; .$$

Here $S, N \in \Sigma X$ are the two 'poles', i.e., the classes of the collapsed subsets $X \times \{0\}$ respectively $X \times \{1\}$. The space Y_2 is contractible to the south pole via the homotopy

$$\begin{aligned} (\Sigma X - \{N\}) \times [0,1] &\longrightarrow \Sigma X - \{N\} \\ ([x,t], s) &\longmapsto [x, ts] \; . \end{aligned}$$

Analogously, Y_1 is contractible to the north pole. The intersection

$$Y_0 \;=\; Y_1 \cap Y_2 \;=\; \Sigma X - \{S, N\}$$

is homeomorphic to $X \times (0,1)$. So the map

$$X \longrightarrow Y_0 \;=\; \Sigma X - \{S,N\} \; , \quad x \longmapsto [x, 1/2]$$

is a homotopy equivalence.

Since Y_1 is contractible, the connecting homomorphism

$$\pi_i(Y_1, Y_0, [x_0, 1/2]) \xrightarrow{\partial} \pi_{i-1}(Y_0, [x_0, 1/2])$$

in the long exact homotopy groups sequence of the pair (Y_1, Y_0) is bijective. Since the map $[-, 1/2] : X \longrightarrow Y_0$ is a homotopy equivalence, the connecting homomorphism

$$\pi_i(Y_1, X, [x_0, 1/2]) \xrightarrow{\partial} \pi_{i-1}(X, x_0)$$

is bijective, where we implicitly identified X with the subspace $X \times \{1/2\}$ of ΣX. Since X was assumed to be $(n-1)$-connected, we conclude that the group $\pi_i(Y_1, Y_0, [x_0, 1/2])$ is trivial for $i \leq n$. The same argument for Y_2 instead of Y_1 shows that the group $\pi_i(Y_2, Y_0, [x_0, 1/2])$ is also trivial for $i \leq n$. We can thus apply the Blakers-Massey theorem with $p = q = n+1$. We conclude that the map

$$\pi_i(Y_2, X, [x_0, 1/2]) \longrightarrow \pi_i(\Sigma X, Y_1, [x_0, 1/2])$$

induced by the inclusion is bijective for $1 \leq i < 2n$ and surjective for $i = 2n$.

Now we identify the inverse

$$\partial^{-1} \; : \; \pi_{i-1}(X, x_0) \longrightarrow \pi_i(Y_2, X, [x_0, 1/2])$$

of the connecting homomorphism as the 'coning map'. Indeed, if $f : S^{i-1} \longrightarrow X$ is a continuous based map, we define the map

$$g : D^i \longrightarrow Y_2 , \quad t \cdot z \longmapsto [f(z), t/2] \tag{17}$$

where $t \in [0,1]$ and $z \in S^{i-1}$. This map is continuous at $0 \in D^i$ and its restriction to S^{i-1} is the composite

$$S^{i-1} \xrightarrow{f} X \xrightarrow{[-,1/2]} Y_2 .$$

So

$$\delta[g] = [f] \quad \text{and hence} \quad \delta^{-1}[f] = [g] .$$

Thus the composite

$$\pi_{i-1}(X, x_0) \xrightarrow{\delta^{-1}} \pi_i(Y_2, X, [x_0, 1/2]) \xrightarrow{\text{incl}_*} \pi_i(\Sigma X, Y_1, [x_0, 1/2])$$

is bijective for $1 \leq i < 2n$ and surjective for $i = 2n$.

Inside the pair $(\Sigma X, Y_1)$, the map g defined in (17) becomes pair homotopic to the map

$$\bar{g} : D^i \longrightarrow \Sigma X , \quad t \cdot z \longmapsto [f(z), t] .$$

This is precisely the formula for the suspension of $[f]$, modulo the homeomorphism

$$\Sigma S^{i-1} \cong D^i / S^{i-1} , \quad [z, t] \longmapsto t \cdot z .$$

This homeomorphism is implicitly used in the long exact homotopy group sequence, so altogether this shows that the following square commutes:

$$\begin{array}{ccc} \pi_{i-1}(X, x_0) & \xrightarrow{\Sigma'} & \pi_i(\Sigma X, [x_0, 1/2]) \\ {\scriptstyle \partial^{-1}} \downarrow \cong & & \cong \downarrow {\scriptstyle \text{incl}_*} \\ \pi_i(Y_1, X, [x_0, 1/2]) & \xrightarrow[\text{incl}_*]{} & \pi_i(\Sigma X, Y_1, [x_0, 1/2]) \end{array}$$

The upper horizontal map is the composite of the suspension isomorphism

$$\Sigma : \pi_{i-1}(X, x_0) \longrightarrow \pi_i(\Sigma X, S)$$

and conjugation by the path

$$[0,1] \longrightarrow \Sigma X , \quad t \longmapsto [x_0, t/2]$$

from the south pol S to the point $[x_0, 1/2]$. Since Y_1 is contractible, the group $\pi_i(\Sigma X, Y_1, [x, 1/2])$ is isomorphic to $\pi_i(\Sigma X, [x, 1/2])$. So the right vertical map is an isomorphism. We conclude that the map Σ', and hence also the suspension homomorphism Σ, is an isomorphism for $1 \leq i < 2n$ and an epimorphism for $i = 2n$. The change of variable $k = i - 1$ then gives the result. □

As a consequence of the Hurewicz theorem we already concluded that the homotopy groups $\pi_n(S^n, *)$ is isomorphic to $\mathbb{Z}$, generated by the class of the identity map, for every $n \geq 1$. Now we can get information about the groups $\pi_{n+1}(S^n, *)$. We recall that the *Hopf map* $\eta : S^3 \longrightarrow S^2$ is the composite

$$S^3 \cong S(\mathbb{C}^2) \xrightarrow{(x,y) \mapsto [x:y]} \mathbb{C}P^1 \cong S^2 .$$

The group $\pi_3(S^2,*)$ is isomorphic to $\mathbb{Z}$, generated by the class of η, by the long exact homotopy group sequence of the locally trivial fiber bundle $\eta : S^3 \longrightarrow S^2$.

Corollary 18. *For every $n \geq 2$ the group $\pi_{n+1}(S^n,*)$ is cyclic and generated by the class $\Sigma^{n-2}\eta$.*

Proof. We argue by induction on n starting with the case $n = 2$ that we already know. For $n \geq 3$ we exploit that S^{n-1} is $(n-2)$-connected, and so the the suspension homomorphism

$$\Sigma \ : \ \pi_k(S^{n-1},*) \ \longrightarrow \ \pi_{k+1}(S^n,*)$$

is an isomorphism for $1 \leq k \leq 2n-4$ and an epimorphism for $k = 2n-3$. In particular, the suspension homomorphism

$$\Sigma \ : \ \pi_3(S^2,*) \ \longrightarrow \ \pi_4(S^3,*)$$

is surjective, so the target group is generated by $\Sigma\eta$. From that point on, i.e., for $n \geq 4$, the suspension homomorphism

$$\Sigma \ : \ \pi_n(S^{n-1},*) \ \longrightarrow \ \pi_{n+1}(S^n,*)$$

is an isomorphism, and this proves the claim. □

Now we claim that twice the suspension of η is null-homotopic, and hence

$$2 \cdot \Sigma\eta \ = \ 0 \quad \text{in } \pi_4(S^3,*).$$

To see this we consider the commutative square

$$\begin{array}{ccccc} (x,y) & S^3 & \xrightarrow{\ \eta\ } & \mathbb{C}\mathrm{P}^1 & [x:y] \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ (\bar{x},\bar{y}) & S^3 & \xrightarrow[\ \eta\ ]{} & \mathbb{C}\mathrm{P}^1 & [\bar{x}:\bar{y}] \end{array}$$

in which the vertical maps are induced by complex conjugation in both coordinates of $\mathbb{C}^2$. The left vertical map has degree 1, so it is homotopic to the identity, whereas complex conjugation on $\mathbb{C}\mathrm{P}^1 \cong S^2$ has degree -1. So $(-1)\circ\eta$ is homotopic to η. Thus the suspension of η is homotopic to the suspension of $(-1)\circ\eta$, which by the following lemma is homotopic to the negative of $\eta \wedge S^1$.

Lemma 19. *Let $f : S^m \longrightarrow S^m$ a based map of degree k, for $m \geq 1$. Then for every homotopy class $x \in \pi_n(S^m,*)$ the classes $f_*(x)$ and $k \cdot x$ become equal in $\pi_{n+1}(S^{m+1},*)$ after one suspension.*

Proof. Let $d_k : S^1 \longrightarrow S^1$ be any pointed map of degree k. Then the maps $f \wedge S^1, S^m \wedge d_k : S^{m+1} \longrightarrow S^{m+1}$ have the same degree k, hence they are based homotopic. Suppose x is represented by $\varphi : S^n \longrightarrow S^m$. Then the suspension of $f_*(x)$ is represented by $(f \wedge S^1)\circ(\varphi \wedge S^1)$ which is homotopic to $(S^m \wedge d_k)\circ(\varphi \wedge S^1) = (\varphi \wedge S^1)\circ(S^n \wedge d_k)$. Precomposition with the degree k map $S^n \wedge d_k$ of S^{n+1} induces multiplication by k, so the last map represents the suspension of $k \cdot x$. □

The conclusion of Lemma 19 does not in general hold without the extra suspension, i.e., $f_*(x)$ need not equal $k \cdot x$ in $\pi_n(S^m, *)$: as we showed above, $(-1) \circ \eta$ is homotopic to η, which is *not* homotopic to $-\eta$ since η generates the infinite cyclic group $\pi_3(S^2, *)$.

So altogether we conclude that for $n \geq 3$ the group $\pi_{n+1}(S^n, *)$ is either trivial or of order 2, depending on whether the appropriate suspension of η becomes null-homotopic or not. In fact, no suspension of η is ever null-homotopic, and so $\pi_{n+1}(S^n, *) \cong \mathbb{Z}/2$ for $n \geq 3$. The standard proof of this fact uses *Steenrod operations* in mod-2 cohomology.

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