Cluster algebras in algebraic Lie theory

Christof Geiss, Bernard Leclerc, Jan Schröer

Abstract. We survey some recent constructions of cluster algebra structures on coordinate rings of unipotent subgroups and unipotent cells of Kac-Moody groups. We also review a quantized version of these results.

Contents

1. Introduction

Cluster algebras were invented by Fomin and Zelevinsky [13] as an abstraction of certain combinatorial structures which they had previously discovered while studying total positivity in semisimple algebraic groups. A cluster algebra is a commutative ring with a distinguished set of generators and a particular type of relations. Although there can be infinitely many generators and relations, they are all obtained from a finite number of them by means of an inductive procedure called mutation. The precise definition of a cluster algebra will be recalled in §2 below.

Several examples arising in Lie theory were already mentioned in [13], like $\mathbb{C}[\mathrm{SL}_4/N]$, $\mathbb{C}[\mathrm{Sp}_4/N]$, $\mathbb{C}[\mathrm{Gr}_{2,n+3}]$. Moreover, Fomin and Zelevinsky [13, p. 498] conjectured that

> *"the above examples can be extensively generalized: for any simply-connected connected semisimple group G, the coordinate rings $\mathbb{C}[G]$ and $\mathbb{C}[G/N]$, as well as coordinate rings of many other interesting varieties related to G, have a natural structure of a cluster algebra. This structure should serve as an algebraic framework for the study of "dual canonical bases" in these coordinate rings and their q-deformations."*

In this survey we shall review the results of a series of papers [20, 23, 25, 27], in which we have implemented (part of) this program for certain varieties associated with elements w of the Weyl group of a Kac-Moody group, namely, the unipotent subgroups $N(w)$ and the unipotent cells N^w.

2. Cluster algebras

We start with a quick (and very incomplete) introduction to the theory of cluster algebras. For more developed accounts we refer to [11, 16, 18, 34].

2.1. Definitions and basic results. Consider the field of rational functions $\mathcal{F} = \mathbb{Q}(x_1, \dots, x_n)$. A *seed* in $\mathcal{F}$ is a pair $\Sigma = (\mathbf{y}, Q)$, where $\mathbf{y} = (y_1, \dots, y_n)$ is a free generating set of $\mathcal{F}$, and Q is a quiver (*i.e.* an oriented graph) with vertices labelled by $\{1, \dots, n\}$. We assume that Q has neither loops nor 2-cycles. For $k = 1, \dots, n$, one defines a new seed $\mu_k(\Sigma) = (\mu_k(\mathbf{y}), \mu_k(Q))$ as follows. First $\mu_k(y_i) = y_i$ for $i \neq k$, and

$$\mu_k(y_k) = \frac{\prod_{i \to k} y_i + \prod_{k \to j} y_j}{y_k}, \tag{1}$$

where the first (*resp.* second) product in the right hand side is over all arrows of Q with target (*resp.* source) k. Next $\mu_k(Q)$ is obtained from Q by

(a) adding a new arrow $i \to j$ for every existing pair of arrows $i \to k$ and $k \to j$;

(b) reversing the orientation of every arrow with target or source equal to k;

(c) erasing every pair of opposite arrows possibly created by (a).

It is easy to check that $\mu_k(\Sigma)$ is a seed, and $\mu_k(\mu_k(\Sigma)) = \Sigma$. The *mutation class* $\mathcal{C}(\Sigma)$ is the set of all seeds obtained from Σ by a finite sequence of mutations μ_k. One can think of the elements of $\mathcal{C}(\Sigma)$ as the vertices of an n-regular tree in which every edge stands for a mutation. If $\Sigma' = ((y'_1, \dots, y'_n), Q')$ is a seed in $\mathcal{C}(\Sigma)$, then the subset $\{y'_1, \dots, y'_n\}$ is called a *cluster*, and its elements are called *cluster variables*.

Now, Fomin and Zelevinsky define the *cluster algebra* $\mathcal{A}_\Sigma$ as the subring of $\mathcal{F}$ generated by all cluster variables. Some important elements of $\mathcal{A}_\Sigma$ are the *cluster monomials*, *i.e.* monomials in the cluster variables supported on a single cluster.

Example 2.1. If $n = 2$ and $\Sigma = ((x_1, x_2), Q)$, where Q is the quiver with a arrows from 1 to 2, then $\mathcal{A}_\Sigma$ is the subring of $\mathbb{Q}(x_1, x_2)$ generated by the rational functions x_k defined recursively by

$$x_{k+1}x_{k-1} = 1 + x_k^a, \qquad (k \in \mathbb{Z}). \tag{2}$$

The clusters of $\mathcal{A}_\Sigma$ are the subsets $\{x_k, x_{k+1}\}$, and the cluster monomials are the special elements of the form

$$x_k^l x_{k+1}^m, \qquad (k \in \mathbb{Z},\ l, m \in \mathbb{N}).$$

It turns out that when $a = 1$, there are only five different clusters and cluster variables, namely

$$x_{5k+1} = x_1,\ \ x_{5k+2} = x_2,\ \ x_{5k+3} = \frac{1+x_2}{x_1},\ \ x_{5k+4} = \frac{1+x_1+x_2}{x_1x_2},\ \ x_{5k} = \frac{1+x_1}{x_2}.$$

For $a \geq 2$ though, the sequence (x_k) is no longer periodic and $\mathcal{A}_\Sigma$ has infinitely many cluster variables.

The next theorem summarizes the first deep results of this theory obtained by Fomin and Zelevinsky.

Theorem 2.2 ([13],[14]). (i) *Every cluster variable of $\mathcal{A}_\Sigma$ is a Laurent polynomial with coefficients in $\mathbb{Z}$ in the cluster variables of any single fixed cluster.*

(ii) *$\mathcal{A}_\Sigma$ has finitely many clusters if and only if the mutation class $\mathcal{C}(\Sigma)$ contains a seed whose quiver is an orientation of a Dynkin diagram of type A, D, E.*

2.2. An example from Lie theory. We illustrate Theorem 2.2 (ii) with a prototypical example. Let $G = SL_4$ and denote by N the subgroup of upper unitriangular matrices. In [2, §2.6] explicit initial seeds for a cluster algebra structure in the coordinate ring of the big cell of the base affine space G/N were described. A simple modification yields initial seeds for $\mathbb{C}[N]$ (see [21]). One of these seeds is

$$((D_{1,2},\ D_{1,3},\ D_{12,23},\ D_{1,4},\ D_{12,34},\ D_{123,234}), Q),$$

where Q is the triangular quiver:

$$\begin{tikzcd}
 & & 1 \arrow[dl] & & \\
 & 2 \arrow[rr] \arrow[dl] & & 3 \arrow[ul] \arrow[dl] & \\
4 \arrow[rr] & & 5 \arrow[ul] \arrow[rr] & & 6 \arrow[ul]
\end{tikzcd}$$

Here, by $D_{I,J}$ we mean the regular function on N assigning to a matrix its minor with row-set I and column-set J. Moreover, the variables

$$x_4 = D_{1,4},\ x_5 = D_{12,34},\ x_6 = D_{123,234}$$

are *frozen*, *i.e.* they cannot be mutated, and therefore they belong to every cluster. If one performs the mutation μ_1, one gets the new cluster variable

$$\mu_1(x_1) = \frac{x_2 + x_3}{x_1} = \frac{D_{1,3} + D_{12,23}}{D_{1,2}} = D_{2,3},$$

and thus the new seed

$$((D_{2,3},\, D_{1,3},\, D_{12,23},\, D_{1,4},\, D_{12,34},\, D_{123,234}), \mu_1(Q)),$$

where $\mu_1(Q)$ is the mutated quiver:

$$\begin{tikzcd}[column sep=small]
 & & 1 \arrow[dr] & & \\
 & 2 \arrow[ur] \arrow[dl] & & 3 \arrow[dl] & \\
4 \arrow[rr] & & 5 \arrow[ul] \arrow[rr] & & 6 \arrow[ul]
\end{tikzcd}$$

The full subquiver of $\mu_1(Q)$ obtained by erasing vertices $4, 5, 6$ corresponding to the frozen variables is a Dynkin quiver of type A_3, hence, by Theorem 2.2, this cluster algebra has finitely many clusters and cluster variables. It is an easy exercise to check that indeed it has 14 clusters and 12 cluster variables if we count the 3 frozen ones. Moreover, it follows from [3] that Lusztig's dual canonical basis of $\mathbb{C}[N]$ coincides with the set of all cluster monomials.

Finally, note that the open subset of N given by the non-vanishing of the 3 frozen variables x_4, x_5, x_6 is equal to

$$N^{w_0} := N \cap (B_- w_0 B_-),$$

where B_- denotes the subgroup of lower triangular matrices in G, and w_0 is the longest element of the Weyl group of G. This is an example of a *unipotent cell*, that is, a stratum of the decomposition of the unipotent group N obtained by intersecting it with the Bruhat decomposition of G associated with the opposite Borel subgroup B_-. The coordinate ring $\mathbb{C}[N^{w_0}]$ is obtained by localizing the polynomial ring $\mathbb{C}[N]$ at the element $x_4x_5x_6$, and we can see that it carries almost the same cluster algebra structure as $\mathbb{C}[N]$, the only difference being that the *coefficient ring* generated by the frozen cluster variables is now the Laurent polynomial ring in x_4, x_5, x_6.

3. Lie theory

We now introduce a class of varieties generalizing the varieties N and N^{w_0} of Example 2.2. These are subvarieties of unipotent subgroups of symmetric Kac-Moody groups, and we first prepare the necessary notation and definitions. For more details, we refer to [37].

3.1. Kac-Moody Lie algebras. Let $C = [c_{ij}]$ be a symmetric $n \times n$ generalized Cartan matrix. Thus $2I - C$ is the adjacency matrix of an unoriented graph Γ with n vertices and no loop. Without loss of generality, we can assume that this graph is connected. We will often denote by $I = [1, n]$ the indexing set of rows and columns of C. Let $\mathfrak{h}$ be a $\mathbb{C}$-vector space of dimension $2n - \mathrm{rank}(C)$. We choose linear independent subsets $\{h_i \mid i \in I\} \subset \mathfrak{h}$ and $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ such that $\alpha_i(h_j) = c_{ij}$. Let $\mathfrak{g}$ be the Kac-Moody Lie algebra over $\mathbb{C}$ with generators e_i, f_i $(i \in I)$, $h \in \mathfrak{h}$, subject to the following relations:

$$\begin{aligned}
&[h, h'] = 0,\ [h, e_i] = \alpha_i(h)e_i,\ [h, f_i] = -\alpha_i(h)f_i, && (h, h' \in \mathfrak{h}),\\
&[e_i, f_j] = \delta_{ij}h_i, && (i, j \in I),\\
&\mathrm{ad}(e_i)^{1-c_{ij}}(e_j) = \mathrm{ad}(f_i)^{1-c_{ij}}(f_j) = 0, && (i \neq j).
\end{aligned}$$

We denote by $\mathfrak{n}_+$ (*resp.* $\mathfrak{n}_-$) the subalgebra generated by e_i $(i \in I)$ (*resp.* f_i $(i \in I)$). For simplicity we shall often write $\mathfrak{n}$ instead of $\mathfrak{n}_+$.

Let W be the subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the reflexions

$$s_i(\alpha) = \alpha - \alpha(h_i)\alpha_i, \qquad (i \in I,\ \alpha \in \mathfrak{h}^*).$$

This is a Coxeter group with length function $w \mapsto \ell(w)$. For $\alpha \in \mathfrak{h}^*$ let

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x,\ h \in \mathfrak{h}\}.$$

We denote by $\Delta := \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\}$ the root system of $\mathfrak{g}$, by $R := \oplus_{i\in I}\mathbb{Z}\alpha_i$ the root lattice, by $R^+ := \oplus_{i\in I}\mathbb{N}\alpha_i$ its positive cone, and by $\Delta^+ := \Delta \cap R^+$ the subset of positive roots. We have $\Delta = \Delta^+ \sqcup (-\Delta^+)$. The Weyl group W acts on Δ, and we define the subset of real roots as the W-orbit of $\{\alpha_i \mid i \in I\}$. For $w \in W$, put

$$\Delta_w := \{\alpha \in \Delta^+ \mid w(\alpha) \in \Delta^-\}.$$

This is a finite set of positive real roots, with cardinality $\ell(w)$. Finally, set

$$\mathfrak{n}(w) := \bigoplus_{\alpha\in\Delta_w} \mathfrak{g}_\alpha \subset \mathfrak{n}_+,$$

a nilpotent subalgebra of $\mathfrak{g}$ of dimension $\ell(w)$.

Example 3.1. Take

$$C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Then $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ is an affine Lie algebra of Dynkin type $\widetilde{A}_1$. We have

$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1\rangle.$$

Let $w = s_2s_1s_2s_1$. Then

$$\Delta_w = \{\alpha_1,\ 2\alpha_1 + \alpha_2,\ 3\alpha_1 + 2\alpha_2,\ 4\alpha_1 + 3\alpha_2\},$$

and

$$\mathfrak{n}(w) = \mathrm{Span}\langle e_1,\ [e_1, [e_2, e_1]], [e_1, [e_2, [e_1, [e_2, e_1]]]],\ [e_1, [e_2, [e_1, [e_2, [e_1, [e_2, e_1]]]]]]\rangle.$$

3.2. Kac-Moody groups. The enveloping algebra $U(\mathfrak{n})$ of the Lie algebra $\mathfrak{n}$ is a cocommutative Hopf algebra, with an R^+-grading given by $\deg(e_i) = \alpha_i$. Let

$$U(\mathfrak{n})^*_{\mathrm{gr}} := \bigoplus_{d \in R^+} U(\mathfrak{n})^*_d$$

be its graded dual. This is a commutative Hopf algebra. Define

$$N := \max \operatorname{Spec}(U(\mathfrak{n})^*_{\mathrm{gr}}) = \operatorname{Hom}_{\mathrm{alg}}(U(\mathfrak{n})^*_{\mathrm{gr}}, \mathbb{C}).$$

The comultiplication of $U(\mathfrak{n})^*_{\mathrm{gr}}$ gives N a group structure. As a group, N can be identified with the pro-unipotent pro-group with Lie algebra

$$\widehat{\mathfrak{n}} = \prod_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

By construction, we can identify $U(\mathfrak{n})^*_{\mathrm{gr}}$ with the coordinate ring $\mathbb{C}[N]$ of N.

For $w \in W$, let $N(w)$ be the subgroup of N with Lie algebra $\mathfrak{n}(w)$. Define also $N'(w)$ to be the subgroup of N with Lie algebra

$$\mathfrak{n}'(w) := \prod_{\alpha \notin \Delta_w} \mathfrak{g}_\alpha \subset \widehat{\mathfrak{n}}.$$

Multiplication yields a bijection $N(w) \times N'(w) \xrightarrow{\sim} N$.

Proposition 3.2 ([25, Proposition 8.2]). *The coordinate ring* $\mathbb{C}[N(w)]$ *is isomorphic to the invariant subring*

$$\mathbb{C}[N]^{N'(w)} = \{f \in \mathbb{C}[N] \mid f(nn') = f(n),\ n \in N,\ n' \in N'(w)\}.$$

Let G be the group attached to $\mathfrak{g}$ by Kac and Peterson [33]. This is an affine ind-variety. It has a refined Tits system

$$(G, \operatorname{Norm}_G(H), N_+, N_-, H),$$

where $\operatorname{Lie}(H) = \mathfrak{h}$, $\operatorname{Lie}(N_+) = \mathfrak{n}_+$, and $\operatorname{Lie}(N_-) = \mathfrak{n}_-$ (see [37]). Note that in general $N \not\subset G$. Both N and G can be regarded as subgroups of a bigger group $G^{\max}$ constructed by Tits. Then $N_+ = N \cap G$.

Example 3.3. (a) If $\mathfrak{g}$ is finite-dimensional of Dynkin type X_n, then $G = G_{X_n}(\mathbb{C})$ is a connected simply-connected algebraic group of type X_n over $\mathbb{C}$, and N_+ is a maximal unipotent subgroup $N_+ = N_{X_n}(\mathbb{C})$. Thus, if $X_n = A_n$, $G = \mathrm{SL}(n+1, \mathbb{C})$, and N_+ is the subgroup of unipotent upper triangular matrices.

(b) If $\mathfrak{g}$ is an affine Lie algebra of affine Dynkin type $\widetilde{X}_n$, then G is a central extension of $G_{X_n}(\mathbb{C}[z, z^{-1}])$ by $\mathbb{C}^*$. Moreover,

$$N_+ \simeq \{g \in G_{X_n}(\mathbb{C}[z]) \mid g|_{z=0} \in N_{X_n}(\mathbb{C})\}.$$

Thus, continuing Example 3.1, if $\mathfrak{g}$ is of type $\widetilde{A}_1$,

$$N_+ = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}[z]) \mid a(0) = d(0) = 1,\ c(0) = 0 \right\}.$$

(c) If $\mathfrak{g}$ is of indefinite type, no "concrete" realization of G is known.

3.3. Generalized minors. We have $\mathrm{Norm}_G(H)/H \cong W$. For $i \in I$, put

$$\overline{s}_i := \exp(f_i)\exp(-e_i)\exp(f_i), \quad \overline{\overline{s}}_i := \exp(-f_i)\exp(e_i)\exp(-f_i).$$

To $w = s_{i_1} \cdots s_{i_r} \in W$ with $\ell(w) = r$, we attach two representatives in $\mathrm{Norm}_G(H)$:

$$\overline{w} = \overline{s}_{i_1} \cdots \overline{s}_{i_r}, \qquad \overline{\overline{w}} = \overline{\overline{s}}_{i_1} \cdots \overline{\overline{s}}_{i_r}.$$

Let $G_0 = N_-HN_+$ be the Zariski open subset of G consisting of elements g having a Birkhoff decomposition. For $g \in G_0$, this unique factorization is written

$$g = [g]_-[g]_0[g]_+, \qquad ([g]_- \in N_-,\ [g]_0 \in H,\ [g]_+ \in N_+).$$

Let $\{\varpi_i \mid i \in I\} \subset \mathfrak{h}^*$ be a fixed choice of fundamental weights, that is,

$$\varpi_i(h_j) = \delta_{ij}, \qquad (i, j \in I).$$

Let $x \mapsto x^{\varpi_i}$ denote the corresponding characters of H. There is a unique regular function $\Delta_{\varpi_i,\varpi_i}$ on G such that

$$\Delta_{\varpi_i,\varpi_i}(g) = [g]_0^{\varpi_i}, \qquad (g \in G_0).$$

Moreover, $G_0 = \{g \in G \mid \Delta_{\varpi_i,\varpi_i}(g) \neq 0,\ i \in I\}$.

Definition 3.4. For $u, v \in W$ and $i \in I$, the generalized minor $\Delta_{u(\varpi_i),v(\varpi_i)}$ is the regular function on G given by

$$\Delta_{u(\varpi_i),v(\varpi_i)}(g) = \Delta_{\varpi_i,\varpi_i}\left(\overline{\overline{u^{-1}}}\, g\, \overline{v}\right), \qquad (g \in G).$$

3.4. Unipotent cells. Let $B_- = N_-H$. The group G has a Bruhat decomposition:

$$G = \bigsqcup_{w \in W} B_-\overline{w}B_-.$$

For $w \in W$, define the *unipotent cell* $N^w := N_+ \cap (B_-\overline{w}B_-)$. Let

$$O_w := \{g \in N(w) \mid \Delta_{\varpi_i,w^{-1}(\varpi_i)}(g) \neq 0,\ i \in I\},$$

an open subset of the affine space $N(w)$.

Proposition 3.5 ([25, Proposition 8.5]). *We have an isomorphism* $O_w \xrightarrow{\sim} N^w$. *It follows that* $\mathbb{C}[N^w]$ *is the localization of* $\mathbb{C}[N(w)] \simeq \mathbb{C}[N]^{N'(w)}$ *at* $\prod_{i \in I} \Delta_{\varpi_i,w^{-1}(\varpi_i)}$.

3.5. Formulas for factorization parameters. Define the one-parameter subgroups of N_+:

$$x_i(t) := \exp(te_i), \qquad (t \in \mathbb{C},\ i \in I).$$

Consider as above a reduced decomposition $w = s_{i_r} \cdots s_{i_1} \in W$. (Note that for reasons which will become clear in §4.3 below, we prefer from now on to number the factors of this decomposition *from right to left*.) Then the image of the map $\mathbf{x}_{i_r,\dots,i_1} : (\mathbb{C}^*)^r \to N$ given by

$$\mathbf{x}_{i_r,\dots,i_1}(t_r,\dots,t_1) := x_{i_r}(t_r)\cdots x_{i_1}(t_1)$$

is a dense subset of N^w. More precisely, $\mathbf{x}_{i_r,\dots,i_1}$ gives a birational isomorphism from $(\mathbb{C}^*)^r$ to N^w.

Thus, if $f \in \mathbb{C}[N^w]$ then $f(\mathbf{x}_{i_r,\dots,i_1}(\mathbf{t}))$ is a rational function of $\mathbf{t} := (t_r,\dots,t_1)$ which completely determines f. Conversely, following Berenstein, Fomin and Zelevinsky [1], one can give explicit formulas in terms of generalized minors for expressing each t_i as a rational function on N^w [26]. These *Chamber Ansatz formulas* will be explained in §6 below.

Example 3.6. We continue Example 3.1. Thus $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ and $w = s_2s_1s_2s_1$. By Example 3.3 (b), we have

$$N_+ = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{C}[z],\ ad-bc = 1,\ a(0) = d(0) = 1,\ c(0) = 0 \right\}.$$

Writing, for $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N_+$,

$$a = 1 + \sum_{k\geq 1} a_k z^k, \quad b = \sum_{k\geq 0} b_k z^k, \quad c = \sum_{k\geq 1} c_k z^k, \quad d = 1 + \sum_{k\geq 1} d_k z^k,$$

we get regular functions $a_k(x), b_k(x), c_k(x), d_k(x)$ on N_+, which by restriction give regular functions on N^w. We have

$$x_1(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad x_2(t) = \begin{pmatrix} 1 & 0 \\ tz & 1 \end{pmatrix},$$

and, putting $\mathbf{x}_{2,1,2,1}(\mathbf{t}) := x_2(t_4)x_1(t_3)x_2(t_2)x_1(t_1)$, we calculate

$$\mathbf{x}_{2,1,2,1}(\mathbf{t}) = \begin{pmatrix} 1 + t_2t_3z & t_1 + t_3 + t_1t_2t_3z \\ (t_2+t_4)z + t_2t_3t_4z^2 & 1 + (t_1t_2 + t_1t_4 + t_3t_4)z + t_1t_2t_3t_4z^2 \end{pmatrix}.$$

Hence

$$\begin{aligned} & a_1(\mathbf{x}_{2,1,2,1}(\mathbf{t})) = t_2t_3, & & \\ & b_0(\mathbf{x}_{2,1,2,1}(\mathbf{t})) = t_1 + t_3, & & b_1(\mathbf{x}_{2,1,2,1}(\mathbf{t})) = t_1t_2t_3, \\ & c_1(\mathbf{x}_{2,1,2,1}(\mathbf{t})) = t_2 + t_4, & & c_2(\mathbf{x}_{2,1,2,1}(\mathbf{t})) = t_2t_3t_4, \\ & d_1(\mathbf{x}_{2,1,2,1}(\mathbf{t})) = t_1t_2 + t_1t_4 + t_3t_4, & & d_2(\mathbf{x}_{2,1,2,1}(\mathbf{t})) = t_1t_2t_3t_4. \end{aligned}$$

It follows that,

$$t_4 = \frac{c_2}{a_1}, \qquad t_3 = \frac{\begin{vmatrix} b_0 & b_1 \\ 1 & a_1 \end{vmatrix}}{a_1}, \qquad t_2 = \frac{\begin{vmatrix} c_1 & c_2 \\ 1 & a_1 \end{vmatrix}}{a_1}, \qquad t_1 = \frac{b_1}{a_1}. \tag{3}$$

In fact, these formulas, although similar in spirit, are *not* the Chamber Ansatz formulas. Indeed the regular functions a_1, b_1, c_2, $b_0a_1 - b_1$, and $c_1a_1 - c_2$, are *not* cluster variables for the cluster algebra structure on $\mathbb{C}[N^w]$. Compare Example 6.2 below.

Example 3.7. For the sake of comparison with Example 3.6, let us describe the groups N, $N(w)$, and $N'(w)$, for the same $w = s_2s_1s_2s_1$ and $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$. First, we have

$$N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{C}[[z]],\ ad - bc = 1,\ a(0) = d(0) = 1,\ c(0) = 0 \right\},$$

where $\mathbb{C}[[z]]$ is the ring of formal power series in z. Next,

$$\begin{aligned} N(w) &= \left\{ \begin{pmatrix} 1 & b_0 + b_1z + b_2z^2 + b_3z^3 \\ 0 & 1 \end{pmatrix} \mid b_0, b_1, b_2, b_3 \in \mathbb{C} \right\}, \\ N'(w) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N \mid b \in z^4\mathbb{C}[[z]] \right\}. \end{aligned}$$

Finally, for $g \in N(w)$, we have

$$\Delta_{\varpi_1, w^{-1}(\varpi_1)}(g) = b_0b_2 - b_1^2, \qquad \Delta_{\varpi_2, w^{-1}(\varpi_2)}(g) = b_1b_3 - b_2^2,$$

so

$$N^w \cong \left\{ \begin{pmatrix} 1 & b_0 + b_1z + b_2z^2 + b_3z^3 \\ 0 & 1 \end{pmatrix} \mid b_i \in \mathbb{C},\ (b_0b_2 - b_1^2)(b_1b_3 - b_2^2) \neq 0 \right\}.$$

4. Categories of modules over preprojective algebras

Seminal works of Ringel and Lusztig have shown that the interaction between Kac-Moody algebras and the representation theory of quivers is essential for understanding the quantum enveloping algebra $U_q(\mathfrak{n})$ and its canonical basis. Following Lusztig, one can also construct the (classical) enveloping algebra $U(\mathfrak{n})$ in terms of the path algebra of a quiver with relations called the preprojective algebra. This will be our basic tool for exploring cluster algebra structures on $\mathbb{C}[N(w)]$ and $\mathbb{C}[N^w]$. In this section we shall introduce the preprojective algebra Λ and its nilpotent representations, and explain how it yields an interesting basis of $\mathbb{C}[N]$ dual to Lusztig's semicanonical basis of $U(\mathfrak{n})$. We will then describe some categories $\mathcal{C}_w$ of Λ-modules introduced and studied in general by Buan, Iyama, Reiten and Scott [5] (and independently in [23], for adaptable Weyl group elements w). This will provide a categorical model for the cluster algebras of §5.

4.1. The preprojective algebra. Let Q be a quiver obtained by orienting the edges of the graph Γ of §3.1. We require Q to be acyclic, that is, Q has no oriented cycle. Let $\overline{Q}$ denote the double quiver obtained from Q by adjoining to every arrow $a\colon i \to j$ an opposite arrow $a^*\colon j \to i$. Consider the element

$$\rho = \sum (aa^* - a^*a)$$

of the path algebra $\mathbb{C}\overline{Q}$ of $\overline{Q}$, where the sum is over all arrows a of Q. Following [28, 46], we define the *preprojective algebra* Λ as the quotient of $\mathbb{C}\overline{Q}$ by the two-sided ideal generated by ρ. It is well-known that Λ is independent of the choice of orientation Q of Γ. Moreover, Λ is finite-dimensional if and only if the Kac-Moody algebra $\mathfrak{g}$ is finite-dimensional, that is, if and only if Γ is a Dynkin diagram of type A, D, E.

We say that a finite-dimensional Λ-module is *nilpotent* if all its composition factors are one-dimensional. Let $\mathrm{nil}(\Lambda)$ denote the category of nilpotent Λ-modules. This is an abelian category with infinitely many isomorphism classes of indecomposable objects, except if $\mathfrak{g}$ has type A_n with $n \leq 4$. It is remarkable that these few exceptional cases coincide precisely with the cases when the cluster algebra $\mathbb{C}[N]$ has finitely many cluster variables. Moreover, it is a nice exercise to verify that the number of indecomposable Λ-modules is then equal to the number of cluster variables. This suggests a close relationship in general between Λ and $\mathbb{C}[N]$. To describe it we start with Lusztig's Lagrangian construction of the enveloping algebra $U(\mathfrak{n})$ [41, 44]. This is a realization of $U(\mathfrak{n})$ as an algebra of $\mathbb{C}$-valued constructible functions over the varieties of nilpotent representations of Λ.

Denote by S_i $(1 \leq i \leq n)$ the one-dimensional Λ-module supported on the vertex i of $\overline{Q}$. Given a sequence $\mathbf{i} = (i_1, \dots, i_d)$ and a nilpotent Λ-module X of dimension d, we introduce the variety $\mathcal{F}_{X,\mathbf{i}}$ of flags of submodules

$$\mathfrak{f} = (0 = F_0 \subset F_1 \subset \cdots \subset F_d = X)$$

such that $F_k/F_{k-1} \cong S_{i_k}$ for $k = 1, \dots, d$. This is a projective variety. Denote by $\Lambda_{\mathbf{d}}$ the affine variety of nilpotent Λ-modules X with a given dimension vector $\mathbf{d} = (d_i)$, where $\sum_i d_i = d$. Consider the constructible function $\chi_{\mathbf{i}}$ on $\Lambda_{\mathbf{d}}$ given by

$$\chi_{\mathbf{i}}(X) = \chi(\mathcal{F}_{X,\mathbf{i}})$$

where χ denotes the Euler-Poincaré characteristic. Let $\mathcal{M}_{\mathbf{d}}$ be the $\mathbb{C}$-vector space spanned by the functions $\chi_{\mathbf{i}}$ for all possible sequences $\mathbf{i}$ of length d, and let

$$\mathcal{M} = \bigoplus_{\mathbf{d} \in \mathbb{N}^n} \mathcal{M}_{\mathbf{d}}.$$

Lusztig has endowed $\mathcal{M}$ with an associative multiplication which formally resembles a convolution product, and he has shown that, if we denote by e_i the Chevalley generators of $\mathfrak{n}$, there is an algebra isomorphism $U(\mathfrak{n}) \xrightarrow{\sim} \mathcal{M}$ mapping the product $e_{i_1} \cdots e_{i_d}$ to $\chi_{\mathbf{i}}$ for every $\mathbf{i} = (i_1, \dots, i_d)$.

Now, following [19, 20], we dualize the picture. Every $X \in \mathrm{nil}(\Lambda)$ determines a linear form δ_X on $\mathcal{M}$ given by

$$\delta_X(f) = f(X), \qquad (f \in \mathcal{M}).$$

Through the isomorphisms $\mathcal{M}_{\rm gr}^* \simeq U(\mathfrak{n})_{\rm gr}^* \simeq \mathbb{C}[N]$, the form δ_X corresponds to an element φ_X of $\mathbb{C}[N]$, and we have thus attached to every object X in $\mathrm{nil}(\Lambda)$ a polynomial function φ_X on N. Unwrapping this definition, we have an explicit formula for evaluating φ_X on an arbitrary product $\mathbf{x_j}(\mathbf{t}) := x_{j_1}(t_1)\cdots x_{j_r}(t_r)$, namely

$$\varphi_X(\mathbf{x_j}(\mathbf{t})) = \sum_{\mathbf{a}\in\mathbb{N}^r} \chi_{\mathbf{j}^\mathbf{a}}(X)\frac{\mathbf{t}^\mathbf{a}}{\mathbf{a}!}, \tag{4}$$

where $\dfrac{\mathbf{t}^\mathbf{a}}{\mathbf{a}!} := \dfrac{t_1^{a_1}\cdots t_r^{a_r}}{a_1!\cdots a_r!}$, and $\mathbf{j}^\mathbf{a} := (j_1,\dots,j_1,j_2,\dots,j_2,\dots,j_r\dots,j_r)$ with each component j_k repeated a_k times.

Example 4.1. If $\mathfrak{g}$ is of type A_3, and if we denote by P_i the projective cover of S_i, one has

$$\varphi_{P_1} = D_{123,234},\quad \varphi_{P_2} = D_{12,34},\quad \varphi_{P_3} = D_{1,4}.$$

More generally, the functions φ_X corresponding to the 12 indecomposable Λ-modules are the 12 cluster variables of $\mathbb{C}[N]$ (see §2.2).

Via the correspondence $X \mapsto \varphi_X$, the ring $\mathbb{C}[N]$ can be regarded as a kind of (dual) Hall algebra of the category $\mathrm{nil}(\Lambda)$. Indeed, the multiplication of $\mathbb{C}[N]$ encodes extensions in $\mathrm{nil}(\Lambda)$, as shown by the following crucial result. Before stating it, we recall that $\mathrm{nil}(\Lambda)$ possesses a remarkable symmetry with respect to extensions, namely, $\mathrm{Ext}^1_\Lambda(X,Y)$ is isomorphic to the dual of $\mathrm{Ext}^1_\Lambda(Y,X)$ functorially in X and Y (see [22]). In particular $\dim \mathrm{Ext}^1_\Lambda(X,Y) = \dim \mathrm{Ext}^1_\Lambda(Y,X)$ for every X, Y.

Theorem 4.2 ([19, 22]). *Let $X, Y \in \mathrm{nil}(\Lambda)$.*

(i) *We have $\varphi_X\varphi_Y = \varphi_{X\oplus Y}$.*

(ii) *Assume that $\dim \mathrm{Ext}^1_\Lambda(X,Y) = 1$, and let*

$$0 \to X \to L \to Y \to 0 \quad \textit{and} \quad 0 \to Y \to M \to X \to 0$$

be non-split short exact sequences. Then $\varphi_X\varphi_Y = \varphi_L + \varphi_M$.

Example 4.3. We illustrate Theorem 4.2 (ii) in type A_2. Take $X = S_1$ and $Y = S_2$. Then we have the non-split short exact sequences

$$0 \to S_1 \to P_2 \to S_2 \to 0 \quad \text{and} \quad 0 \to S_2 \to P_1 \to S_1 \to 0,$$

which imply the relation

$$\varphi_{S_1}\varphi_{S_2} = \varphi_{P_2} + \varphi_{P_1},$$

that is, the elementary determinantal relation

$$D_{1,2}D_{2,3} = D_{1,3} + D_{12,23}$$

on the unitriangular subgroup of $\mathrm{SL}(3,\mathbb{C})$.

4.2. The dual semicanonical basis. We can now introduce the dual semicanonical basis of the vector space $\mathbb{C}[N]$. Let $\mathbf{d} = (d_i)$ be a dimension vector. The variety $\overline{E}_{\mathbf{d}}$ of representations of $\mathbb{C}\overline{Q}$ with dimension vector $\mathbf{d}$ is a vector space with a natural symplectic structure. Lusztig [41] has shown that $\Lambda_{\mathbf{d}}$ is a Lagrangian subvariety of $\overline{E}_{\mathbf{d}}$, whose number of irreducible components is equal to the dimension of the degree $\mathbf{d}$ homogeneous component of $U(\mathfrak{n})$. Let Z be an irreducible component of $\Lambda_{\mathbf{d}}$. Since $\varphi\colon X \mapsto \varphi_X$ is a constructible map on $\Lambda_{\mathbf{d}}$, it is constant on a Zariski dense open subset of Z. Let φ_Z denote this generic value of φ on Z. Then, if we denote by $\mathcal{Z}$ the collection of all irreducible components of all varieties $\Lambda_{\mathbf{d}}$, one can easily check that

$$\mathcal{S}^* := \{\varphi_Z \mid Z \in \mathcal{Z}\} \subset \mathbb{C}[N]$$

is dual to the basis $\mathcal{S} = \{f_Z \mid Z \in \mathcal{Z}\}$ of $\mathcal{M} \cong U(\mathfrak{n})$ constructed by Lusztig in [44], and called by him the semicanonical basis.

For example, suppose that $X \in \mathrm{nil}(\Lambda)$ is *rigid*, *i.e.* that $\mathrm{Ext}^1_\Lambda(X, X) = 0$. Then X is a generic point of the unique irreducible component Z on which it sits, that is, $\varphi_X = \varphi_Z$ belongs to $\mathcal{S}^*$.

4.3. Categories attached to Weyl group elements. Let $w \in W$ be of length r and choose a reduced decomposition $w = s_{i_r} \cdots s_{i_1}$ (note that again, we number the factors from right to left). It is well-known that Δ_w consists of the roots

$$\beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), \qquad (1 \le k \le r).$$

The following sequence of R^+ will also play an important role:

$$\gamma_k := \varpi_{i_k} - s_{i_1} \cdots s_{i_k}(\varpi_{i_k}), \qquad (1 \le k \le r).$$

From now on, we shall freely identify dimension vectors $\mathbf{d} = (d_i)$ with elements of R^+ via

$$\mathbf{d} \equiv \sum_{i \in I} d_i \alpha_i.$$

For $k = 1, \ldots, r$, one can show that there is a unique $V_k \in \mathrm{nil}(\Lambda)$ (up to isomorphism) whose socle is S_{i_k} and whose dimension vector is γ_k. Let us write $\mathbf{i} = (i_r, \ldots, i_1)$, and define

$$V_{\mathbf{i}} := \bigoplus_{k=1}^{r} V_k \in \mathrm{nil}(\Lambda).$$

Up to duality, this Λ-module is the same as the one introduced by Buan, Iyama, Reiten, and Scott [5]. Following [5], we consider the full subcategory of $\mathrm{nil}(\Lambda)$ whose objects are factor modules of direct sums of finitely many copies of $V_{\mathbf{i}}$. It turns out that this category depends only on w (and not on the choice of a reduced expression $\mathbf{i}$), so we may denote it by $\mathcal{C}_w$.

For $j \in I$, let $k_j := \max\{1 \le k \le r \mid i_k = j\}$. The submodule $\bigoplus_{j \in I} V_{k_j}$ of $V_{\mathbf{i}}$ also depends only on w, and we denote it by I_w.

Example 4.4. We consider again $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ and $w = s_2 s_1 s_2 s_1$, so $\mathbf{i} = (2, 1, 2, 1)$. We take for Q the Kronecker quiver:

$$\begin{tikzcd} 1 \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & 2 \end{tikzcd}$$

The following pictures describe the indecomposable direct summands of $V_{\mathbf{i}}$. The numbers 1 and 2 in the pictures are basis vectors of the modules V_k. The solid edges show how the arrows a and b of $\overline{Q}$ act on these vectors, and the dotted edges illustrate the actions of a^* and b^*. The arrows a and b^* are pointing south east, and the arrow a^* and b are pointing south west.

$$V_1 = 1 \qquad V_2 = \begin{tikzcd}[column sep=small, row sep=small] 1 \arrow[dr] & & 1 \arrow[dl] \\ & 2 & \end{tikzcd}$$

$$V_3 = \begin{tikzcd}[column sep=small, row sep=small] 1 \arrow[dr] & & 1 \arrow[dl] \arrow[dr] & & 1 \arrow[dl] \\ & 2 \arrow[dr, dotted] & & 2 \arrow[dl, dotted] & \\ & & 1 & & \end{tikzcd} \qquad V_4 = \begin{tikzcd}[column sep=small, row sep=small] 1 \arrow[dr] & & 1 \arrow[dl] \arrow[dr] & & 1 \arrow[dl] \arrow[dr] & & 1 \arrow[dl] \\ & 2 \arrow[dr, dotted] & & 2 \arrow[dl, dotted] \arrow[dr, dotted] & & 2 \arrow[dl, dotted] & \\ & & 1 \arrow[dr] & & 1 \arrow[dl] & & \\ & & & 2 & & & \end{tikzcd}$$

Here, $I_w = V_3 \oplus V_4$. The modules

$$M_3 = \begin{tikzcd}[column sep=small, row sep=small] 1 \arrow[dr] & & 1 \arrow[dl] \arrow[dr] & & 1 \arrow[dl] \\ & 2 & & 2 & \end{tikzcd} \qquad M_4 = \begin{tikzcd}[column sep=small, row sep=small] 1 \arrow[dr] & & 1 \arrow[dl] \arrow[dr] & & 1 \arrow[dl] \arrow[dr] & & 1 \arrow[dl] \\ & 2 & & 2 & & 2 & \end{tikzcd}$$

are factor modules of V_3 and V_4 respectively, so they are objects of $\mathcal{C}_w$. On the other hand the Λ-module

$$X = \begin{tikzcd}[column sep=small, row sep=small] 2 \arrow[dr, dotted] & \\ & 1 \end{tikzcd}$$

cannot be obtained as a factor module of (a direct sum of copies) of $V_{\mathbf{i}}$, hence it does not belong to $\mathcal{C}_w$.

4.4. Frobenius categories. Let us recall some definitions from homological algebra. Let $\mathcal{C}$ be a subcategory of the category of modules over an algebra A, which is closed under extensions. Clearly, we have

$$\mathrm{Ext}^1_{\mathcal{C}}(X, Y) = \mathrm{Ext}^1_A(X, Y)$$

for all modules X and Y in $\mathcal{C}$. An A-module C in $\mathcal{C}$ is called $\mathcal{C}$-*projective* (resp. $\mathcal{C}$-*injective*) if $\mathrm{Ext}^1_A(C, X) = 0$ (resp. $\mathrm{Ext}^1_A(X, C) = 0$) for all $X \in \mathcal{C}$. If C is $\mathcal{C}$-projective and $\mathcal{C}$-injective, then C is also called $\mathcal{C}$-*projective-injective*. We say that $\mathcal{C}$ has *enough projectives* (resp. *enough injectives*) if for each $X \in \mathcal{C}$ there exists a short exact sequence $0 \to Y \to C \to X \to 0$ (resp. $0 \to X \to C \to Y \to 0$) where C is $\mathcal{C}$-projective (resp. $\mathcal{C}$-injective) and $Y \in \mathcal{C}$. If $\mathcal{C}$ has enough projectives and enough injectives, and if these coincide (i.e. an object is $\mathcal{C}$-projective if and only if it is $\mathcal{C}$-injective), then $\mathcal{C}$ is called a *Frobenius category*.

For such a Frobenius category $\mathcal{C}$ we can define its *stable category* $\underline{\mathcal{C}}$. The objects of $\underline{\mathcal{C}}$ are the same as the objects of $\mathcal{C}$, and the morphism spaces $\mathrm{Hom}_{\underline{\mathcal{C}}}(X,Y)$ are the morphism spaces in $\mathcal{C}$ modulo morphisms factoring through $\mathcal{C}$-projective-injective objects. The category $\underline{\mathcal{C}}$ is a triangulated category in a natural way [30], where the shift is given by the relative inverse syzygy functor $\Omega^{-1}\colon \underline{\mathcal{C}} \to \underline{\mathcal{C}}$. (Recall that the syzygy functor Ω maps an object to the kernel of its projective cover. This induces an auto-equivalence of the stable category.) For all X and Y in $\mathcal{C}$ there is a functorial isomorphism

$$\mathrm{Ext}^1_{\mathcal{C}}(X,Y) \cong \mathrm{Hom}_{\underline{\mathcal{C}}}(X,\Omega^{-1}(Y)). \tag{5}$$

The category $\mathcal{C}$ is called *stably* 2*-Calabi-Yau* if the stable category $\underline{\mathcal{C}}$ is a 2*-Calabi-Yau category*, that is, if for all $X,Y \in \mathcal{C}$ there is a functorial isomorphism

$$\mathrm{Ext}^1_{\mathcal{C}}(X,Y) \cong \mathrm{D}\mathrm{Ext}^1_{\mathcal{C}}(Y,X), \tag{6}$$

where D denotes the duality for vector spaces. We can now state some important properties of the categories $\mathcal{C}_w$.

Theorem 4.5 ([5, 23]). *The categories $\mathcal{C}_w$ are Frobenius categories. They are stably* 2*-Calabi-Yau. The indecomposable $\mathcal{C}_w$-projective-injective modules, are the indecomposable direct summands of I_w.*

4.5. Cluster-tilting modules. Let $\mathcal{C}$ be a subcategory of $\mathrm{nil}(\Lambda)$ closed under extensions, direct sums and direct summands. For an object T of $\mathcal{C}$ we denote by $\mathrm{add}(T)$ the *additive envelope* of T, that is, the full subcategory whose objects are finite direct sums of direct summands of T. We say that

- T is $\mathcal{C}$*-maximal rigid* if $\mathrm{Ext}^1_\Lambda(T\oplus X, X) = 0$ with $X \in \mathcal{C}$ implies $X \in \mathrm{add}(T)$;
- T is a $\mathcal{C}$*-cluster-tilting module* if $\mathrm{Ext}^1_\Lambda(T,X) = 0$ with $X \in \mathcal{C}$ implies $X \in \mathrm{add}(T)$.

Since $\mathrm{Ext}^1_\Lambda(T\oplus X, X) \cong \mathrm{Ext}^1_\Lambda(T,X)\oplus\mathrm{Ext}^1_\Lambda(X,X)$, we see that the second property is a priori stronger than the first.

Theorem 4.6 ([5, 23]). *For a rigid Λ-module T in $\mathcal{C}_w$ the following are equivalent:*

(i) *T has r pairwise non-isomorphic indecomposable direct summands;*

(ii) *T is $\mathcal{C}_w$-maximal rigid;*

(iii) *T is a $\mathcal{C}_w$-cluster-tilting module.*

Let $T = T_1 \oplus \cdots \oplus T_r$ be a $\mathcal{C}_w$-cluster-tilting module, with each summand T_i indecomposable. Consider the endomorphism algebra $A_T := \mathrm{End}_\Lambda(T)^{\mathrm{op}}$. This is a basic algebra, with indecomposable projective modules

$$P_{T_i} := \mathrm{Hom}_\Lambda(T,T_i), \qquad (1 \le i \le r).$$

The simple A_T-modules are the heads S_{T_i} of the projectives P_{T_i}. One defines a quiver Γ_T with vertex set $\{1, \dots, r\}$, and d_{ij} arrows from i to j, where

$$d_{ij} := \dim \operatorname{Ext}^1_{A_T}(S_{T_i}, S_{T_j}).$$

This is known as the *Gabriel quiver* of A_T.

Let now $\mathbf{i} = (i_r, \dots, i_1)$ be a reduced expression for w. Following [2], we define a quiver $\Gamma_{\mathbf{i}}$ as follows. The vertex set of $\Gamma_{\mathbf{i}}$ is equal to $\{1, \dots, r\}$. For $1 \le k \le r$, let

$$\begin{aligned} k^- &:= \max\left(\{0\} \cup \{1 \le s \le k-1 \mid i_s = i_k\}\right), \\ k^+ &:= \min\left(\{k+1 \le s \le r \mid i_s = i_k\} \cup \{r+1\}\right). \end{aligned}$$

For $1 \le s,\, t \le r$ such that $i_s \ne i_t$, there are $|c_{i_s,i_t}|$ arrows from s to t provided $t^+ \ge s^+ > t > s$. (Here, as in §3.1, the c_{ij}'s are the entries of the Cartan matrix.) These are called the *ordinary arrows* of $\Gamma_{\mathbf{i}}$. Furthermore, for each $1 \le s \le r$ there is an arrow $s \to s^-$ provided $s^- > 0$. These are the *horizontal arrows* of $\Gamma_{\mathbf{i}}$.

Theorem 4.7 ([5, 23]). *The module $V_{\mathbf{i}}$ is a $\mathcal{C}_w$-cluster-tilting module, and we have $\Gamma_{V_{\mathbf{i}}} = \Gamma_{\mathbf{i}}$.*

Example 4.8. We continue Example 4.4. For $\mathbf{i} = (2, 1, 2, 1)$, the Gabriel quiver $\Gamma_{\mathbf{i}}$ of $\operatorname{End}_\Lambda(V_{\mathbf{i}})^{\mathrm{op}}$ is:

$$\begin{tikzcd}
4 \arrow[rr] & & 2 \arrow[dl, shift left] \arrow[dl, shift right] & \\
& 3 \arrow[ul, shift left] \arrow[ul, shift right] \arrow[rr] & & 1 \arrow[ul, shift left] \arrow[ul, shift right]
\end{tikzcd}$$

Note that the $\mathcal{C}_w$-projective summands of $V_{\mathbf{i}}$ correspond to the leftmost vertices of each row of $\Gamma_{\mathbf{i}}$.

4.6. Mutations of cluster-tilting modules.

Let $T = T_1 \oplus \cdots \oplus T_r$ be a $\mathcal{C}_w$-cluster-tilting module, and let Γ_T be the corresponding quiver defined in §4.5. Define

$$b_{ij} := (\text{number of arrows } j \to i \text{ in } \Gamma_T) - (\text{number of arrows } i \to j \text{ in } \Gamma_T).$$

Clearly each indecomposable $\mathcal{C}_w$-projective-injective module is a direct summand of T. In the sequel we will not need the arrows of Γ_T between the vertices corresponding to these projective-injective summands.

Theorem 4.9 ([5, 23]). *Let T_k be a non-projective indecomposable direct summand of T.*

(i) *There exists a unique indecomposable module $T_k^* \ncong T_k$ such that $(T/T_k) \oplus T_k^*$ is a $\mathcal{C}_w$-cluster-tilting module. We call $(T/T_k) \oplus T_k^*$ the* mutation *of T in direction k, and denote it by $\mu_k(T)$.*

(ii) *The quiver $\Gamma_{\mu_k(T)}$ is equal to the mutation μ_k of Γ_T in the sense of Fomin and Zelevinsky (ignoring the arrows between projective-injective vertices).*

(iii) *We have* $\dim \operatorname{Ext}^1_\Lambda(T_k, T_k^*) = 1$. *Let*

$$0 \to T_k \to T_k' \to T_k^* \to 0 \quad \textit{and} \quad 0 \to T_k^* \to T_k'' \to T_k \to 0$$

be non-split short exact sequences. Then

$$T_k' \cong \bigoplus_{b_{jk}<0} T_j^{-b_{jk}}, \qquad T_k'' \cong \bigoplus_{b_{jk}>0} T_j^{b_{jk}}.$$

Remark 4.10. Let $\underline{T}$ be any cluster-tilting object of the stable category $\underline{\mathcal{C}}_w$, coming from a cluster-tilting module T in the mutation class of a module $V_{\mathbf{i}}$. It was shown by Buan, Iyama, Reiten and Smith [6] that the endomorphism algebra of $\underline{T}$ is the Jacobian algebra of a quiver with potential. Jacobian algebras and their mutations have been introduced by Derksen, Weyman and Zelevinsky [9], and they have been used in [10] to prove several important conjectures on cluster algebras by Fomin and Zelevinsky [15].

4.7. A distinguished sequence of mutations. For $1 \le k \le l \le r$ such that $i_k = i_l = i$, we have a natural embedding of Λ-modules $V_{k^-} \to V_l$. Following [25, §9.8], we define $M[l,k]$ as the cokernel of this embedding, that is,

$$M[l,k] := V_l / V_{k^-}.$$

In particular, we set $M_k := M[k,k]$, and

$$M_{\mathbf{i}} := M_1 \oplus \cdots \oplus M_r.$$

We will use the convention that $M[l,k] = 0$ if $k > l$. Every module $M[l,k]$ is indecomposable and rigid. But note that $M_{\mathbf{i}}$ is *not* rigid. Define

$$k_{\min} := \min\{1 \le s \le r \mid i_s = i_k\},$$
$$k_{\max} := \max\{1 \le s \le r \mid i_s = i_k\}.$$

Then $V_k = M[k, k_{\min}]$ corresponds to an *initial interval*. The direct sum of all modules $M[k_{\max}, k]$ corresponding to *final intervals* is also a $\mathcal{C}_w$-cluster-tilting module, denoted by $T_{\mathbf{i}}$. We number the summands of $T_{\mathbf{i}}$ as follows:

$$T_k := \begin{cases} V_k & \text{if } k^+ = r+1, \\ M[k_{\max}, k^+] & \text{otherwise.} \end{cases}$$

This numbering ensures that, for a non-projective V_k we have $\Omega_w^{-1}(V_k) = T_k$, where Ω_w denotes the relative syzygy functor of $\underline{\mathcal{C}}_w$.

Example 4.11. We continue Example 4.4 and Example 4.8. In this case we have $M_1 = V_1$, $M_2 = V_2$, and the modules M_3 and M_4 were already introduced in Example 4.4. The indecomposable direct summands of $T_{\mathbf{i}}$ are then

$$T_1 = M_3, \quad T_2 = M_4, \quad T_3 = V_3, \quad T_4 = V_4,$$

and the quiver $\Gamma_{T_{\mathbf{i}}}$ is

$$\begin{tikzcd}
2 \arrow[rr] & & 4 \arrow[dl, shift left] \arrow[dl, shift right] & \\
& 1 \arrow[ul, shift left] \arrow[ul, shift right] \arrow[rr] & & 3 \arrow[ul, shift left] \arrow[ul, shift right]
\end{tikzcd}$$

Note that this quiver is the same as $\Gamma_{V_{\mathbf{i}}}$, but the vertices corresponding to the $\mathcal{C}_w$-projective summands are now at the right end of each row.

It was shown in [25, §13.1] that there is an explicit sequence of mutations of $\mathcal{C}_w$-cluster-tilting modules starting from $V_{\mathbf{i}}$ and ending in $T_{\mathbf{i}}$. This sequence of mutations plays an important role in several of our constructions, so we want to describe it in some detail. For $j \in I$ and $1 \leq k \leq r+1$, we set

$$k[j] := |\{1 \leq s \leq k-1 \mid i_s = j\}|, \qquad t_j := (r+1)[j].$$

Thus t_j is the number of occurences of j in $\mathbf{i}$. Our sequence consists of

$$\sum_{j \in I} \frac{t_j(t_j-1)}{2}$$

mutations, which we conveniently group into r steps. We start from $V_{\mathbf{i}}$ and its quiver $\Gamma_{\mathbf{i}}$. Note that $\Gamma_{\mathbf{i}}$ is naturally displayed on n rows, where all vertices k such that $i_k = j$ are sitting on row j, with their labels increasing from right to left (see Example 4.8). At Step 1, we perform $t_{i_1} - 1 - 1[i_1] = t_{i_1} - 1$ mutations at consecutive vertices sitting on row i_1 of $\Gamma_{\mathbf{i}}$, starting from the rightmost vertex. Next, at Step $k = 2, \dots, r$, we perform $t_{i_k} - 1 - k[i_k]$ mutations at consecutive vertices sitting on row i_k, starting from the rightmost vertex. Now we claim:

Theorem 4.12 ([25, §13]). (i) *The above sequence of mutations applied to $V_{\mathbf{i}}$ gives $T_{\mathbf{i}}$.*

(ii) *All the indecomposable direct summands of the $\mathcal{C}_w$-cluster-tilting modules occuring in this sequence are of the form $M[l,k]$.*

(iii) *Each step of this sequence consists of the mutation of a module $M[d^-,b]$ into a module $M[d,b^+]$, for some $1 \leq b < d \leq r$ with $i_b = i_d = i$. The corresponding pair of short exact sequences is*

$$0 \to M[d^-,b] \to M[d^-,b^+] \oplus M[d,b] \to M[d,b^+] \to 0,$$

$$0 \to M[d,b^+] \to \bigoplus_{j \neq i} M[d^-(j), b^+(j)]^{\oplus |c_{ij}|} \to M[d^-,b] \to 0,$$

where for $1 \leq k \leq r$, we set

$$k^-(j) := \max\{0, s < k \mid i_s = j\}, \quad k^+(j) := \min\{r+1, k < s \mid i_s = j\}.$$

(iv) *Every module $M[l,k]$ with $1 \leq k \leq l \leq r$ and $i_k = i_l$ arises in this sequence.*

Example 4.13. We continue Example 4.11. We have $t_1 = t_2 = 2$, hence the sequence consists of only $1 + 1 = 2$ mutations.

Step 1: we perform mutation μ_1. The two short exact sequences are

$$0 \to V_1 \to V_3 \to T_1 \to 0, \qquad 0 \to T_1 \to V_2^{\oplus 2} \to V_1 \to 0.$$

Note that we have

$$V_1 = M[3^-, 1], \quad T_1 = M[3, 1^+], \quad V_3 = M[3, 1], \quad V_2 = M[3^-(2), 1^+(2)].$$

Step 2: we perform mutation μ_2. The two short exact sequences are

$$0 \to V_2 \to V_4 \to T_2 \to 0, \qquad 0 \to T_2 \to T_1^{\oplus 2} \to V_2 \to 0.$$

Note that we have

$$V_2 = M[4^-, 2], \quad T_2 = M[4, 2^+], \quad V_4 = M[4, 2], \quad T_1 = M[4^-(1), 2^+(1)].$$

For more complicated examples, we refer to [25, §13].

We conclude this section by noting that the functions $\varphi_{M[l,k]}$ associated with the modules $M[l, k]$ are restrictions to N_+ of some generalized minors in the sense of §3.3. In particular in type A_n, they are nothing but ordinary minors of a unitriangular matrix of size $n + 1$. Using Theorem 4.2, we can convert the above mutation sequence into a sequence of determinantal identities, and thus recover certain identities of Fomin and Zelevinsky [12].

5. Cluster structures on coordinate rings

We can now use the categories $\mathcal{C}_w$ of §4.3 to produce some cluster algebras, and show that they are isomorphic to the coordinate rings of $N(w)$ and N^w. Our construction readily implies that the cluster monomials are contained in the dual semicanonical basis $\mathcal{S}^*$ of $\mathbb{C}[N]$.

5.1. From categories to cluster algebras. We say that a $\mathcal{C}_w$-cluster-tilting module is *reachable* if it can be obtained from $V_{\mathbf{i}}$ as the result of a (finite) sequence of mutations. It is known [5, Lemma II.4.2] that if $\mathbf{j}$ is another reduced word for w, then $V_{\mathbf{j}}$ is reachable from $V_{\mathbf{i}}$, hence this notion does not depend on the choice of $\mathbf{i}$. It is an open problem whether every $\mathcal{C}_w$-cluster-tilting module is reachable. More generally, we call a Λ-module *reachable* if it is isomorphic to a direct summand of a reachable $\mathcal{C}_w$-cluster-tilting module.

Let $\mathcal{R}(\mathcal{C}_w)$ be the subalgebra of $\mathbb{C}[N]$ generated by the φ_{T_k} $(1 \le k \le r)$ where $T = T_1 \oplus \cdots \oplus T_r$ runs over all reachable $\mathcal{C}_w$-cluster-tilting modules. Let $\mathcal{A}_{\mathbf{i}}$ denote the cluster algebra defined by the initial seed $((y_1, \ldots, y_r), \Gamma_{\mathbf{i}})$, in which the variables y_k corresponding to $\mathcal{C}_w$-projective-injective vertices are frozen.

Theorem 5.1 ([25]). (i) *There is a unique algebra isomorphism ι from $\mathcal{A}_{\mathbf{i}}$ to $\mathcal{R}(\mathcal{C}_w)$ such that*

$$\iota(y_k) = \varphi_{V_k}, \qquad (1 \le k \le r).$$

(ii) *If we identify the two algebras $\mathcal{A}_{\mathbf{i}}$ and $\mathcal{R}(\mathcal{C}_w)$ via ι, then the clusters of $\mathcal{A}_{\mathbf{i}}$ are identified with the r-tuples $(\varphi_{T_1}, \dots, \varphi_{T_r})$, where T runs over all reachable $\mathcal{C}_w$-cluster-tilting modules. Moreover, all cluster monomials belong to the dual semicanonical basis $\mathcal{S}^*$ of $\mathbb{C}[N]$.*

Theorem 5.1 gives a Lie-theoretic realization of a large class of cluster algebras. Its proof relies on Theorem 4.2 and Theorem 4.9. As an application, we can compute the Euler characteristics $\chi(\mathcal{F}_{X,\mathbf{i}})$ for all modules X in the additive closure of a reachable $\mathcal{C}_w$-cluster-tilting module, and all composition series types $\mathbf{i}$, using the Fomin-Zelevinsky mutation formula (1). Equivalently, we have an algorithm for computing the elements φ_X of $\mathcal{S}^*$ corresponding to any reachable rigid module.

The next theorem describes $\mathbb{C}$-bases of the above cluster algebras, and shows that they are polynomial rings. Recall from §4.7 the Λ-module

$$M_{\mathbf{i}} = M_1 \oplus \cdots \oplus M_r \in \mathcal{C}_w.$$

Theorem 5.2 ([25]). (i) *The cluster algebra $\mathcal{R}(\mathcal{C}_w)$ is a polynomial ring in r variables. More precisely, we have*

$$\mathcal{R}(\mathcal{C}_w) = \mathbb{C}[\varphi_{M_1}, \dots, \varphi_{M_r}].$$

(ii) *The set $\mathcal{P}^*_{\mathbf{i}} := \{\varphi_M \mid M \in \mathrm{add}(M_{\mathbf{i}})\}$ is a $\mathbb{C}$-basis of $\mathcal{R}(\mathcal{C}_w)$.*

(iii) *The subset $\mathcal{S}^*_w := \mathcal{S}^* \cap \mathcal{R}(\mathcal{C}_w)$ of the dual semicanonical basis of $\mathbb{C}[N]$ is a $\mathbb{C}$-basis of $\mathcal{R}(\mathcal{C}_w)$ containing all cluster monomials.*

The bases given by Theorem 5.2 (ii) and (iii) are called *dual PBW basis* and *dual semicanonical basis* of $\mathcal{R}(\mathcal{C}_w)$, respectively. The proof of this theorem uses Theorem 4.12 for showing that $\mathbb{C}[\varphi_{M_1}, \dots, \varphi_{M_r}] \subseteq \mathcal{R}(\mathcal{C}_w)$. The reverse inclusion is obtained by proving that $\mathcal{P}^*_{\mathbf{i}}$ is a subset of a basis of $\mathbb{C}[N]$ dual to a Poincaré-Birkhoff-Witt basis of $U(\mathfrak{n})$ (see [25, §15]).

5.2. Coordinate rings of unipotent subgroups and unipotent cells. Using the fact that $\mathcal{P}^*_{\mathbf{i}}$ is a subset of an appropriate dual PBW-basis, one can show that the functions φ_{M_k} are $N'(w)$-invariant. It then follows from Proposition 3.2 that we can relate the subalgebra $\mathcal{R}(\mathcal{C}_w)$ of $\mathbb{C}[N]$ to the coordinate ring of $N(w)$, as explained in the next theorem.

Theorem 5.3 ([25]). *The cluster algebra $\mathcal{R}(\mathcal{C}_w)$ coincides with the invariant subring $\mathbb{C}[N]^{N'(w)}$, and is naturally isomorphic to $\mathbb{C}[N(w)]$.*

As a result, we have obtained that the coordinate ring $\mathbb{C}[N(w)]$ has the structure of a cluster algebra. Moreover, its clusters are in one-to-one correspondence with reachable $\mathcal{C}_w$-cluster-tilting objects. The frozen cluster variables are the φ_{I_i}, where I_i is the indecomposable $\mathcal{C}_w$-projective-injective module with simple socle S_i. It turns out that

$$\varphi_{I_i} = D_{\varpi_i,\, w^{-1}(\varpi_i)}, \qquad (i \in I),$$

where we denote by $D_{u(\varpi_i),v(\varpi_i)}$ the restriction to N_+ of the generalized minor $\Delta_{u(\varpi_i),v(\varpi_i)}$. The coefficient ring of the cluster algebra $\mathbb{C}[N(w)]$ is therefore the polynomial ring in the frozen variables $D_{\varpi_i,w^{-1}(\varpi_i)}$.

By Proposition 3.5, these frozen variables are precisely the functions that need to be inverted in order to pass from $\mathbb{C}[N(w)]$ to the coordinate ring $\mathbb{C}[N^w]$ of the unipotent cell N^w. Therefore $\mathbb{C}[N^w]$ has almost the same cluster algebra structure as $\mathbb{C}[N(w)]$. The only difference is in the coefficient ring, which in the case of N^w should be taken as the ring of *Laurent polynomials* in the generalized minors $D_{\varpi_i,\, w^{-1}(\varpi_i)}$. Hence we have

Theorem 5.4 ([25]). *The cluster algebra $\widetilde{\mathcal{R}}(\mathcal{C}_w)$ obtained from $\mathcal{R}(\mathcal{C}_w)$ by formally inverting all the frozen variables is naturally isomorphic to $\mathbb{C}[N^w]$.*

When $\mathfrak{g}$ is finite-dimensional, the cluster algebra structure on the unipotent cell N^w given by Theorem 5.4 was already obtained by Berenstein, Fomin and Zelevinsky [2], by a completely different method. In fact they treated the more general case of *double Bruhat cells*

$$G^{v,w} := (BvB) \cap (B_-wB_-), \qquad (v, w \in W).$$

These varieties $G^{v,w}$ carry a free action of the torus H by left (or right) multiplication, and quotienting this action gives the *reduced double Bruhat cells*

$$L^{v,w} := (NvN) \cap (B_-wB_-), \qquad (v, w \in W).$$

One can then show that going from $G^{e,w}$ to $L^{e,w} = N^w$ only modifies the coefficient ring of the cluster algebra. However, our approach shows that $\mathbb{C}[N^w]$ is a genuine cluster algebra (not only an *upper* cluster algebra in the sense of [2]), and it shows that cluster monomials belong to the dual semicanonical basis.

In the general Kac-Moody case, Theorem 5.4 was conjectured by Buan, Iyama, Reiten and Scott [5, Conjecture III.3.1]. Theorem 5.4 has also been extended to the non simply-laced case and adaptable Weyl group element w by Demonet [8].

Example 5.5. We continue Example 3.7. The full subquiver obtained by erasing vertices 3 and 4 of the quiver of Example 4.8 is the Kronecker quiver with two arrows. It follows that $\mathbb{C}[N(w)]$ is a rank 2 acyclic cluster algebra of affine type $\widetilde{A}_1$, that is, a version of the cluster algebra $\mathcal{A}_\Sigma$ of Example 2.1, with $a = 2$ and two additional frozen variables. It has infinitely many clusters and cluster variables.

Using the notation of Example 4.4 and Example 4.11, the dual PBW generators φ_{M_i} evaluated at an element $x \in N(w)$ are expressed, in terms of the coordinate functions b_k on $\mathbb{C}[N(w)]$ introduced in Example 3.7, as

$$\varphi_{M_1} = b_0, \quad \varphi_{M_2} = -b_1, \quad \varphi_{M_3} = b_2, \quad \varphi_{M_4} = -b_3. \qquad (7)$$

Our initial cluster for $\mathbb{C}[N(w)]$ is $(\varphi_{V_1}, \varphi_{V_2}, \varphi_{V_3}, \varphi_{V_4})$, where

$$\varphi_{V_1} = b_0, \quad \varphi_{V_2} = -b_1, \quad \varphi_{V_3} = b_0 b_2 - b_1^2, \quad \varphi_{V_4} = b_1 b_3 - b_2^2. \tag{8}$$

In agreement with Example 4.13, one can check the exchange relations corresponding to a mutation at vertex 1 followed by a mutation at vertex 2:

$$\varphi_{V_1}\varphi_{M_3} = \varphi_{V_3} + \varphi_{V_2}^2, \qquad \varphi_{V_2}\varphi_{M_4} = \varphi_{V_4} + \varphi_{M_3}^2,$$

If instead we start mutating at vertex 2, we get the new cluster variable

$$\frac{b_0^2(b_1b_3 - b_2^2) + (b_0b_2 - b_1)^2}{(-b_1)} = 2b_0b_1b_2 - b_1^3 - b_0^2 b_3.$$

We may also evaluate φ_{V_i} and φ_{M_i} at an arbitrary point $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of N_+. We obtain the following determinantal expressions in terms of the coordinate functions b_k and d_k of Example 3.6:

$$\varphi_{M_1} = \varphi_{V_1} = b_0, \qquad \varphi_{M_2} = \varphi_{V_2} = \begin{vmatrix} b_0 & b_1 \\ 1 & d_1 \end{vmatrix},$$

$$\varphi_{V_3} = \begin{vmatrix} b_0 & b_1 & b_2 \\ 1 & d_1 & d_2 \\ 0 & b_0 & b_1 \end{vmatrix}, \qquad \varphi_{M_3} = \begin{vmatrix} b_0 & b_1 & b_2 \\ 1 & d_1 & d_2 \\ 0 & 1 & d_1 \end{vmatrix},$$

$$\varphi_{V_4} = \begin{vmatrix} b_0 & b_1 & b_2 & b_3 \\ 1 & d_1 & d_2 & d_3 \\ 0 & b_0 & b_1 & b_2 \\ 0 & 1 & d_1 & d_2 \end{vmatrix}, \quad \varphi_{M_4} = \begin{vmatrix} b_0 & b_1 & b_2 & b_3 \\ 1 & d_1 & d_2 & d_3 \\ 0 & 1 & d_1 & d_2 \\ 0 & 0 & 1 & d_1 \end{vmatrix}.$$

(These formulas give back equations (7), (8), if we specialize $d_1 = d_2 = d_3 = 0$.) Using the description of $N'(w)$ from Example 3.7, it is easy to check that these determinantal expressions are $N'(w)$-invariant, as claimed in Theorem 5.3.

For more general determinantal and combinatorial evaluations of functions φ_X in the case of $\widehat{\mathfrak{sl}}_2$, see [48].

6. Chamber Ansatz

We now come back to the problem of computing the factorization parameters t_k of a point $x_{\mathbf{i}}(\mathbf{t})$ of the unipotent cell N^w (see §3.5). To do so, we introduce a new distinguished cluster-tilting object of $\mathcal{C}_w$. Let Ω_w be the syzygy functor of the stable category $\underline{\mathcal{C}}_w$. Thus, for a module $X \in \mathcal{C}_w$, we have a short exact sequence

$$0 \to \Omega_w(X) \to P(X) \to X \to 0,$$

where $P(X)$ denotes the projective cover of X in $\mathcal{C}_w$. Note that for a $\mathcal{C}_w$-projective-injective object X, $\Omega_w(X) = 0$. Define

$$W_{\mathbf{i}} := \Omega_w(V_{\mathbf{i}}) \oplus I_w.$$

Since Ω_w is an auto-equivalence of $\underline{\mathcal{C}}_w$, we see that $W_{\mathbf{i}}$ is indeed cluster-tilting.

Recall from §4.7 that we have another $\mathcal{C}_w$-cluster-tilting module $T_{\mathbf{i}}$, related to $V_{\mathbf{i}}$ by a distinguished sequence of mutations. It is easy to check that

$$V_{\mathbf{i}} = \Omega_w(T_{\mathbf{i}}) \oplus I_w.$$

This implies that, given a reachable $\mathcal{C}_w$-cluster-tilting module M, we have an "algorithm" for producing a sequence of mutations from M to $\Omega_w(M) \oplus I_w$ (see [25, Proposition 13.4]). In particular, $\Omega_w(M) \oplus I_w$ is again reachable. Hence $W_{\mathbf{i}}$ is reachable, and the functions φ_{W_k} associated with its indecomposable direct summands W_k $(1 \leq k \leq r)$ form a new cluster of the cluster algebra $\mathbb{C}[N^w]$. For $k = 1, \dots, r$, define

$$\varphi'_{V_k} := \frac{\varphi_{W_k}}{\varphi_{P(V_k)}}, \tag{9}$$

a Laurent monomial in the φ_{W_k} (since $\mathrm{add}(W_{\mathbf{i}})$ contains all $\mathcal{C}_w$-projective-injective modules), and put

$$C_k := \frac{1}{\varphi'_{V_k} \varphi'_{V_{k^-(i_k)}}} \cdot \prod_{j \neq i_k} \left(\varphi'_{V_{k^-(j)}} \right)^{|c_{i_k,j}|}. \tag{10}$$

Here we recall that $k^-(j) := \max\{0, 1 \leq s \leq k-1 \mid i_s = j\}$ and V_0 is by convention the zero module. We can now state the Chamber Ansatz formula, which expresses the rational function t_k as an explicit Laurent monomial in the cluster variables φ_{W_k}.

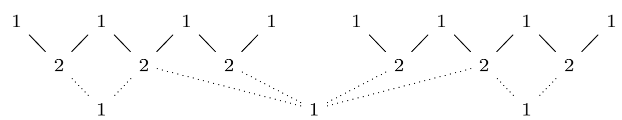
Theorem 6.1. *For $1 \leq k \leq r$ and $\mathbf{x}_{\mathbf{i}}(\mathbf{t}) = x_{i_r}(t_r) \cdots x_{i_1}(t_1)$ we have*

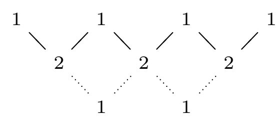
$$t_k = C_k(\mathbf{x}_{\mathbf{i}}(\mathbf{t})).$$

Example 6.2. We continue Example 3.6. First we compute $W_{\mathbf{i}}$. We have short exact sequences

$$0 \to W_1 \to P(V_1) \to V_1 \to 0 \qquad \text{and} \qquad 0 \to W_2 \to P(V_2) \to V_2 \to 0.$$

where $P(V_1) \cong V_3^{\oplus 3}$ and $P(V_2) \cong V_3^{\oplus 2}$. Thus, using the same convention as in Example 4.4, we can represent W_1 and W_2 as

$W_1 =$ 

$W_2 =$ 

Putting $\mathbf{x_i}(\mathbf{t}) = x_2(t_4)x_1(t_3)x_2(t_2)x_1(t_1)$, one can then calculate

$$\begin{aligned}
\varphi_{V_1}(\mathbf{x_i}(\mathbf{t})) &= t_3 + t_1,\\
\varphi_{V_2}(\mathbf{x_i}(\mathbf{t})) &= t_4(t_3^2 + 2t_3t_1 + t_1^2) + t_2t_1^2,\\
\varphi_{V_3}(\mathbf{x_i}(\mathbf{t})) &= t_3t_2^2t_1^3,\\
\varphi_{V_4}(\mathbf{x_i}(\mathbf{t})) &= t_4t_3^2t_2^3t_1^4,\\
\varphi_{W_1}(\mathbf{x_i}(\mathbf{t})) &= t_3^3t_2^6t_1^8,\\
\varphi_{W_2}(\mathbf{x_i}(\mathbf{t})) &= t_3^2t_2^3t_1^4.
\end{aligned}$$

Noting that

$$\varphi_{W_1} = \varphi'_{V_1}\varphi^3_{V_3}, \quad \varphi_{W_2} = \varphi'_{V_2}\varphi^2_{V_3}, \quad \varphi_{W_3} = \varphi_{V_3} = \frac{1}{\varphi'_{V_3}}, \quad \varphi_{W_4} = \varphi_{V_4} = \frac{1}{\varphi'_{V_4}},$$

we thus get, in agreement with Theorem 6.1,

$$\begin{aligned}
t_1 &= \frac{1}{\varphi'_{V_1}}(\mathbf{x_i}(\mathbf{t})) &&= \frac{\varphi^3_{W_3}}{\varphi_{W_1}}(\mathbf{x_i}(\mathbf{t})),\\
t_2 &= \frac{(\varphi'_{V_1})^2}{\varphi'_{V_2}}(\mathbf{x_i}(\mathbf{t})) &&= \frac{\varphi^2_{W_1}}{\varphi_{W_2}\varphi^4_{W_3}}(\mathbf{x_i}(\mathbf{t})),\\
t_3 &= \frac{(\varphi'_{V_2})^2}{\varphi'_{V_3}\varphi'_{V_1}}(\mathbf{x_i}(\mathbf{t})) &&= \frac{\varphi^2_{W_2}}{\varphi_{W_1}}(\mathbf{x_i}(\mathbf{t})),\\
t_4 &= \frac{(\varphi'_{V_3})^2}{\varphi'_{V_4}\varphi'_{V_2}}(\mathbf{x_i}(\mathbf{t})) &&= \frac{\varphi_{W_4}}{\varphi_{W_2}}(\mathbf{x_i}(\mathbf{t})).
\end{aligned}$$

As already mentioned in Example 3.6, the numerators and denominators of these rational functions differ from those of (3), which are *not* cluster monomials, and which are *not* $N'(w)$-invariant. For example the denominator a_1 of (3) is the restriction to N^w of the function φ_X of $\mathbb{C}[N]$, where X is the 2-dimensional Λ-module described at the end of Example 4.4, which is not an object of $\mathcal{C}_w$.

7. Quantum cluster structures on quantum coordinate rings

The coordinate ring $\mathbb{C}[N(w)]$ has a quantum analogue $U_q(\mathfrak{n}(w))$ introduced by De Concini, Kac and Procesi. On the other hand, Berenstein and Zelevinsky [4] have introduced quantum analogues of cluster algebras. In this section we explain that $U_q(\mathfrak{n}(w))$ has a quantum cluster algebra structure obtained by q-deforming in an appropriate way the cluster algebra structure of $\mathbb{C}[N(w)]$.

7.1. Quantum cluster algebras. The guiding principle is to replace the Laurent polynomial rings generated by cluster variables of any given cluster by *quantum tori*, that is, to require that the corresponding quantum cluster variables are pairwise q-commutative. A certain compatibility condition then ensures that all these quantum tori can be glued together to form a flat deformation of the original cluster algebra. It is important to note that a given cluster algebra may have several non-isomorphic q-deformations, and it may also have no q-deformation at all (if its exchange matrix does not have maximal rank).

Instead of repeating the definition of a quantum cluster algebra (for which we refer to [4]), let us describe an example constructed from the Frobenius category $\mathcal{C}_w$ of §4.3. For $M, N \in \mathcal{C}_w$ let us write for short $[M, N] := \dim \mathrm{Hom}_\Lambda(M, N)$. Recall the $\mathcal{C}_w$-cluster-tilting object $V_{\mathbf{i}} = V_1 \oplus \cdots \oplus V_r$, and define

$$\lambda_{ij} := [V_i, V_j] - [V_j, V_i], \qquad (1 \le i, j \le r).$$

Let q denote an indeterminate over $\mathbb{Q}$. We introduce the quantum torus

$$\mathcal{T} := \mathbb{Q}(q)\langle Y_1^{\pm 1}, \dots, Y_r^{\pm 1}\rangle$$

whose generators Y_k obey the q-commutation relations

$$Y_i Y_j = q^{\lambda_{ij}} Y_j Y_i.$$

For $R = V_1^{a_1} \oplus \cdots \oplus V_r^{a_r} \in \mathrm{add}(V_{\mathbf{i}})$, set $Y_R := q^{-\alpha(R)} Y_1^{a_1} \cdots Y_r^{a_r}$, where

$$\alpha(R) := \sum_{1 \le i < j \le r} a_i a_j [V_i, V_j] + \sum_{1 \le i \le r} \frac{a_i(a_i - 1)}{2} [V_i, V_i].$$

In particular, $Y_{V_i} = Y_i$. It is easy to check that for any modules $R, S \in \mathrm{add}(V_{\mathbf{i}})$, we have

$$Y_R Y_S = q^{[R,S]} Y_{R \oplus S}.$$

Theorem 7.1 ([27]). (i) *There exists a unique collection $Y_R \in \mathcal{T}$ indexed by all reachable rigid objects R of $\mathcal{C}_w$, such that:*

(a) *if $R, S \in \mathrm{add}(T)$ for some (reachable) $\mathcal{C}_w$-cluster-tilting module T, then*

$$Y_R Y_S = q^{[R,S]} Y_{R \oplus S} = q^{[R,S]-[S,R]} Y_S Y_R \tag{11}$$

(b) *if M and L are indecomposable rigid modules related by a mutation, with corresponding non-split short exact sequences $0 \to M \to E' \to L \to 0$ and $0 \to L \to E'' \to M \to 0$, then*

$$Y_L Y_M = q^{[L,M]}(q^{-1} Y_{E'} + Y_{E''}). \tag{12}$$

(ii) *The subalgebra $\mathcal{A}_q(\mathcal{C}_w)$ of $\mathcal{T}$ generated by the Y_R's is a quantum cluster algebra, in the sense of Berenstein-Zelevinsky. The Y_R's are its quantum cluster monomials (up to rescaling by some powers of q).*

Observe that (11) and (12) are q-analogues of Theorem 4.2 (i) and (ii), but in contrast with the classical case where we have defined a regular function φ_X for every object X of $\mathcal{C}_w$, here we only have elements Y_M of the quantum cluster algebra for the (reachable) *rigid* modules M.

Example 7.2. We consider again the category $\mathcal{C}_w$ associated with $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ and $w = s_2s_1s_2s_1$. For the module $V_{\mathbf{i}}$ described in Example 4.4, we calculate the matrix

$$L := [\lambda_{ij}] = \begin{pmatrix} 0 & -2 & -2 & -4 \\ 2 & 0 & 0 & -2 \\ 2 & 0 & 0 & -4 \\ 4 & 2 & 4 & 0 \end{pmatrix}$$

encoding the q-commutation relations of the variables $Y_i = Y_{V_i}$. The two mutations discussed in Example 4.13 give rise to the following quantum exchange formulas:

$$Y_{V_1}Y_{T_1} = q^{-2}Y_{V_2}^2 + Y_{V_3}, \qquad Y_{V_2}Y_{T_2} = q^{-2}Y_{T_1}^2 + Y_{V_4}.$$

7.2. Quantum coordinate rings. Let $U_q(\mathfrak{g})$ be the Drinfeld-Jimbo quantized enveloping algebra of $\mathfrak{g}$ over $\mathbb{Q}(q)$, with its subalgebra $U_q(\mathfrak{n})$. For a dominant weight λ, and $u, v \in W$ such that $u(\lambda) \le v(\lambda)$ for the Bruhat ordering, we have introduced [27, §5.2] a *unipotent quantum minor*

$$D^q_{u(\lambda),v(\lambda)} \in U_q(\mathfrak{n}).$$

Following Lusztig, and De Concini, Kac, and Procesi, there is a well-defined subalgebra $U_q(\mathfrak{n}(w))$ of $U_q(\mathfrak{n})$ generated by certain quantum root vectors $E(\beta)$ labelled by the roots $\beta \in \Delta_w$. As suggested by the notation, this is a q-analogue of the enveloping algebra of the nilpotent algebra $\mathfrak{n}(w)$. Putting

$$\lambda_k = s_{i_1} \cdots s_{i_k}(\varpi_{i_k}), \qquad (1 \le k \le r),$$

the element $E(\beta_k)$ is in fact equal to $D^q_{\lambda_{k^-},\lambda_k}$ up to a scaling factor ([27, Proposition 7.4]). Thus $U_q(\mathfrak{n}(w))$ can also be regarded as a q-analogue of the coordinate ring $\mathbb{C}[N(w)]$ of the unipotent group $N(w)$. We can now state a quantum analogue of Theorem 5.3:

Theorem 7.3 ([27]). *Let $M_{\mathbf{i}} = M_1 \oplus \cdots \oplus M_r$ be as in §4.7. The assignment*

$$Y_{M_k} \mapsto D^q_{\lambda_{k^-},\lambda_k}, \qquad (1 \le k \le r),$$

extends to an algebra isomorphism ι from the quantum cluster algebra $\mathcal{A}_q(\mathcal{C}_w)$ to the quantum coordinate ring $U_q(\mathfrak{n}(w))$. Moreover we have

$$\iota(Y_{V_k}) = D^q_{\varpi_{i_k},\lambda_k}, \qquad (1 \le k \le r).$$

The proof uses again in a crucial manner the explicit sequence of mutations from $V_{\mathbf{i}}$ to $T_{\mathbf{i}}$ given by Theorem 4.12. The corresponding sequence of mutations for the unipotent quantum minors is similar to a q-deformation of a T-system, like those appearing in the representation theory of quantum affine algebras (see [38]).

When $\mathfrak{g}$ is a simple Lie algebra of type A, D, E, and $w = w_0$ the longest element of W, Theorem 7.3 shows that $U_q(\mathfrak{n}) = U_q(\mathfrak{n}(w_0))$ has the structure of a quantum cluster algebra. More generally if $w = w_0w_0^K$, where w_0^K is the longest element of some parabolic subgroup W_K of W, then $U_q(\mathfrak{n}(w_0 w_0^K))$ can be regarded as the quantum coordinate ring of a big cell in the corresponding partial flag variety, and by Theorem 7.3 it also carries a quantum cluster algebra structure.

Theorem 7.3 can be seen as an important step towards a conjecture of Berenstein and Zelevinsky [2, Conjecture 10.10], which gives a candidate for a quantum cluster structure on the quantum coordinate ring of any double Bruhat cell $G^{v,w}$ for an arbitrary semisimple group G (not necessarily simply-laced). However one should pay attention to the fact that the relation between $G^{e,w}$ and its reduced counterpart $L^{e,w} = N^w$ is not as straightforward in the quantum case as it was in the classical case. Indeed, to pass from the quantum coordinate ring of N^w to that of $G^{e,w}$, we need to replace our *unipotent* quantum minors $D^q_{u(\lambda),v(\lambda)}$ by ordinary ones $\Delta^q_{u(\lambda),v(\lambda)}$, which satisfy slightly different commutation relations (see [27, §11.1]).

Before [27], only a few examples of "concrete" quantum cluster algebras had appeared in the literature. Grabowski and Launois [29] showed that the quantum coordinate rings of the Grassmannians $\mathrm{Gr}(2,n)$ $(n \geq 2)$, $\mathrm{Gr}(3,6)$, $\mathrm{Gr}(3,7)$, and $\mathrm{Gr}(3,8)$ have a quantum cluster algebra structure. Lampe [39, 40] proved two particular instances of Theorem 7.3, namely when $\mathfrak{g}$ has type A_n or $A_1^{(1)}$ and $w = c^2$ is the square of a Coxeter element. The existence of a quantum cluster structure on every algebra $U_q(\mathfrak{n}(w))$ in the general Kac-Moody case had been conjectured by Kimura [35, Conj.1.1].

Example 7.4. As in Example 7.2, we consider the quantum cluster algebra $\mathcal{A}_q(\mathcal{C}_w)$ for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ and $w = s_2 s_1 s_2 s_1$. It was shown by Lampe [39] that in this case the images under ι of all the quantum cluster variables belong to Lusztig's dual canonical basis of $U_q(\mathfrak{n}(w))$.

Generalizing this example, Kimura and Qin [36] have recently shown that when w is the square of a Coxeter element and $\mathfrak{g}$ is an arbitrary symmetric Kac-Moody algebra, the quantum cluster monomials of $U_q(\mathfrak{n}(w))$ belong to the dual canonical basis.

8. Related topics

In this final section, we briefly mention a few additional topics not covered in this survey, and refer the reader to the relevant references.

8.1. Partial flag varieties. Let us assume that $\mathfrak{g}$ is finite-dimensional, so that G is a complex simple and simply-connected algebraic group. Let B_- denote the Borel subgroup with unipotent radical N_-. We fix a non-empty subset J of I and we denote its complement by $K = I \setminus J$. Let B_-^K be the standard parabolic subgroup of G generated by B_- and the one-parameter subgroups

$$x_k(t), \qquad (k \in K,\, t \in \mathbb{C}).$$

Consider the *partial flag variety* $B_-^K\backslash G$. For example, when $\mathfrak{g}$ is of type A_n and $J = \{j\}$, then $B_-^K\backslash G$ is the Grassmannian variety parametrizing j-dimensional subspaces of $\mathbb{C}^{n+1}$. When $\mathfrak{g}$ is of type D_n and $J = \{n\}$, then $B_-^K\backslash G$ is a smooth quadric in $\mathbb{P}^{2n-1}(\mathbb{C})$.

With our assumption, Λ is a preprojective algebra of Dynkin type, hence Λ is a finite-dimensional Frobenius algebra, and $\mathrm{nil}(\Lambda) = \mathrm{mod}(\Lambda)$. Let Q_j be the indecomposable injective Λ-module with socle S_j. Put $Q_J := \oplus_{j\in J} Q_j$, and denote by $\mathrm{Sub}(Q_J)$ the full subcategory of $\mathrm{mod}(\Lambda)$ whose objects are submodules of direct sums of finitely many copies of Q_J. This is a Frobenius stably 2-Calabi-Yau category.

In analogy with §5, it is shown in [24] that $\mathrm{Sub}(Q_J)$ provides a categorical model for a cluster algebra structure on the coordinate ring of an open cell of $B_-^K\backslash G$ (see also [25, §17], [5]). Moreover, by suitably extending the coefficient ring, we can lift it to a cluster algebra structure on the multi-homogeneous coordinate ring of any type A partial flag variety. This generalizes previous results of Gekhtman, Shapiro, Vainshtein [17], of Scott [47] (for type A Grassmannians), and of Berenstein, Fomin, Zelevinsky [2] (for complete flag varieties $B_-\backslash G$).

8.2. Total positivity. As already mentioned, Lusztig's theory of total positivity for real algebraic groups [42, 43] is one of the initial motivations for introducing cluster algebras [11, 16]. The basic idea is that if X is a variety having a totally positive part $X_{>0}$ in Lusztig's sense, then the coordinate ring of X should have a cluster structure such that each cluster gives rise to a positive coordinate system.

For instance, this has been verified for double Bruhat cells $G^{u,v}$ in [2] (see [16, Remark 4.8]). As a consequence, it holds for complete flag varieties $B_-\backslash G$. It was also proved for Grassmannians in [47]. Recently, using the results of [24, 25], Chevalier [7] has extended this result to the partial flag varieties $B_-^K\backslash G$ in simply-laced type.

8.3. Canonical bases. The precise relation between cluster algebras and dual canonical bases coming from the theory of quantum groups is still elusive, and remains a subject of active research. It is expected that when an algebra has both a cluster structure and a dual canonical basis, like the coordinate ring $\mathbb{C}[N]$ of a maximal unipotent subgroup of a semisimple group, then the cluster monomials should form a subset of the dual canonical basis [13].

In our setting, Theorem 5.1 shows that all cluster monomials belong to the dual *semi*canonical basis of $\mathbb{C}[N(w)]$. But it is known that in general the canonical basis differs from the semicanonical one [19]. To prove that the conjecture of Fomin and Zelevinsky holds in this case, one would have to understand better the intersection of these two bases. In this direction, we have formulated the *open orbit conjecture*. Let Z be an irreducible component of $\Lambda_{\mathbf{d}}$. The group $\mathrm{GL}_{\mathbf{d}} := \prod_{i\in I} GL(d_i,\mathbb{C})$ acts naturally on $\Lambda_{\mathbf{d}}$ (its orbits are in natural one-to-one correspondence with isoclasses of Λ-modules of dimension vector $\mathbf{d}$).

Conjecture 8.1 ([25]). *Let Z be an irreducible component of $\Lambda_{\mathbf{d}}$, and let φ_Z be the associated dual semicanonical basis vector. If Z contains an open $\mathrm{GL}_{\mathbf{d}}$-orbit,* *then φ_Z belongs to the dual canonical basis of* $\mathbb{C}[N]$.

Here, by dual canonical basis we mean the specialization at $q = 1$ of the dual canonical basis of $U_q(\mathfrak{n})$. In view of §7.2, one may also conjecture that the quantum cluster monomials of $U_q(\mathfrak{n}(w))$ belong to the dual canonical basis of $U_q(\mathfrak{n})$ (see *e.g.* Example 7.4).

8.4. Grothendieck rings of quantum affine algebras. Let us assume again that $\mathfrak{g}$ is finite-dimensional. Let $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ be the loop algebra of $\mathfrak{g}$, and let $U_q(L\mathfrak{g})$ denote the quantum analogue of its enveloping algebra, introduced by Drinfeld and Jimbo. Here we assume that $q \in \mathbb{C}^*$ is not a root of unity.

In [31], Hernandez and Leclerc have introduced a cluster algebra structure on the Grothendieck ring of a certain tensor category $\mathcal{C}_1$ of finite-dimensional $U_q(L\mathfrak{g})$-modules. This cluster algebra has finitely many cluster variables, and its cluster type (in the sense of Theorem 2.2 (ii)) coincides with the Lie type of $\mathfrak{g}$. Moreover, the cluster monomials coincide with the classes of simple objects of $\mathcal{C}_1$. This was shown in [31] for type A_n and D_4, and later extended to all A, D, E types by Nakajima [45]. It follows that the fusion rules of $\mathcal{C}_1$ are completely encoded by the combinatorics of cluster variables.

It is conjectured in [31] that similar results hold for more general tensor categories $\mathcal{C}_\ell$ parametrized by arbitrary integers $\ell > 0$. Moreover, it follows from [27, 32] that the quantum Grothendieck rings of the categories $\mathcal{C}_\ell$ ($\ell \leq h/2 - 1$), where h is the Coxeter number of $\mathfrak{g}$, have the structure of a quantum cluster algebra.

References

[1] A. Berenstein, S. Fomin and A. Zelevinsky, *Parametrizations of canonical bases and totally positive matrices*, Adv. Math. 122 (1996), 49–149.

[2] A. Berenstein, S. Fomin, A. Zelevinsky, *Cluster algebras III: Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), 1–52.

[3] A. Berenstein, A. Zelevinsky, *String bases for quantum groups of type A_r*, in I.M. Gelfand seminar, Adv. Soviet Math. **16**, pp. 51–89, AMS 1993.

[4] A. Berenstein, A. Zelevinsky, *Quantum cluster algebras*, Adv. Math. **195** (2005), 405–455.

[5] A. Buan, O. Iyama, I. Reiten, J. Scott, *Cluster structures for 2-Calabi-Yau categories and unipotent groups*, Compos. Math. **145** (2009), 1035–1079.

[6] A. Buan, O. Iyama, I. Reiten, D. Smith, *Mutation of cluster-tilting objects and potentials*, Amer. J. Math. **133** (2011), 835–887.

[7] N. Chevalier, *Total positivity criteria for partial flag varieties*, J. Algebra **348** (2011), 402–415.

[8] L. Demonet, *Categorification of skew-symmetrizable cluster algebras*, Algebr. Represent. Theory **14** (2011), 1087–1162.

[9] H. Derksen, J. Weyman, A. Zelevinsky, *Quivers with potentials and their representations I: mutations*, Selecta Math. **14** (2008), 59–119.

[10] H. Derksen, J. Weyman, A. Zelevinsky, *Quivers with potentials and their representations II: applications to cluster algebras*, J. Amer. Math. Soc. **23** (2010), 749–790.

[11] S. Fomin, *Total positivity and cluster algebras*, Proceedings of the International Congress of Mathematicians, Volume II, Hindustan Book Agency, 2010, 125–145.

[12] S. Fomin, A. Zelevinsky, *Double Bruhat cells and total positivity*, J. Amer. Math. Soc. 12 (1999), 335–380.

[13] S. Fomin, A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), 497–529.

[14] S. Fomin, A. Zelevinsky, *Cluster algebras II: Finite type classification*, Invent. Math. **154** (2003), 63–121.

[15] S. Fomin, A. Zelevinsky, *Cluster algebras IV: Coefficients*, Compos. Math. **143** (2007), 112–164.

[16] S. Fomin, A. Zelevinsky, *Cluster algebras: notes for the CDM-*03 *conference*, Current developments in mathematics, 2003, Int. Press, Somerville, MA 2003, 1–34.

[17] M. Gehktman, M. Shapiro, A. Vainshtein, *Cluster algebras and Poisson geometry*, Moscow Math. J. **3** (2003), 899–934.

[18] M. Gehktman, M. Shapiro, A. Vainshtein, *Cluster algebras and Poisson geometry*, Mathematical surveys and monographs **167**, AMS 2010.

[19] C. Geiss, B. Leclerc and J. Schröer, *Semicanonical bases and preprojective algebras*, Ann. Scient. Éc. Norm. Sup. **38** (2005), 193–253.

[20] C. Geiss, B. Leclerc and J. Schröer, *Rigid modules over preprojective algebras*, Invent. Math., **165** (2006), 589–632.

[21] C. Geiss, B. Leclerc, J. Schröer, *Auslander algebras and initial seeds for cluster algebras*, J. London Math. Soc., **75** (2007), 718–740.

[22] C. Geiss, B. Leclerc, J. Schröer, *Semicanonical bases and preprojective algebras II: A multiplication formula*, Compositio Math., **143** (2007), 1313–1334.

[23] C. Geiss, B. Leclerc and J. Schröer, *Cluster algebra structures and semicanonical bases for unipotent groups*, arXiv:math/0703039 (2007), 121p.

[24] C. Geiss, B. Leclerc, J. Schröer, *Partial flag varieties and preprojective algebras*, Ann. Inst. Fourier (Grenoble) **58** (2008), 825–876.

[25] C. Geiss, B. Leclerc, J. Schröer, *Kac-Moody groups and cluster algebras*, Adv. Math., **228** (2011), 329–433.

[26] C. Geiss, B. Leclerc, J. Schröer, *Generic bases of cluster algebras and the Chamber Ansatz*, J. Amer. Math. Soc. **25** (2012), 21–76.

[27] C. Geiss, B. Leclerc, J. Schröer, *Cluster structures on quantum coordinate rings*, arXiv:1104.0531 (2011).

[28] I. M. Gelfand, V.A. Ponomarev, *Model algebras and representations of graphs*, Functional Anal. Appl. **13** (1979), 157–165.

[29] J. Grabowski, S. Launois, *Quantum cluster algebra structures on quantum Grassmannians and their quantum Schubert cells: the finite-type cases*, Int. Math. Res. Not. IMRN **10** (2011), 2230–2262.

[30] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, 119. Cambridge University Press, Cambridge, 1988. x+208 pp.

[31] D. Hernandez, B. Leclerc, *Cluster algebras and quantum affine algebras*, Duke Math. J. **154** (2010), 265–341.

[32] D. Hernandez, B. Leclerc, *Quantum Grothendieck rings and derived Hall algebras*, arXiv:1109.0862 (2011).

[33] V. Kac and D. Peterson, *Regular functions on certain infinite-dimensional groups*, Arithmetic and geometry, Vol II, 141–166, Progr. Math., 36, Birkhäuser Boston, Boston, MA, 1983.

[34] B. Keller, *Cluster algebras and derived categories*, arXiv:1202.4161, (2012), 60 p.

[35] Y. Kimura, *Quantum unipotent subgroup and dual canonical basis*, Kyoto J. Math. **52** (2012), 277–331.

[36] Y. Kimura, F. Qin, *Graded quiver varieties, quantum cluster algebras and dual canonical basis*, arXiv:1205.2066.

[37] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, 204. Birkhäuser Boston, Inc., Boston, MA, 2002.

[38] A. Kuniba, T. Nakanishi, J. Suzuki, *T-systems and Y-systems in integrable systems*, J. Phys. A **44** (2011), 146 p.

[39] P. Lampe, *A quantum cluster algebra of Kronecker type and the dual canonical basis*, Int. Math. Res. Not. IMRN **13** (2011), 2970–3005.

[40] P. Lampe, *Quantum cluster algebras of type A and the dual canonical basis*, arXiv:1101.0580, (2010), 46 p.

[41] G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), 365–421.

[42] G. Lusztig, *Total positivity in reductive groups*, In *Lie theory and geometry*, Progr. Math., **123**, Birkhäuser Boston, 1994, 531–568.

[43] G. Lusztig, *Total positivity in partial flag manifolds*, Represent. Theory **2** (1998), 70–78.

[44] G. Lusztig, *Semicanonical bases arising from enveloping algebras*, Adv. Math. **151** (2000), 129–139.

[45] H. Nakajima, *Quiver varieties and cluster algebras*, Kyoto J. Math. 51 (2011), 71–126.

[46] C.M. Ringel, *The preprojective algebra of a quiver*, Algebras and modules, II (Geiranger, 1996), 467–480, CMS Conf. Proc., **24**, Amer. Math. Soc., Providence, RI, 1998.

[47] J. Scott, *Grassmannians and cluster algebras*, Proc. London Math. Soc. **92** (2006), 345–380.

[48] J. Scott, *Block-Toeplitz determinants, chess tableaux, and the type $\hat{A}_1$ Geiss-Leclerc-Schröer φ-map*, arXiv:0707.3046, (2007), 23 p.

1. Christof Geiss, Instituto de Matemáticas, Universidad Nacional Autónoma de México, Ciudad Universitaria, 04510 México D. F. México

E-mail: christof@math.unam.mx

2. Bernard Leclerc, LMNO UMR 6139, Université de Caen Basse-Normandie, CNRS, Campus 2, F-14032 Caen cedex, France. Institut Universitaire de France
E-mail: bernard.leclerc@unicaen.fr

3. Jan Schröer, Mathematisches Institut, Universität Bonn, Endenicher Allee 60, D-53115 Bonn, Germany
E-mail: schroer@math.uni-bonn.de