# Conjectural relation of twisted differentials to Pixton's cycle

Johannes Schmitt

ETH Zürich

February 8, 2017

# Pixton's cycle

Fix a genus $g \geq 0$, a number $n \geq 0$ of markings, an integer $k \geq 0$ and a partition $\widetilde{\mu} = (\widetilde{m}_1, \ldots, \widetilde{m}_n)$ of $k(2g - 2 + n)$.

# Pixton's cycle

Fix a genus $g \geq 0$, a number $n \geq 0$ of markings, an integer $k \geq 0$ and a partition $\widetilde{\mu} = (\widetilde{m}_1, \ldots, \widetilde{m}_n)$ of $k(2g - 2 + n)$.

**Note**

We can obtain the partition $\widetilde{\mu}$ above from a partition $\mu = (m_1, \ldots, m_n)$ of $k(2g - 2)$ by

$$\widetilde{\mu} = (m_1 + k, \ldots, m_n + k)$$

# Pixton's cycle

Fix a genus $g \geq 0$, a number $n \geq 0$ of markings, an integer $k \geq 0$ and a partition $\widetilde{\mu} = (\widetilde{m}_1, \ldots, \widetilde{m}_n)$ of $k(2g-2+n)$. Then for every degree $d \geq 0$, Pixton constructs a tautological class

$$P_g^{d,k}(\widetilde{\mu}) \in R^d(\overline{M}_{g,n}) \subset A^d(\overline{M}_{g,n}).$$

## Note

We can obtain the partition $\widetilde{\mu}$ above from a partition $\mu = (m_1, \ldots, m_n)$ of $k(2g-2)$ by

$$\widetilde{\mu} = (m_1 + k, \ldots, m_n + k)$$

## Summary : Pixton's cycle

The tautological class $P_g^{d,k}(\widetilde{\mu})$ is defined as an explicit sum over dual graphs $\Gamma$ and additional combinatorial data of terms of the form

$$\xi_{\Gamma *}\left(\text{polynomial in } \kappa \text{ and } \psi\text{-classes on } \prod_{v \in V(\Gamma)} \overline{M}_{g(v),n(v)}\right).$$

# Conjecture A

**Conjecture A ($k = 1$ Janda-Pandharipande-Pixton-Zvonkine [FP-Appendix], $k \geq 1$ S.)**

For $k \geq 1$ and $\mu$ not of the form $\mu = k\mu'$ for a nonnegative partition $\mu'$ of $2g - 2$, we have

# Conjecture A

**Conjecture A ($k = 1$ Janda-Pandharipande-Pixton-Zvonkine [FP-Appendix], $k \geq 1$ S.)**

For $k \geq 1$ and $\mu$ not of the form $\mu = k\mu'$ for a nonnegative partition $\mu'$ of $2g - 2$, we have

$$\sum_{\Gamma \text{ star graph}} \sum_{I} \frac{1}{|\mathrm{Aut}(\Gamma)|} C_{\Gamma,I} =$$

where

$$C_{\Gamma,I} = (\xi_\Gamma)_* \Big[ \Big[\overline{\mathcal{H}}^k_{g(v_0)}(\mu[v_0], -I[v_0] - k)\Big] \cdot \prod_{v \in V_{\mathrm{out}}(\Gamma)} \Big[\overline{\mathcal{H}}^1_{g(v)}\left(\frac{\mu[v]}{k}, \frac{I[v] - k}{k}\right)\Big] \Big]$$

# Conjecture A

**Conjecture A ($k=1$ Janda-Pandharipande-Pixton-Zvonkine [FP-Appendix], $k \geq 1$ S.)**

For $k \geq 1$ and $\mu$ not of the form $\mu = k\mu'$ for a nonnegative partition $\mu'$ of $2g-2$, we have

$$\sum_{\Gamma \text{ star graph}} \sum_{I} \frac{\prod_{e \in E(\Gamma)} I(e)}{k^{|V_{\text{out}}(\Gamma)|}} \frac{1}{|\text{Aut}(\Gamma)|} C_{\Gamma,I} =$$

where

$$C_{\Gamma,I} = (\xi_\Gamma)_* \Big[ \left[\overline{\mathcal{H}}^k_{g(v_0)}(\mu[v_0], -I[v_0]-k)\right] \cdot \prod_{v \in V_{\text{out}}(\Gamma)} \left[\overline{\mathcal{H}}^1_{g(v)}\left(\frac{\mu[v]}{k}, \frac{I[v]-k}{k}\right)\right] \Big]$$

# Conjecture A

**Conjecture A ($k = 1$ Janda-Pandharipande-Pixton-Zvonkine [FP-Appendix], $k \geq 1$ S.)**

For $k \geq 1$ and $\mu$ not of the form $\mu = k\mu'$ for a nonnegative partition $\mu'$ of $2g-2$, we have

$$\sum_{\Gamma \text{ star graph}} \sum_{I} \frac{\prod_{e \in E(\Gamma)} I(e)}{k^{|V_{\text{out}}(\Gamma)|}} \frac{1}{|\text{Aut}(\Gamma)|} C_{\Gamma,I} = 2^{-g} P_g^{g,k}(\tilde{\mu})$$

where

$$C_{\Gamma,I} = (\xi_\Gamma)_* \Big[ \left[\overline{\mathcal{H}}^k_{g(v_0)}(\mu[v_0], -I[v_0] - k)\right] \cdot \prod_{v \in V_{\text{out}}(\Gamma)} \left[\overline{\mathcal{H}}^1_{g(v)}\left(\frac{\mu[v]}{k}, \frac{I[v] - k}{k}\right)\right] \Big]$$

# Application: Recursion for $[\overline{\mathcal{H}}_g^k(\mu)]$

**Theorem (FP-Appendix, S.)**

Conjecture A effectively determines the classes $[\overline{\mathcal{H}}_g^k(\mu)]$ for

$$\left\{ \right.$$

# Application: Recursion for $[\overline{\mathcal{H}}_g^k(\mu)]$

**Theorem (FP-Appendix, S.)**

Conjecture A effectively determines the classes $[\overline{\mathcal{H}}_g^k(\mu)]$ for

$$\begin{cases} k \geq 1 \text{ and } \mu \neq k\mu' \text{ with } \mu' \geq 0 \text{ (codim } g\text{)} \\ \end{cases}$$

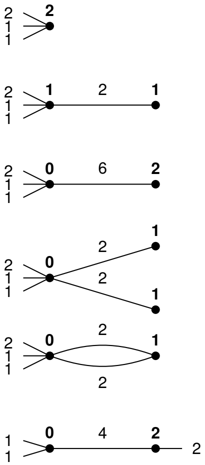
# Application: Recursion for $[\overline{\mathcal{H}}_g^k(\mu)]$

**Theorem (FP-Appendix, S.)**

Conjecture A effectively determines the classes $[\overline{\mathcal{H}}_g^k(\mu)]$ for

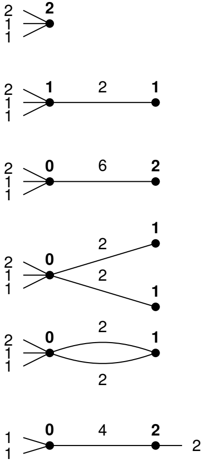
$$\begin{cases} k \geq 1 \text{ and } \mu \neq k\mu' \text{ with } \mu' \geq 0 \text{ (codim } g) \\ k = 1 \text{ and } \mu \geq 0 \text{ (codim } g-1) \end{cases}$$

# Example: $[\overline{\mathcal{H}}_2^2(2,1,1)]$



# Example: $[\overline{\mathcal{H}}_2^2(2,1,1)]$

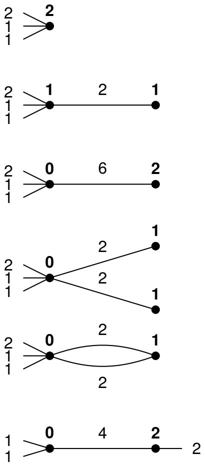
$$\left[\overline{\mathcal{H}}_2^2(2,1,1)\right]$$



# Example: $[\overline{\mathcal{H}}_2^2(2,1,1)]$

$$\left[\overline{\mathcal{H}}_2^2(2,1,1)\right]$$

$$+ (\xi_{\Gamma_2})_* \quad \left[[\overline{\mathcal{H}}_1^2(2,1,1,-4)] \cdot [\overline{M}_{1,1}]\right]$$

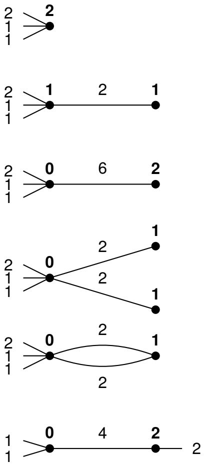


# Example: $[\overline{\mathcal{H}}_2^2(2,1,1)]$

$$\left[\overline{\mathcal{H}}_2^2(2,1,1)\right]$$

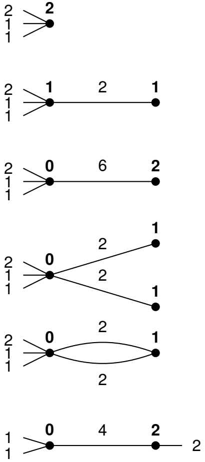
$$+ (\xi_{\Gamma_2})_* \quad \left[[\overline{\mathcal{H}}_1^2(2,1,1,-4)] \cdot [\overline{M}_{1,1}]\right]$$

$$+ 3(\xi_{\Gamma_3})_* \quad \left[[\overline{M}_{0,4}] \cdot [\overline{\mathcal{H}}_2^1(2)]\right]$$



# Example: $[\overline{\mathcal{H}}_2^2(2,1,1)]$

$$\begin{aligned}
& \left[\overline{\mathcal{H}}_2^2(2,1,1)\right] \\
+ (\xi_{\Gamma_2})_* \quad & \left[[\overline{\mathcal{H}}_1^2(2,1,1,-4)] \cdot [\overline{M}_{1,1}]\right] \\
+ 3(\xi_{\Gamma_3})_* \quad & \left[[\overline{M}_{0,4}] \cdot [\overline{\mathcal{H}}_2^1(2)]\right] \\
+ \tfrac{1}{2}(\xi_{\Gamma_4})_* \quad & \left[[\overline{M}_{0,5}] \cdot [\overline{M}_{1,1}] \cdot [\overline{M}_{1,1}]\right] \\
+ (\xi_{\Gamma_5})_* \quad & \left[[\overline{M}_{0,5}] \cdot [\overline{M}_{1,2}]\right] \\
+ 2(\xi_{\Gamma_6})_* \quad & \left[[\overline{M}_{0,3}] \cdot [\overline{\mathcal{H}}_2^1(1,1)]\right] \\
= \quad & \tfrac{1}{4} P_2^{2,2}(4,3,3)
\end{aligned}$$



# Example: $[\overline{\mathcal{H}}_2^1(2)]$

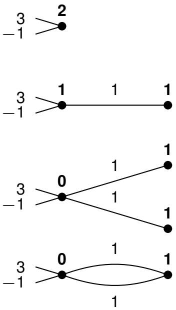
# Example: $[\overline{\mathcal{H}}_2^1(2)]$

Take Conjecture A for $\mu^+ = (3, -1)$ on $\overline{M}_{2,2}$

# Example: $[\overline{\mathcal{H}}_2^1(2)]$

Take Conjecture A for $\mu^+ = (3, -1)$ on $\overline{M}_{2,2}$

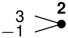
$$
\begin{aligned}
& \left[\overline{\mathcal{H}}_2^1(3,-1)\right] \\
+ \quad & (\xi_{\Gamma_2})_* \left[[\overline{\mathcal{H}}_1^1(3,-1,-2)] \cdot [\overline{M}_{1,1}]\right] \\
+\tfrac{1}{2} \quad & (\xi_{\Gamma_3})_* \left[[\overline{M}_{0,4}] \cdot [\overline{M}_{1,1}] \cdot [\overline{M}_{1,1}]\right] \\
+\tfrac{1}{2} \quad & (\xi_{\Gamma_4})_* \left[[\overline{M}_{0,4}] \cdot [\overline{M}_{1,2}]\right] \\
+3 \quad & (\xi_{\Gamma_5})_* \left[[\overline{M}_{0,3}] \cdot [\overline{\mathcal{H}}_2^1(2)]\right] \\
= \quad & \tfrac{1}{4} \quad P_2^{2,1}(4,0)
\end{aligned}
$$



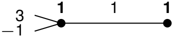
# Example: $[\overline{\mathcal{H}}_2^1(2)]$

Take Conjecture A for $\mu^+ = (3, -1)$ on $\overline{M}_{2,2}$ and push forward under the forgetful map $\epsilon : \overline{M}_{2,2} \to \overline{M}_{2,1}$ of the second point

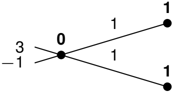
$$\left[\overline{\mathcal{H}}_2^1(3,-1)\right]$$



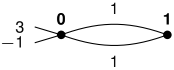

$$+ \quad (\xi_{\Gamma_2})_* \quad \left[[\overline{\mathcal{H}}_1^1(3,-1,-2)] \cdot [\overline{M}_{1,1}]\right]$$



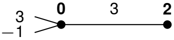

$$+\frac{1}{2} \quad (\xi_{\Gamma_3})_* \quad \left[[\overline{M}_{0,4}] \cdot [\overline{M}_{1,1}] \cdot [\overline{M}_{1,1}]\right]$$




$$+\frac{1}{2} \quad (\xi_{\Gamma_4})_* \quad \left[[\overline{M}_{0,4}] \cdot [\overline{M}_{1,2}]\right]$$




$$+3 \quad (\xi_{\Gamma_5})_* \quad \left[[\overline{M}_{0,3}] \cdot [\overline{\mathcal{H}}_2^1(2)]\right]$$

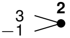



$$= \quad \frac{1}{4} \quad P_2^{2,1}(4,0)$$

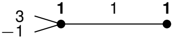
# Example: $[\overline{\mathcal{H}}_2^1(2)]$

Take Conjecture A for $\mu^+ = (3, -1)$ on $\overline{M}_{2,2}$ and push forward under the forgetful map $\epsilon : \overline{M}_{2,2} \to \overline{M}_{2,1}$ of the second point

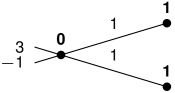
$$\epsilon_* \quad \left[\overline{\mathcal{H}}_2^1(3,-1)\right]$$



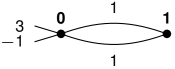

$$+\,\epsilon_*(\xi_{\Gamma_2})_* \quad \left[[\overline{\mathcal{H}}_1^1(3,-1,-2)] \cdot [\overline{M}_{1,1}]\right]$$



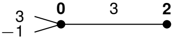

$$+\tfrac{1}{2}\epsilon_*(\xi_{\Gamma_3})_* \quad \left[[\overline{M}_{0,4}] \cdot [\overline{M}_{1,1}] \cdot [\overline{M}_{1,1}]\right]$$




$$+\tfrac{1}{2}\epsilon_*(\xi_{\Gamma_4})_* \quad \left[[\overline{M}_{0,4}] \cdot [\overline{M}_{1,2}]\right]$$



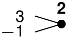

$$+\,3\epsilon_*(\xi_{\Gamma_5})_* \quad \left[[\overline{M}_{0,3}] \cdot [\overline{\mathcal{H}}_2^1(2)]\right]$$



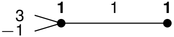

$$= \quad \tfrac{1}{4}\epsilon_* P_2^{2,1}(4,0)$$

# Example: $[\overline{\mathcal{H}}_2^1(2)]$

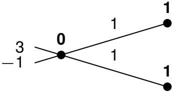
Take Conjecture A for $\mu^+ = (3,-1)$ on $\overline{M}_{2,2}$ and push forward under the forgetful map $\epsilon : \overline{M}_{2,2} \to \overline{M}_{2,1}$ of the second point



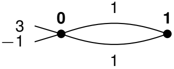

$$\epsilon_*(\xi_{\Gamma_2})_* \quad \left[[\overline{\mathcal{H}}_1^1(3,-1,-2)] \cdot [\overline{M}_{1,1}]\right]$$



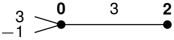

$$+\frac{1}{2}\epsilon_*(\xi_{\Gamma_3})_* \quad \left[[\overline{M}_{0,4}] \cdot [\overline{M}_{1,1}] \cdot [\overline{M}_{1,1}]\right]$$




$$+\frac{1}{2}\epsilon_*(\xi_{\Gamma_4})_* \quad \left[[\overline{M}_{0,4}] \cdot [\overline{M}_{1,2}]\right]$$



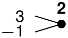

$$+\,3\epsilon_*(\xi_{\Gamma_5})_* \quad \left[[\overline{M}_{0,3}] \cdot [\overline{\mathcal{H}}_2^1(2)]\right]$$



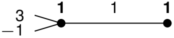

$$= \quad \frac{1}{4}\epsilon_* P_2^{2,1}(4,0)$$

# Example: $[\overline{\mathcal{H}}_2^1(2)]$

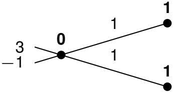
Take Conjecture A for $\mu^+ = (3, -1)$ on $\overline{M}_{2,2}$ and push forward under the forgetful map $\epsilon : \overline{M}_{2,2} \to \overline{M}_{2,1}$ of the second point



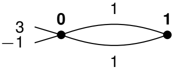

$$\epsilon_*(\xi_{\Gamma_2})_* \quad \left[[\overline{\mathcal{H}}_1^1(3,-1,-2)] \cdot [\overline{M}_{1,1}]\right]$$



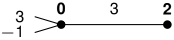

$$+\frac{1}{2}\epsilon_*(\xi_{\Gamma_3})_* \quad \left[[\overline{M}_{0,4}] \cdot [\overline{M}_{1,1}] \cdot [\overline{M}_{1,1}]\right]$$




$$+\frac{1}{2}\epsilon_*(\xi_{\Gamma_4})_* \quad \left[[\overline{M}_{0,4}] \cdot [\overline{M}_{1,2}]\right]$$




$$+3 \quad \left[\overline{\mathcal{H}}_2^1(2)\right]$$




$$= \frac{1}{4}\epsilon_* P_2^{2,1}(4,0)$$

# Evidence for Conjecture A

Conjecture A is true for

# Evidence for Conjecture A

Conjecture A is true for

- $g = 0$ trivial $(1 = 1)$

# Evidence for Conjecture A

Conjecture A is true for

- $g = 0$ trivial ($1 = 1$)
- $g = 1$ (FP-Appendix)

# Evidence for Conjecture A

Conjecture A is true for

- $g = 0$ trivial $(1 = 1)$
- $g = 1$ (FP-Appendix)
- $g = 2$
  - $k = 1$ and $\mu = (3, -1), (2, 1, -1)$ (FP-Appendix)

# Evidence for Conjecture A

Conjecture A is true for

- $g = 0$ trivial $(1 = 1)$
- $g = 1$ (FP-Appendix)
- $g = 2$
  - $k = 1$ and $\mu = (3, -1), (2, 1, -1)$ (FP-Appendix)
    **Note:** $\mathcal{H}_2^1(\mu) = \emptyset$ in both cases

# Evidence for Conjecture A

Conjecture A is true for

- $g = 0$ trivial $(1 = 1)$
- $g = 1$ (FP-Appendix)
- $g = 2$
  - $k = 1$ and $\mu = (3, -1), (2, 1, -1)$ (FP-Appendix)
    **Note:** $\mathcal{H}_2^1(\mu) = \emptyset$ in both cases
  - $k = 2$ and $\mu = (3, 1), (2, 1, 1)$ (S)

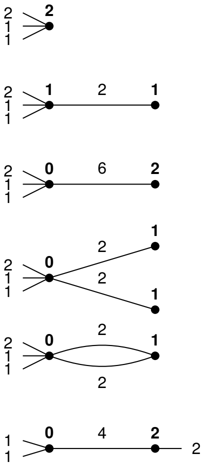
# Evidence for Conjecture A

Conjecture A is true for

- $g = 0$ trivial $(1 = 1)$
- $g = 1$ (FP-Appendix)
- $g = 2$
  - $k = 1$ and $\mu = (3, -1), (2, 1, -1)$ (FP-Appendix)
    **Note:** $\mathcal{H}_2^1(\mu) = \emptyset$ in both cases
  - $k = 2$ and $\mu = (3, 1), (2, 1, 1)$ (S)
    - $\mathcal{H}_2^2(3, 1) = \emptyset$
    - $\mathcal{H}_2^2(2, 1, 1) = \left\{ (C, q, p_1, p_2) : \begin{array}{l} q \text{ Weierstrass point,} \\ p_1, p_2 \text{ hyperelliptic conjugate} \end{array} \right\}$

# Example: $[\overline{\mathcal{H}}_2^2(2,1,1)]$

$$
\begin{aligned}
& \left[\overline{\mathcal{H}}_2^2(2,1,1)\right] \\
+(\xi_{\Gamma_2})_* \quad & \left[[\overline{\mathcal{H}}_1^2(2,1,1,-4)] \cdot [\overline{M}_{1,1}]\right] \\
+3(\xi_{\Gamma_3})_* \quad & \left[[\overline{M}_{0,4}] \cdot [\overline{\mathcal{H}}_2^1(2)]\right] \\
+\tfrac{1}{2}(\xi_{\Gamma_4})_* \quad & \left[[\overline{M}_{0,5}] \cdot [\overline{M}_{1,1}] \cdot [\overline{M}_{1,1}]\right] \\
+(\xi_{\Gamma_5})_* \quad & \left[[\overline{M}_{0,5}] \cdot [\overline{M}_{1,2}]\right] \\
+2(\xi_{\Gamma_6})_* \quad & \left[[\overline{M}_{0,3}] \cdot [\overline{\mathcal{H}}_2^1(1,1)]\right] \\
= \quad & \tfrac{1}{4} P_2^{2,2}(4,3,3)
\end{aligned}
$$



# Conjecture A′

What if $\mu = k\mu'$ with $\mu' \geq 0$?

$$\left[\overline{\mathcal{H}}_g^k(\mu)\right] = \left[\overline{\mathcal{H}}_g^1(\frac{\mu}{k})\right] \quad + \left[\overline{\mathcal{H}}_g^k(\mu)'\right]$$

# Conjecture A$'$

What if $\mu = k\mu'$ with $\mu' \geq 0$?

$$\left[\overline{\mathcal{H}}_g^k(\mu)\right] = \left[\overline{\mathcal{H}}_g^1(\frac{\mu}{k})\right]^{\text{vir}} + \left[\overline{\mathcal{H}}_g^k(\mu)'\right] \in A^g(\overline{M}_{g,n})$$

# Conjecture A′

What if $\mu = k\mu'$ with $\mu' \geq 0$?

$$\left[\overline{\mathcal{H}}_g^k(\mu)\right] = \left[\overline{\mathcal{H}}_g^1(\frac{\mu}{k})\right]^{\text{vir}} + \left[\overline{\mathcal{H}}_g^k(\mu)'\right] \in A^g(\overline{M}_{g,n})$$

(1) Formula from Conjecture A for $\mu, k$:

$$\left[\overline{\mathcal{H}}_g^1(\frac{\mu}{k})\right]^{\text{vir}} + \left[\overline{\mathcal{H}}_g^k(\mu)'\right] + \sum_{(\Gamma, I)\text{ nontrivial}} \text{Cont}_{g,\mu}^k(\Gamma, I) = 2^{-g} P_g^{g,k}(\widetilde{\mu})$$

# Conjecture A′

What if $\mu = k\mu'$ with $\mu' \geq 0$?

$$\left[\overline{\mathcal{H}}_g^k(\mu)\right] = \left[\overline{\mathcal{H}}_g^1(\frac{\mu}{k})\right]^{\text{vir}} + \left[\overline{\mathcal{H}}_g^k(\mu)'\right] \in A^g(\overline{M}_{g,n})$$

(1) Formula from Conjecture A for $\mu, k$:

$$\left[\overline{\mathcal{H}}_g^1(\frac{\mu}{k})\right]^{\text{vir}} + \left[\overline{\mathcal{H}}_g^k(\mu)'\right] + \sum_{(\Gamma,I)\text{ nontrivial}} \text{Cont}_{g,\mu}^k(\Gamma, I) = 2^{-g} P_g^{g,k}(\widetilde{\mu})$$

(2) Formula from Conjecture A for $\frac{\mu}{k}, 1$:

$$\left[\overline{\mathcal{H}}_g^1(\frac{\mu}{k})\right]^{\text{vir}} + \sum_{(\Gamma',I')\text{ nontrivial}} \text{Cont}_{g,\mu/k}^1(\Gamma', I') = 2^{-g} P_g^{g,1}(\widetilde{\mu/k})$$

# Conjecture A′

What if $\mu = k\mu'$ with $\mu' \geq 0$?

$$\left[\overline{\mathcal{H}}_g^k(\mu)\right] = \left[\overline{\mathcal{H}}_g^1(\frac{\mu}{k})\right]^{\text{vir}} + \left[\overline{\mathcal{H}}_g^k(\mu)'\right] \in A^g(\overline{M}_{g,n})$$

(1) Formula from Conjecture A for $\mu, k$:

$$\left[\overline{\mathcal{H}}_g^1(\frac{\mu}{k})\right]^{\text{vir}} + \left[\overline{\mathcal{H}}_g^k(\mu)'\right] + \sum_{(\Gamma,I)\text{ nontrivial}} \text{Cont}_{g,\mu}^k(\Gamma, I) = 2^{-g} P_g^{g,k}(\widetilde{\mu})$$

(2) Formula from Conjecture A for $\frac{\mu}{k}, 1$:

$$\left[\overline{\mathcal{H}}_g^1(\frac{\mu}{k})\right]^{\text{vir}} + \sum_{(\Gamma',I')\text{ nontrivial}} \text{Cont}_{g,\mu/k}^1(\Gamma', I') = 2^{-g} P_g^{g,1}(\widetilde{\mu/k})$$

Idea: Take (2) as a definition of $\left[\overline{\mathcal{H}}_g^1(\frac{\mu}{k})\right]^{\text{vir}}$

$\implies$ Conjecture A′ = (1) for $k > 1$, $\mu = k\mu'$ with $\mu' \geq 0$

# Evidence for Conjecture A′

# Evidence for Conjecture A′

- $g = 0$ no nontrivial examples possible

# Evidence for Conjecture A′

- $g = 0$ no nontrivial examples possible
- $g = 1$ implies $\mu = (0, \ldots, 0)$, check on formulas

# Evidence for Conjecture A′

- $g = 0$ no nontrivial examples possible
- $g = 1$ implies $\mu = (0, \ldots, 0)$, check on formulas
- $g = 2, k = 2$ and $\mu = (4), (2, 2)$

# Evidence for Conjecture A′

- $g = 0$ no nontrivial examples possible
- $g = 1$ implies $\mu = (0, \ldots, 0)$, check on formulas
- $g = 2, k = 2$ and $\mu = (4), (2, 2)$
  - $\mathcal{H}_2^2(4)' = \emptyset$

# Evidence for Conjecture A′

- $g = 0$ no nontrivial examples possible
- $g = 1$ implies $\mu = (0, \ldots, 0)$, check on formulas
- $g = 2, k = 2$ and $\mu = (4), (2, 2)$
  - $\mathcal{H}_2^2(4)' = \emptyset$
  - $\mathcal{H}_2^2(2,2)' = \{(C, p, q) : p, q \text{ Weierstrass points}\}$
    Class computed by Tarasca in 2015

# Evidence for Conjecture A′

- $g = 0$ no nontrivial examples possible
- $g = 1$ implies $\mu = (0, \dots, 0)$, check on formulas
- $g = 2, k = 2$ and $\mu = (4), (2, 2)$
  - $\mathcal{H}_2^2(4)' = \emptyset$
  - $\mathcal{H}_2^2(2,2)' = \{(C, p, q) : p, q \text{ Weierstrass points}\}$
    Class computed by Tarasca in 2015

Again, Conjecture A′ gives an effective way to compute the classes $\left[\overline{\mathcal{H}}_g^k(\mu)'\right]$.

Thank you for your attention.

# Pixton's cycle

## Definition

Let $\Gamma$ be a stable graph of genus $g$ with $n$ legs. A *$k$-weighting mod $r$* of $\Gamma$ is a function on the set of half-edges

$$w : H(\Gamma) \to \{0, 1, \ldots, r-1\}$$

satisfying

# Pixton's cycle

## Definition

Let $\Gamma$ be a stable graph of genus $g$ with $n$ legs. A *$k$-weighting mod $r$* of $\Gamma$ is a function on the set of half-edges

$$w : H(\Gamma) \to \{0, 1, \ldots, r-1\}$$

satisfying

i) for the leg $h_i$ corresponding to marking $i$

$$w(h_i) = \widetilde{m}_i \mod r,$$

# Pixton's cycle

## Definition

Let $\Gamma$ be a stable graph of genus $g$ with $n$ legs. A *$k$-weighting mod $r$* of $\Gamma$ is a function on the set of half-edges

$$w : H(\Gamma) \to \{0, 1, \ldots, r-1\}$$

satisfying

i) for the leg $h_i$ corresponding to marking $i$

$$w(h_i) = \widetilde{m}_i \mod r,$$

ii) for all half-edges $h, h'$ forming an edge

$$w(h) + w(h') = 0 \mod r$$

# Pixton's cycle

## Definition

Let $\Gamma$ be a stable graph of genus $g$ with $n$ legs. A *$k$-weighting mod $r$* of $\Gamma$ is a function on the set of half-edges

$$w : H(\Gamma) \to \{0, 1, \ldots, r-1\}$$

satisfying

i) for the leg $h_i$ corresponding to marking $i$

$$w(h_i) = \widetilde{m}_i \mod r,$$

ii) for all half-edges $h, h'$ forming an edge

$$w(h) + w(h') = 0 \mod r$$

ii) for all vertices v of $\Gamma$

$$\sum_{v(h)=v} w(h) = k(2g(v) - 2 + n(v)) \mod r$$

# Pixton's cycle

## Definition

For each positive integer $r$, let $P_g^{d,r,k}(\widetilde{\mu})$ be the degree $d$ component of the tautological class

$$\sum_{\Gamma,w} \frac{1}{|\mathrm{Aut}(\Gamma)|} \frac{1}{r^{h^1(\Gamma)}} \xi_{\Gamma *}\Big[ \prod_{v \in V(\Gamma)} e^{-k^2 \kappa_1(v)} \prod_{i=1}^{n} e^{\widetilde{m}_i^2 \psi_{h_i}} \prod_{e=(h,h') \in E(\Gamma)} \frac{1 - e^{-w(h)w(h')(\psi_h + \psi_{h'})}}{\psi_h + \psi_{h'}} \Big]$$

# Pixton's cycle

## Definition

For each positive integer $r$, let $P_g^{d,r,k}(\widetilde{\mu})$ be the degree $d$ component of the tautological class

$$\sum_{\Gamma,w} \frac{1}{|\mathrm{Aut}(\Gamma)|} \frac{1}{r^{h^1(\Gamma)}} \xi_{\Gamma*}\Big[ \prod_{v \in V(\Gamma)} e^{-k^2\kappa_1(v)} \prod_{i=1}^{n} e^{\widetilde{m}_i^2 \psi_{h_i}} \prod_{e=(h,h') \in E(\Gamma)} \frac{1 - e^{-w(h)w(h')(\psi_h + \psi_{h'})}}{\psi_h + \psi_{h'}} \Big]$$

## Proposition/Definition (Pixton)

$P_g^{d,r,k}(\widetilde{\mu}) \in R^d(\overline{M}_{g,n})$ is polynomial in $r$ for $r \gg 0$.
Let $P_g^{d,k}(\widetilde{\mu}) \in R^d(\overline{M}_{g,n})$ be the value of this polynomial at $r = 0$.