The moduli space of curves

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The following are the lecture notes for a course about the moduli space of stable curves given in the Summer of 2020 at the University of Bonn. In the course, we define the moduli space and discuss its basic properties. We present many of the basic constructions related to it, e.g. of gluing and forgetful maps, and explain a number of examples of these moduli spaces in more details. Towards the end of the course we also cover the basics about the intersection theory on the space and the definition of tautological classes inside its cohomology groups.

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1 Introduction and Motivation

This course is an introduction to the moduli spaces of algebraic curves. The idea behind these spaces is that they allow us to classify algebraic curves up to isomorphism.

To get some intuition what this could mean, let's start with a more basic and familiar example: finite-dimensional vector spaces V over a field K. These mathematical objects are uniquely classified by their dimension, since $V \cong V'$ if and only if $\dim V = \dim V'$. Thus it makes sense to write the following:

$$\{\text{n. dim. vector spaces over } K\}/\text{iso} \xrightarrow{\sim} \{0, 1, 2, \ldots\}; \qquad [V] \mapsto \dim V. \tag{1}$$

This is a great classification: the left side looks complicated and scary but we understand the right-hand side (the natural numbers) very well!

The above was an example from linear algebra, but for an algebraic geometer, a natural question is to try classifying algebraic varieties up to isomorphism. In this course we will look at algebraic curves, i.e. algebraic varieties of dimension one. Why dimension one? It turns out that varieties of dimension zero are too easy: over an algebraically closed field K they are disjoint unions of points (i.e. $\mathrm{Spec}(K)$). On the other hand, varieties of dimension greater than one turn out to be more complicated to classify, so dimension one is the natural place to start.

To fix some conventions, we will say that a curve is a variety of pure dimension 1 and a variety is a reduced, separated scheme of finite type over our base field (not necessarily irreducible). For now, let us fix the base field to be the complex numbers $\mathbb{C}$ and let us try to classify smooth, irreducible, projective curves over $\mathbb{C}$, i.e. look at the set

$$\mathcal{M} = \{C \text{ smooth, irreducible, complex projective curve}\}/\text{iso}. \tag{2}$$

Why did we require C to have this list of properties? Some motivation:

- We ask C to be smooth since singularities add complications and can destroy some of the nice structures that smooth curves have, which will help us classify them. We will relax this condition below and allow so-called nodal singularities.

- Once we assume C to be smooth, being irreducible is equivalent to being connected.[1] A possibly disconnected curve is the union of connected ones, so it makes sense to classify these first.

- The assumption that C is complex is mostly for our convenience, much can be extended to working over algebraically closed fields (sometimes of characteristic 0). However, having $\mathbb{C}$ as the base field will allow us to draw some nice pictures, see below.

- As for the assumption that the curve is projective, note that for every smooth, irreducible, not-necessarily projective curve C^0 there exists an embedding $C^0 \hookrightarrow C$ into a smooth, irreducible and projective curve C (see [Vak17, Theorem 17.4.2]). By dimension reasons, the complement $C \setminus C^0$ is a finite union of points and it will be easier to simply classify the data of C together with these points (from which we can reconstruct C^0). Again, this is something we will see later, for today we just consider curves without the additional data of marked points.

[1] For a connected, reducible scheme there must be a point contained in two irreducible components (otherwise all these components would be disjoint, contradicting connectedness). Then one checks that such a point cannot be smooth. This statement is true in more generality, see [Sta13, Tag 033M].

What can we say about $\mathcal{M}$? Is it finite or infinite? Can we hope to give a list of elements, as in (1)?

Well, let's start getting our hands on some elements of $\mathcal{M}$. Below we will use that apart from the Zariski topology on a complex curve C, the set $C(\mathbb{C})$ of complex (or closed) points of the scheme C also has a complex[2] topology. We'll treat this in more detail later, so if you haven't seen it, dont worry; for now it just allows us to draw some nice pictures of curves.

Example 1.1 (The projective line). The most basic example is the curve $\mathbb{P}^1$, the projective line. It is covered by $U_0 = \mathbb{P}^1 \setminus \{[0:1]\} = \mathbb{A}^1$ and $U_1 = \mathbb{P}^1 \setminus \{[1:0]\} = \mathbb{A}^1$, overlapping in $U_0 \cap U_1 = \mathbb{A}^1 \setminus \{0\}$. This allows us to see that its complex points $\mathbb{P}^1(\mathbb{C})$ are exactly given by the two-sphere S^2 (see Figure 1).

Figure 1: The complex points of the projective line and its chart U_0

A second source of examples are curves which are subvarieties of the projective plane $\mathbb{P}^2$ cut out by a homogeneous equation. It turns out that smooth curves cut out by equations of degree one and two (lines and quadrics) are actually isomorphic to $\mathbb{P}^1$, so they don't give new examples. Thus, let's go to degree three.

Example 1.2 (Plane cubics) Consider the family E_t of cubic curves in $\mathbb{P}^2$ defined by

$$E_t = \{[X:Y:Z] \in \mathbb{P}^2 : Y^2Z + X(X-Z)(X-tZ) = 0\} \quad ; \quad t \in \mathbb{C}: \tag{3}$$

One checks that for $t \notin \{0;1\}$, these curves are indeed smooth and with some work one can show that their complex points $E_t(\mathbb{C})$ are isomorphic to a torus $T = S^1 \times S^1$, for all values $t \in \mathbb{C} \setminus \{0;1\} = U$ (see Figure 2).

In the above examples, we saw that the complex points of the curves we considered had a nice structure, being a sphere and a torus. This generalizes to arbitrary curves C in $\mathcal{M}$: the complex points $C(\mathbb{C})$ are a smooth, oriented compact real surface (also known as the surface of a donut with g holes). This number g of holes[3] is called the genus of the surface (and also of the corresponding curve C). See Figure 3 for an illustration.

[2] The idea behind the complex topology is that for $U \subset \mathbb{A}^n$ an affine variety, the complex topology on $U(\mathbb{C}) \subset \mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n$ is the relative topology coming from $\mathbb{C}^n$. For a general complex variety X with an affine cover of U_i, the set $X(\mathbb{C})$ is covered by the $U_i(\mathbb{C})$ and their complex topologies glue together.

[3] To make this mathematically precise, one can define g as half of the dimension of the first singular cohomology group of the surface.

Figure 2: The complex points of E_t

Figure 3: The complex points a curve C of genus g

From an algebraic geometry point of view, the genus of a curve $C \in \mathcal{M}$ can be defined as the dimension[4]

$$g(C) = \dim H^0(C; \Omega^1_C) \tag{4}$$

of the space of sections of the cotangent line bundle Ω^1_C of C.

Exercise 1.3. Use this definition to verify that the genus of $\mathbb{P}^1$ is 0 and the genus of E_t is 1.

For the problem of classifying curves up to isomorphism, the genus is important since two curves of different genera clearly cannot be isomorphic (in particular $\mathbb{P}^1 \not\cong E_t$). Indeed we can write down a well-defined map

$$\mathcal{M} \to \{0, 1, 2, \ldots\}, \quad [C] \mapsto g(C) \tag{5}$$

and given $g \in \{0, 1, 2, \ldots\}$ define $\mathcal{M}_g$ as the preimage of g under this map. In other words, we have that $\mathcal{M}_g$ is the set of isomorphism classes of genus g curves.

Returning to our examples above, it turns out that in genus zero our classification is already complete:

Fact 1.4. Every smooth, irreducible complex projective curve C of genus 0 is isomorphic to $\mathbb{P}^1$.

In other words $\mathcal{M}_0$ is the set with unique element $[\mathbb{P}^1]$. However, the situation is already more complicated in genus one:

[4] Silly remark: note how we use the classification (1) via dimension for the vector space $H^0(C; \Omega^1_C)$!

Fact 1.5. Every smooth, irreducible complex projective curve C of genus 1 is isomorphic to one of the curves E_t ($t \in \mathbb{C} \setminus \{0,1\} = U$). Moreover, for $t_1, t_2 \in U$ we have $E_{t_1} \cong E_{t_2}$ if and only if t_2 satisfies

$$t_2 \in \left\{ t_1, \frac{1}{t_1}, 1 - t_1, \frac{1}{1-t_1}, \frac{t_1 - 1}{t_1}, \frac{t_1}{t_1 - 1} \right\}. \tag{6}$$

So while the topological space $E_t(\mathbb{C})$ is independent of t, it turns out that the algebraic curves E_t are not all isomorphic. In fact, we see that there are uncountably infinitely many isomorphism classes of curves M_1! It turns out that the story continues similarly for higher genus: for each $g \geq 2$ the set M_g is uncountable (of the same cardinality as $\mathbb{C}$).

So are we done yet? Is the course complete at this point? After all, we have classified M_g as abstract sets. It turns out that we can ask for more!

For this, we look back at the family E_t parametrized by $t \in U$. This is a very nice family of curves (e.g. the equations defining E_t depend continuously, in fact algebraically, on the parameter t). Fact 1.5 tells us that every genus 1 curve appears as a member of this family and gives a criterion which members of the family are pairwise isomorphic. Starting with the family over U, we can try to get rid of this redundancy. One can define an action of the symmetric group S_3 on U such that the generators $(12), (23) \in S_3$ act by

$$(12) \cdot t = \frac{1}{t}; \quad (23) \cdot t = 1 - t. \tag{7}$$

Then the orbit of $t_1 \in U$ under S_3 is exactly the set given in (6). Thus we have a natural identification $M_1 = U/S_3$, where U/S_3 is the set of orbits in U under the S_3-action. But this shows that we can hope to have more structure on M_1! After all, the set $U = \mathbb{C} \setminus \{0,1\}$ is naturally a topological space, and in fact the set of complex points of the scheme $\mathbb{A}^1 \setminus \{0,1\}$. Also, the action of S_3 is algebraic (e.g. the map

$$\mathbb{A}^1 \setminus \{0,1\} \to \mathbb{A}^1 \setminus \{0,1\}; \; t \mapsto 1/t$$

is an algebraic morphism) and in fact it turns out that the quotient $(\mathbb{A}^1 \setminus \{0,1\})/S_3$ also makes sense as an scheme. This scheme is given by the affine line $\mathbb{A}^1$ and the quotient morphism

$$j : \mathbb{A}^1 \setminus \{0,1\} \to \mathbb{A}^1; \; t \mapsto 2^8 \frac{(t^2 - t + 1)^3}{t^2(t-1)^2} \tag{8}$$

is known as the j-invariant[5]. This is an algebraic morphism and the fibres (of closed points in $\mathbb{A}^1$) are exactly the orbits of the action of S_3 on U.

Looking back, we see that the scheme $\mathbb{A}^1$ together with the data of the j-invariant and the family E_t is a much more satisfying answer to the problem of "classifying genus 1 curves" than just saying that M_1 is an infinite set. In particular it allows us to make the following statements about curves of genus $g = 1$:

> There exists an algebraic variety $U = U_g$ and a family C_t of genus g curves parametrized by $t \in U_g$ such that every smooth, irreducible projective genus g curve appears as a fibre C_t.
>
> The variety U_g is smooth and connected, so any two genus g curves can be deformed into each other using this family.
>
> There exists a variety M_g and a surjective morphism $U_g \to M_g$ which identifies two closed points t_1, t_2 if $C_{t_1} \cong C_{t_2}$, so the closed points of M_g are in bijection with the smooth, irreducible projective genus g curves up to isomorphism.

[5] There are good reasons for putting the factor 2^8, see the reference section below.

It turns out that all of these statements are still true for $g \geq 2$. The spaces M_g appearing in the statement then deserve to be called the moduli spaces of curves and those are the objects of study in this course. However, there is also a surprising set of things which go wrong or unexpected:

> For $g \geq 2$, the variety M_g is not smooth. For $g \geq 4$ its singular locus corresponds to the isomorphism classes of curves C having a nontrivial automorphism.
>
> For $g \geq 1$, the family C_t, $t \in U_g$, above does not descend to a family over M_g. In other words, we cannot write down a (reasonable) family of curves parametrized by M_g such that the member of the family associated to $[C] \in M_g$ is isomorphic to C. However, there does exist such a family over the complement of the locus of curves having a nontrivial automorphism.

Thus in both cases automorphisms of curves are the source of trouble. In the course, we will see how the theory of *algebraic stacks*, a generalization of schemes, resolves this problem by viewing M_g as a stack instead of a scheme.

The first part of our course will be to make precise what we mean by words like "moduli space" and "reasonable family of curves" above and then to study the spaces M_g that can be defined this way. We will see how modern tools of algebraic geometry (e.g. deformation theory) can be used in this study (e.g. to determine the dimension of M_g).

The second part of the course will focus on studying the cohomology groups of the moduli spaces of curves. Here, we will encounter a second issue, related to a quote by Angelo Vistoli:

> "Working with noncompact spaces is like trying to
> keep change with holes in your pockets."

Indeed, for $g \geq 1$ the spaces M_g are not proper, so their sets of $\mathbb{C}$-points are not compact. This means that their cohomology groups lack some nice properties, such as Poincaré duality, which hold for compact spaces. We already saw an example above: identifying M_1 with $\mathbb{A}^1$, its complex points are $\mathbb{C}$, which is contractible and thus has the cohomology of a point.

This issue can be resolved by finding a *compactification* of M_g, i.e. a space $\overline{M}_g$ containing M_g as an open subset and working on $\overline{M}_g$ instead. For this compactification to be helpful, it should itself be a moduli space of geometric objects generalizing the smooth genus g curves which M_g classifies.

Going back to the family E_t of genus 1 curves parametrized by $t \in \mathbb{C} \setminus \{0, 1\}$ can give us a hint what these more general geometric objects could be. Indeed, why did we restrict to $t \neq 0, 1$ in this family? The reason was that putting $t = 0, 1$ in the defining equation of E_t would give us a singular curve in $\mathbb{P}^2$. For instance, for $t = 0$ we obtain

$$E_0 = \{[X : Y : Z] \in \mathbb{P}^2 : Y^2Z + X^2(X - Z) = 0\}.$$

The singular point is $[X : Y : Z] = [0 : 0 : 1]$. Let's see how the curve looks like in a neighborhood of this point: going to the chart $\{Z \neq 0\}$ of $\mathbb{P}^2$ with coordinates $[X : Y : Z] = [x : y : 1]$, the singular point is $(x, y) = (0, 0)$ and the equation becomes

$$y^2 + x^2(x - 1) = 0.$$

Around $(0, 0)$ we have $x - 1 \neq 1$ and so in a small neighbourhood of this point[6] the equation looks like

$$y^2 - x^2 = 0 \iff (y - x)(y + x) = 0,$$

Figure 4: A neighbourhood of the singular point in E_0 looks like the union of the lines $y = x$ and $y = \quad x$

which is simply two lines meeting transversally.

Such a singular point is called a *nodal singularity*. Using the pictures of real surfaces from above, we can illustrate how the set of complex points $E_t(C)$ changes as t approaches 0.

Figure 5: The degeneration of the surface $E_t(C)$ as $t \to 0$

Generalizing the moduli space M_g of smooth curves C we will study a moduli space $\overline{M}_g$ which allows the curve C to have such nodal singularities. This space $\overline{M}_g$ then turns out to be proper (or compact), fixing the holes that we had in our pockets.

The goal of the second half of this lecture series is to show the beautiful structures that appear in the cohomology groups of the space $\overline{M}_g$. This is an area of active research, with connections to many parts of mathematics such as graph theory, enumerative geometry and the theory of integrable systems (to name but a few).

References and further reading (and watching)

A nice and mostly elementary discussion of the Facts about curves of genus zero and one stated above can be found in Sections 19.3 and 19.9 of [Vak17]. The result that the set of complex points of a plane cubic is isomorphic to a torus (which can and should be seen as a quotient $\mathbb{C}/\quad$ of the complex numbers by a lattice $\quad \subset \mathbb{C}$) has a beautiful connection to the so-called Weierstrass $\wp$-function, e.g. explained in [Hai08, Section 5].

There is a video recording of a great one-hour introductory talk to the moduli spaces of curves given by my PhD advisor Rahul Pandharipande at the ICM in Rio in 2018. It does not only cover essentially everything that we are going to treat in this course, but it

[6] This can be made precise using formal algebraic geometry and completions of local rings, let's not worry about this now.

also contains the Portuguese translation of a quote by Riemann (at 10:53) and a picture of a passion ower taken by a PhD brother of mine (at 13:39), among other things.

2 Fine and coarse moduli spaces

2.1 Motivation

Before we discuss the moduli space of curves, we should first make sure we understand what we even mean by "moduli space". To start off, let us consider a familiar example: the projective space $\mathbb{P}^n$. It classifies lines through the origin (i.e. sub-vector spaces of dimension 1) in $\mathbb{C}^{n+1}$. However, just writing

$$\mathbb{P}^n = \{ \ell : \ell \text{ line through the origin in } \mathbb{C}^{n+1} \} \tag{9}$$

is not enough: the right-hand side of (9) is a set, equal to the set $\mathbb{P}^n(\mathbb{C})$ of $\mathbb{C}$-points of $\mathbb{P}^n$, but it does not have the structure of a scheme.

So what makes the usual scheme structure on $\mathbb{P}^n$ the right one to make it a moduli space of lines in $\mathbb{C}^{n+1}$? Let's start with the topology: intuitively, for the lines

$$\ell_n = \begin{matrix} 1 \\ 1/n \end{matrix} \quad ; \quad \ell_0 = \begin{matrix} 1 \\ 0 \end{matrix}$$

the sequence ℓ_n of lines in $\mathbb{C}^2$ "converges" to the line ℓ_0.

Figure 6: The sequence of lines ℓ_n converging to the line ℓ_0

And indeed, the (complex) topology on $\mathbb{P}^n(\mathbb{C})$ satisfies $\ell_n \xrightarrow{n\to\infty} \ell_0$. Similarly, for the scheme structure on $\mathbb{P}^n$ note that the family of lines

$$L_t = \begin{matrix} 1 \\ t \end{matrix} \quad : t \in \mathbb{A}^1$$

parametrized by the variety $\mathbb{A}^1$ has defining equations which are algebraic in the coordinate t on $\mathbb{A}^1$. Because of this, we expect that the scheme structure on $\mathbb{P}^2$ should satisfy that the map

$$\mathbb{A}^1 \to \mathbb{P}^2; t \mapsto L_t \tag{10}$$

is an algebraic morphism, which again is true for the standard structure on $\mathbb{P}^2$. Generalizing this example from $X = \mathbb{A}^1$ to arbitrary schemes X, there should be a bijective correspondence

$$\begin{aligned} \text{Families of lines parametrized by } X &\to \text{Morphisms } X \to \mathbb{P}^n; \\ (L_t : t \in X) &\to (X \to \mathbb{P}^n; t \mapsto L_t): \end{aligned} \tag{11}$$

The crucial insight is that knowing the sets of morphisms $X \to \mathbb{P}^n$ for every scheme X uniquely determines the scheme structure on $\mathbb{P}^n$ (see Lemma 2.1 below). Thus, by the equivalence (11) above, we can uniquely specify the scheme $\mathbb{P}^n$ by making precise what we mean with a "family of lines parametrized by X".

2.2 Moduli functors and fine moduli spaces

The correct way to do this is using category theory. Denote by Sch_C the category of schemes over C, where for schemes $X, Y \in Ob(Sch_C)$ the set $Mor(X, Y)$ of morphisms from X to Y is the set of algebraic morphisms $X \to Y$ over C. Then, for any scheme $M \in Ob(Sch_C)$, we have a functor

$$h^M : Sch_C^{op} \to Sets, \quad X \mapsto Mor(X, M). \tag{12}$$

Why the "op" above? This is because $Mor(\ , M)$ is a contravariant functor, i.e. given $g : X' \to X$ we get a natural map $h^M(X) \to h^M(X')$ in the opposite direction, given by

$$Mor(X, M) \to Mor(X', M), \quad (X \xrightarrow{f} M) \mapsto (X' \xrightarrow{f \circ g} X \xrightarrow{f} M).$$

For $M = \mathbb{P}^n$, the set $h^M(X)$ is exactly the right-hand side of (11). The statement of the Yoneda Lemma is that the functor h^M uniquely determines the scheme M.

Lemma 2.1 (Yoneda's Lemma) The functor

$$\begin{aligned} h \ : Sch_C &\to Functors(Sch_C^{op} \to Sets), \\ M &\mapsto h^M \end{aligned} \tag{13}$$

is a fully faithful embedding. In other words

- given schemes M, N, the morphisms $M \to N$ are in bijection with natural transformations $h^M \to h^N$,
- in particular $M = N \iff h^M = h^N$.

By this Lemma, we can see the category of schemes as sitting inside the category of functors $Sch_C^{op} \to Sets$. Thus we will give these functors their own name.

Definition 2.2. A moduli functor h is a functor $h : Sch_C^{op} \to Sets$. In other words, the data we need to specify is the following:

- for every scheme X over C a set $h(X)$ (the families parametrized by/over X),
- for every morphism $f : X' \to X$ a map $h(f) : h(X) \to h(X')$ (the pullback of families under f),
- satisfying that $h(id_X) = id_{h(X)}$ and that for $X'' \xrightarrow{g} X' \xrightarrow{f} X$ we have that the composition $h(X) \xrightarrow{h(f)} h(X') \xrightarrow{h(g)} h(X'')$ equals $h(X) \xrightarrow{h(f \circ g)} h(X'')$ (the compatibility of pullback for identity and compositions).

Definition 2.3. A moduli functor h is called *representable* if it is of the form $h = h^M$ for a scheme M. This scheme M (which is unique up to isomorphism) is then called a *fine moduli space* for h.

Example 2.4. We return to the example of projective space $\mathbb{P}^n$, showing that it is a fine moduli space for "lines through the origin in $\mathbb{C}^{n+1}$". Define a moduli functor $h : \mathrm{Sch}_{\mathbb{C}}^{op} \to \mathrm{Sets}$ by the following data:

for every scheme X over $\mathbb{C}$ the set $h(X)$ is given by

$$h(X) = \left\{ \begin{tikzcd} L \arrow[r, "i"] \arrow[d] & X \times \mathbb{C}^{n+1} \arrow[dl] \\ X & \end{tikzcd} \right\} = \tag{14}$$

where $L \to X$ is (the total space of) a line bundle on X which is a subbundle of the trivial bundle $X \times \mathbb{C}^{n+1} \to X$ (i.e. there is an injective map $i : L \to X \times \mathbb{C}^{n+1}$ of vector bundles such that the quotient is also a vector bundle). We take this set up to isomorphisms, i.e. $(L \xrightarrow{i} X \times \mathbb{C}^{n+1}) \sim (L' \xrightarrow{i'} X \times \mathbb{C}^{n+1})$ if there is an isomorphism $L \to L'$ of line bundles on X making the obvious diagram commute,

for every morphism $f : X' \to X$ we define the pullback by

$$h(f) : h(X) \to h(X');$$
$$(L \xrightarrow{i} X \times \mathbb{C}^{n+1}) \mapsto (f^* L \xrightarrow{f^* i} f^*(X \times \mathbb{C}^{n+1}) = X' \times \mathbb{C}^{n+1}):$$

The compatibility conditions of the pullback are satisfied since clearly $h(\mathrm{id}_X) = \mathrm{id}_{h(X)}$ and since for $X'' \xrightarrow{g} X' \xrightarrow{f} X$ we have a canonical isomorphism $g^* f^* L = (g \circ f)^* L$.

Finally we want to show that h is representable by $\mathbb{P}^n$, i.e. that we have a natural equivalence $h = h^{\mathbb{P}^n}$ of functors. This exactly makes precise the equivalence (11) we claimed before. Recall that to specify a natural equivalence $h = h^{\mathbb{P}^n}$ we must give, for every scheme X, a bijection $h(X) \to h^{\mathbb{P}^n}(X) = \mathrm{Mor}(X; \mathbb{P}^n)$ such that for $f : X' \to X$ the diagram

$$\begin{tikzcd} h(X) \arrow[r] \arrow[d, "h(f)"] & \mathrm{Mor}(X; \mathbb{P}^n) \arrow[d, "h^{\mathbb{P}^n}(f)"] \\ h(X') \arrow[r] & \mathrm{Mor}(X'; \mathbb{P}^n) \end{tikzcd} \tag{15}$$

commutes. What should the map $h(X) \to \mathrm{Mor}(X; \mathbb{P}^n)$ do again? Given

$$(i : L \to X \times \mathbb{C}^{n+1}) \in h(X); \tag{16}$$

for any $x \in X$ we have that $i(L_x) \subset \{x\} \times \mathbb{C}^{n+1}$ is a line through the origin of $\mathbb{C}^{n+1}$ and so (16) should be sent to the morphism $X \to \mathbb{P}^n$ mapping x to $[i(L_x)] \in \mathbb{P}^n$. But a priori it is only clear what to do at closed points $x \in X$ and we should find a more algebraic way to phrase this, allowing us to deal with more complicated X.

We take a small but interesting detour to do this. Fix a scheme X and an element $(L \xrightarrow{i} X \times \mathbb{C}^{n+1})$ in $h(X)$. The injective map i of total spaces of vector bundles (whose cokernel is a vector bundle) corresponds to a short exact sequence

$$0 \to L \xrightarrow{i} \mathcal{O}_X^{n+1} \to Q \to 0$$

of locally free sheaves on X[7]. Taking the dual, this is equivalent to an exact sequence

$$0 \leftarrow L \xleftarrow{i^\vee} \mathcal{O}_X^{n+1} \leftarrow Q \leftarrow 0$$

[7] Alternatively, we can say it corresponds to a locally split injective morphism $i : L \to \mathcal{O}_X^{n+1}$.

of locally free sheaves. Since the kernel of a map of locally free sheaves is automatically locally free, this is equivalent to just specifying the surjection $\mathcal{O}_X^{n+1} \to \mathcal{L}$. Such a surjection is determined by specifying sections $s_0, \dots, s_n \in H^0(X; \mathcal{L})$ without common zero, i.e. not vanishing simultaneously anywhere on X. Then writing $M = \mathcal{L}$ and defining

$$h^0(X) = \left\{ (M; s_0, \dots, s_n) : \begin{array}{c} M \text{ a locally free sheaf on } X \\ s_0, \dots, s_n \in H^0(X; M) \text{ sections} \\ \text{without common zero} \end{array} \right\} / \cong \tag{17}$$

we have just described a map $h(X) \to h^0(X)$. One checks that this map is a bijection, that h^0 is in fact a moduli functor and that the maps $h(X) \to h^0(X)$ define a natural equivalence $h = h^0$. Thus we have reduced the problem to showing that $h^0 = h^{\mathbb{P}^n}$.

The fact that a map $\varphi : X \to \mathbb{P}^n$ is equivalent to the data of a line bundle M on X together with $n+1$ sections without common zero (up to isomorphism) is proven in many textbooks on Algebraic Geometry (see e.g. [Sch17, Corollary 12.10], [Har77, II, Theorem 7.1], [Vak17, Important Theorem 16.4.1.]). Let us recall the argument: starting with an element $(M; s_0, \dots, s_n) \in h^0(X)$ we can write down a map $X \to \mathbb{P}^n$ by

$$\varphi : X \to \mathbb{P}^n, \; x \mapsto [s_0(x) : s_1(x) : \dots : s_n(x)]. \tag{18}$$

A priori, the expression $[s_0(x) : s_1(x) : \dots : s_n(x)]$ does not make sense, since the $s_j(x)$ are not functions but elements in the fibre M_x of the line bundle M on X. But we can choose any cover $X = \bigcup_i U_i$ of X trivializing the line bundle and any isomorphism $M|_{U_i} \cong \mathcal{O}_{U_i}$. Then on U_i we can identify the sections s_j with regular functions and the element $[s_0(x) : s_1(x) : \dots : s_n(x)]$ does not depend on the choice of trivialization. Indeed, a different choice of trivialization differs by multiplying all components of $[s_0(x) : s_1(x) : \dots : s_n(x)]$ with the same nonvanishing function, so the corresponding element of $\mathbb{P}^n$ does not change.

Conversely, given a map $\varphi : X \to \mathbb{P}^n$ we define $M = \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$. For the sections, observe that $\mathcal{O}_{\mathbb{P}^n}(1)$ has a space of global sections of rank $n+1$ given by

$$H^0(\mathbb{P}^n; \mathcal{O}(1)) = \langle x_0, \dots, x_n \rangle$$

and we define $s_j = \varphi^*(x_j) \in H^0(X; M)$. One checks that the two correspondences between $(M; s_0, \dots, s_n)$ (up to isomorphism) and $\varphi : X \to \mathbb{P}^n$ are inverse to each other and "functorial" (i.e. making the diagrams (15) commute). Thus they define a natural equivalence $h^0 = h^{\mathbb{P}^n}$, finishing the proof.

Remark 2.5. Instead of constructing the equivalence $h = h^{\mathbb{P}^n}$ via the functor h^0, a direct construction is also possible: given an element $(L \xrightarrow{i} X \times \mathbb{C}^{n+1})$ in $h(X)$ let $L^0 \subset L$ be the open set that is the complement of the zero section. Due to the assumption that i is injective, it maps L^0 to $X \times (\mathbb{C}^{n+1} \setminus 0)$. Consider the diagram

$$\begin{tikzcd} L^0 \arrow[r, "i"] \arrow[d] & X \times (\mathbb{C}^{n+1} \setminus 0) \arrow[r] & \mathbb{P}^n \\ X \arrow[urr, dashed, "\varphi"'] & & \end{tikzcd}$$

where φ is the composition of the projection $X \times (\mathbb{C}^{n+1} \setminus 0) \to \mathbb{C}^{n+1} \setminus 0$ with the quotient map $\mathbb{C}^{n+1} \setminus 0 \to \mathbb{P}^n$. One checks that the composition $\varphi \circ i : L^0 \to \mathbb{P}^n$ is constant on the fibres of $L^0 \to X$. Intuitively, this should allow us to define a map $\varphi : X \to \mathbb{P}^n$ completing the diagram above by $\varphi(x) = (\varphi \circ i)(\ell)$ for any $\ell \in L^0$ mapping to x. This intuition indeed works, using the so-called fpqc descent (see [Sta13, Tag 023Q]). So starting from $(L \xrightarrow{i} X \times \mathbb{C}^{n+1}) \in h(X)$ we constructed $\varphi \in \mathrm{Mor}(X; \mathbb{P}^n)$ and one checks that this induces the same natural transformation $h = h^{\mathbb{P}^n}$ as in Example 2.4.

Definition 2.6. Given a moduli functor h and a natural isomorphism $h = h^M$ for some scheme M, define the universal family $U \in h(M)$ to be the element in $h(M)$ corresponding to the canonical element $id_M \in Mor(M; M) = h^M(M)$.

Exercise 2.7. Let h be a moduli functor with fine moduli space M and universal family $U \in h(M)$.

a) Show that for any scheme X and any family $F \in h(X)$ parametrized by X there exists a unique morphism $f : X \to M$ such that the pullback of U under f is F, i.e. $h(f)(U) = F$.

b) Show that for the moduli functor h from Example 2.4 with moduli space $\mathbb{P}^n$, the universal family is given by the "tautological" line bundle

$$\begin{tikzcd} \{(\ell; v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : v \in \ell\} = L \arrow[r, "i"] \arrow[d] & \mathbb{P}^n \times \mathbb{C}^{n+1} \arrow[dl] \\ \mathbb{P}^n & \end{tikzcd}$$

The line bundle L above is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-1)$.

c) Show that for the moduli functor h^0 from Example 2.4 with moduli space $\mathbb{P}^n$, the universal family is given by

$$(\mathcal{O}_{\mathbb{P}^n}(1); x_0; \ldots; x_n).$$

Easy exercise 2.8 (Fibre products as moduli spaces) Let $\varphi_X : X \to Z$, $\varphi_Y : Y \to Z$ be morphisms of schemes. For any scheme S define

$$h(S) = (h^X \times_{h^Z} h^Y)(S) = \{(\varphi_X; \varphi_Y) : \begin{array}{l} \varphi_X : S \to X;\ \varphi_Y : S \to Y \\ \text{such that } \varphi_X \circ \varphi_X = \varphi_Y \circ \varphi_Y \end{array}\}$$

a) Show that h defines a moduli functor.

b) Prove that the fibre product $X \times_Z Y$ is a fine moduli space for h. What is its universal family?

2.3 Application : The Picard scheme

Until now we only used the theory of moduli spaces to interpret known schemes as a fine moduli space for some functor. Now let's use it to actually define a new space. For this, recall the definition of the Picard group of a scheme[8] X:

$$\mathrm{Pic}(Y) = \{L : L \text{ an invertible sheaf on } Y\}/\text{iso}. \tag{19}$$

It classifies line bundles up to isomorphisms. A priori, the definition (19) only makes sense as a set (or a group with respect to tensor product). But it turns out that in many situations we can find a nice scheme structure underlying $\mathrm{Pic}(Y)$ by seeing it as a fine moduli space.

What should be the moduli functor? Given a scheme X we need to define what we mean by a "family of line bundles on Y over the base X". The natural thing is to take the product $X \times Y$ (the trivial family with fibre Y over X) and look at the data of line bundles up to isomorphism on $X \times Y$.

[8] In this section, all schemes will be schemes over $\mathbb{C}$, all morphisms will be morphisms over $\mathbb{C}$, all fibre products will be fibre products over $\mathbb{C}$ etc.

Definition 2.9 (Picard functor - first attempt). We define the (absolute) Picard functor $\mathrm{Pic}_Y : \mathrm{Sch}^{op} \to \mathrm{Sets}$ of the scheme Y as

$$\mathrm{Pic}_Y(X) = \mathrm{Pic}(X \times Y).$$

Given $f : X' \to X$ the corresponding morphism $\mathrm{Pic}_Y(X) \to \mathrm{Pic}_Y(X')$ is given by the pullback $L \mapsto f^* L$ of line bundles under $f \times \mathrm{id}_Y$.

This is a perfectly nice and reasonable moduli functor, except for the fact that it is never representable (for Y nonempty).

Proposition 2.10. For $Y \neq \emptyset$, the functor Pic_Y is not representable.

Proof. Let X be any scheme with a nontrivial line bundle M (e.g. $X = \mathbb{P}^1$ with $M = \mathcal{O}_{\mathbb{P}^1}(1)$). Let $\pi_X : X \times Y \to X$ be the projection, then we have the object

$$M_X = \pi_X^* M \in \mathrm{Pic}(X \times Y) = \mathrm{Pic}_Y(X).$$

Note that this object is nontrivial (i.e. not isomorphic to $\mathcal{O}_{X \times Y}$). Indeed, since Y is nonempty (and since $\mathbb{C}$ is algebraically closed) it has a $\mathbb{C}$-point and so π_X has a section. Then the pullback of M_X under this section is M, which is not trivial.

Assume that Pic_Y were representable by a scheme P with a universal object $U \in \mathrm{Pic}_Y(P)$. Then by definition there exists a unique morphism $g : X \to P$ with $(g \times \mathrm{id}_Y)^* U = M_X$. Similarly, there is a unique morphism $p : \mathrm{pt} = \mathrm{Spec}(\mathbb{C}) \to P$ associated to the trivial line bundle $\mathcal{O}_{\mathrm{pt} \times Y}$ i.e. satisfying $(p \times \mathrm{id}_Y)^* U = \mathcal{O}_{\mathrm{pt} \times Y}$.

Now let $X = \bigcup_i U_i$ be an open cover on which M is trivial, then the pullbacks M_{U_i} of M_X to $U_i \times Y$ are trivial. In other words, they are pullbacks of $\mathcal{O}_{\mathrm{pt} \times Y}$ under the maps $U_i \to \mathrm{pt}$ to a point. This implies that the unique map $U_i \to P$ associated to M_{U_i} must factor through $p : \mathrm{pt} \to P$. We then have a diagram of maps as follows

$$\begin{tikzcd} X \arrow[r, "g"] & P \\ \bigcup_i U_i \arrow[u, "\iota"] \arrow[r] \arrow[ur, dashed, "g'"] & \mathrm{pt} \arrow[u, "p"'] \end{tikzcd} \tag{20}$$

Since the restrictions of the map $g : X \to P$ to the covering opens U_i all factor through p, the map g itself must factor through this morphism. But this is impossible since then we have

$$M_X = (g \times \mathrm{id}_Y)^* U = (g' \times \mathrm{id}_Y)^* (p \times \mathrm{id}_Y)^* U = (g' \times \mathrm{id}_Y)^* \mathcal{O}_{\mathrm{pt} \times Y} = \mathcal{O}_{X \times Y};$$

a contradiction to the statement that M_X is not trivial. □

Remark 2.11. The fact that Pic_Y does not have a moduli space is a first example of a general slogan[9] in the theory of moduli spaces:

For moduli functors h, the presence of nontrivial automorphisms often prevents the existence of a fine moduli space. (21)

As above, the rough idea is that automorphisms allow us to find a scheme X with an open cover $X = \bigcup_i U_i$ such that we can construct a nontrivial family $F_X \in h(X)$ by gluing trivial families $F_{U_i} \in h(U_i)$ along nontrivial automorphisms $F_{U_i}|_{U_i \setminus U_j} \cong F_{U_j}|_{U_i \setminus U_j}$. We then reach a contradiction by using a variant of the diagram (20). As we will see later, the theory of algebraic stacks was specifically invented to be able to deal with these kinds of moduli problems.

[9] As with many one-sentence slogans, this is a bit too general to be true in all contexts and hard to make precise anyway. It is more a guideline to build some first intuitions.

In the proof of Proposition 2.10 we were able to reach a contradiction for the representability of Pic_Y by using the nontrivial family $M \times_X 2 \mathrm{Pic}(X \times Y)$ obtained by pulling back a line-bundle $M 2 \mathrm{Pic}(X)$ from the base. Surprisingly, in this case we can solve all our problems by just dividing out such pullback bundles in our moduli functor.

De nition 2.12 (Picard functor - second attempt). We de ne the (relative) Picard functor $\mathrm{Pic}_{Y=C} : \mathrm{Sch}^{op} ! \mathrm{Sets}$ of the scheme Y as

$$\mathrm{Pic}_{Y=C}(X) = \mathrm{Pic}(X \times Y)= \pi_X \mathrm{Pic}(X);$$

where $\pi_X : X \times Y ! X$ is the projection on the rst factor and $\pi_X \mathrm{Pic}(X) \subset \mathrm{Pic}(X \times Y)$ is the set of line bundles that are pullbacks from X. Given $f : X^0 ! X$ the corresponding morphism $\mathrm{Pic}_{Y=C}(X) ! \mathrm{Pic}_{Y=C}(X^0)$ is still given by the pullback $L 7! f \times L$ of line bundles under $f \times \mathrm{id}_Y$ (and this respects the equivalence relation we divided out).

Theorem 2.13. If Y is an integral, projective variety over C, then the functor $\mathrm{Pic}_{Y=C}$ is representable by a separated, locally nite type scheme $\mathrm{Pic}_{Y=C}$.

Proof. See e.g. [Kle05, Theorem 4.8]. □

Remark 2.14. Assume we are given Y such that $\mathrm{Pic}_{Y=C}$ is representable by a scheme $\mathrm{Pic}_{Y=C}$.

a) A line bundle L on $\mathrm{Pic}_{Y=C} \times Y$ representing the universal family on the moduli space is called a *Poincaré line bundle*. Given a C-point $[M] 2 \mathrm{Pic}_{Y=C}$, the restriction of L to $[M] \times Y \subset \mathrm{Pic}_{Y=C} \times Y$ is isomorphic to M. The line bundle L is unique up to tensoring with line bundles pulled back from $\mathrm{Pic}_{Y=C}$.

b) The set $\mathrm{Pic}(Y)$ of line bundles on Y has a natural structure of an abelian group, where multiplication is given by tensor product of line bundles. From this one can show that $\mathrm{Pic}_{Y=C}$ has the structure of an *algebraic group*, i.e. the natural maps

$$\mathrm{Pic}_{Y=C} \times \mathrm{Pic}_{Y=C} ! \mathrm{Pic}_{Y=C}; \qquad \mathrm{Pic}_{Y=C} ! \mathrm{Pic}_{Y=C}$$
$$([L_1]; [L_2]) 7! [L_1 \otimes L_2] \qquad L 7! L^{-}$$

are algebraic morphisms which together with the inclusion $[\mathcal{O}_Y] 2 \mathrm{Pic}_{Y=C}$ de ne multiplication, inverse and neutral element of a group-structure on $\mathrm{Pic}_{Y=C}$.

c) We denote by $\mathrm{Pic}^0_{Y=C} \subset \mathrm{Pic}_{Y=C}$ the connected component of $\mathrm{Pic}_{Y=C}$ containing the trivial line bundle $[\mathcal{O}_Y]$. It gives a subgroup of $\mathrm{Pic}_{Y=C}$. For $Y = C$ a smooth, projective algebraic curve, the scheme $\mathrm{Pic}^0_{C=C} = \mathrm{Jac}(C)$ is called the *Jacobian* of C.

Example 2.15. a) For a point $Y = pt = \mathrm{Spec}(C)$ we have that for any scheme S we obtain

$$\mathrm{Pic}_{pt=C}(S) = \mathrm{Pic}(pt \times S)=\mathrm{Pic}(S) = f[\mathcal{O}_S]g:$$

This implies that

$$\mathrm{Pic}_{pt=C} = \mathrm{Spec}(C):$$

However, surprisingly for $Y = A^n$ the same is not true (even though $\mathrm{Pic}(A^n) = f\mathcal{O}_{A^n}g$). This follows from the existence [Liu] of a scheme X such that the pullback $\mathrm{Pic}(X) ! \mathrm{Pic}(X \times A^1)$ is not an isomorphism.

b) For any $n \geq 1$ we have that

$$\mathrm{Pic}_{P^n=C} = \bigsqcup_{m2Z} f\mathcal{O}_{P^n}(m)g$$

is a countable union of isolated points.

c) Given an elliptic curve $(E; p_0)$, we have

$$\mathrm{Pic}^0_{E=C} = E;$$

and for $E \quad E$ the diagonal, the line bundle

$$L = \mathcal{O}_{E \quad E}(\quad E \quad p_0) \; 2 \; \mathrm{Pic}(\mathrm{Pic}^0_{E=C} \quad E)$$

is a Poincare line bundle over $\mathrm{Pic}^0_{E=C}$.

2.4 Coarse moduli spaces

As we have seen, even reasonable functors like Pic_Y can fail to have ne moduli spaces. While in this particular example we were able to x this by considering the relative Picard functor, in general (and in particular for the moduli functors of curves that we will consider soon) a ne moduli space is too much to ask. Thus we need a weaker notion for a scheme to "approximately represent" a moduli functor.

De nition 2.16. Given a moduli functor h, a coarse moduli space is a pair $(M;)$ of a scheme M together with a natural transformation $: h \; ! \; h^M$ such that

a) $(M;)$ is initial among all such pairs, i.e. for any other scheme M^0 and natural transformation $^0 : h \; ! \; h^{M^0}$ there exists a unique natural transformation $: h^M \; ! \; h^{M^0}$ such that the following diagram commutes

$$\begin{tikzcd} h \arrow[rr] \arrow[dr, "{}^0"'] & & h^M \arrow[dl, "9!"] \\ & h^{M^0} & \end{tikzcd} \tag{22}$$

b) the map induces a bijection

$$(\mathrm{Spec}(C)) : h(\mathrm{Spec}(C)) \; ! \; h^M(\mathrm{Spec}(C)) = M(C) \tag{23}$$

on C-points.

Easy exercise 2.17. a) Show that every ne moduli space is also a coarse moduli space (in particular, make precise what this statement means).

b) Show that given a moduli functor h, if a pair $(M;)$ satisfying condition a) in De nition 2.16 exists, this pair is unique up to isomorphism. In particular, coarse moduli spaces are unique up to isomorphism.

In the next section we are going to introduce the moduli functors for families of curves and we will see that they have a coarse moduli space, but not a ne one. A second example of this are the absolute Picard functors Pic_C introduced in Section 2.3.

Proposition 2.18. For any scheme Y such that the relative Picard functor $\mathrm{Pic}_{Y=C}$ has a ne moduli space $\mathrm{Pic}_{Y=C}$, the natural map $: \mathrm{Pic}_Y \; ! \; \mathrm{Pic}_{Y=C} = h^{\mathrm{Pic}_{Y=C}}$ makes $(\mathrm{Pic}_{Y=C};)$ into a coarse moduli space for Pic_C.

Proof. To show property a) from Definition 2.16, let $\ ^0 : \mathrm{Pic}_Y \to h^{M^0}$ be a second natural transformation. We need to show that given X, the map $\mathrm{Pic}_Y(X) \to \mathrm{Mor}(X; M^0)$ induced by $\ ^0$ factors through $\mathrm{Pic}_Y(X) = \ _X \mathrm{Pic}(X)$. So let $L_1; L_2$ be line bundles on $X \ \ Y$ with $L_2 = L_1 \ \ _X M$ for a line bundle M on X. Let $g_1; g_2 : X \to M^0$ be the maps induced from $L_1; L_2$ via $\ ^0$. We are finished if we can show $g_1 = g_2$.

For this note that given for any open cover $X = \ \bigcup_i U_i$ trivializing M on X, the pullbacks of $L_1; L_2$ to $U_i \ \ Y$ coincide for all i. By functoriality of $\ ^0$, this implies that $g_1 j_{U_i} = g_2 j_{U_i}$. But since morphisms are determined by their restriction to an open cover, we conclude $g_1 = g_2$.

Property b) of Definition 2.16 follows since

$$\begin{aligned}\mathrm{Pic}_Y(\mathrm{Spec}(\mathbb{C})) &= \mathrm{Pic}(\mathrm{Spec}(\mathbb{C}) \ \ Y) \\ &= \mathrm{Pic}(\mathrm{Spec}(\mathbb{C}) \ \ Y) = \mathrm{Pic}(\mathrm{Spec}(\mathbb{C})) = \mathrm{Pic}_{Y=\mathbb{C}}(\mathrm{Spec}(\mathbb{C})) = \mathrm{Pic}(\mathbb{C}). \qquad \square\end{aligned}$$

Challenge 2.19. Prove or disprove that $pt = \mathrm{Spec}(\mathbb{C})$ is a coarse moduli space for $\mathrm{Pic}_{\mathbb{A}^1=\mathbb{C}}$. See also my question on math.stackexchange.
Update: The question has been answered affirmatively within a day by two students from the class! You can check out their answers under the link above.

Exercise 2.20. We want to study the moduli problem of classifying \two points on $\mathbb{P}^1$, not necessarily distinct, up to projective linear transformations". For this, consider the moduli functor

$$h : \mathrm{Sch}^{op} \to \mathrm{Sets}; X \mapsto \{ s : X \to \mathbb{P}^1 \ \ \mathbb{P}^1 \} = \ ; \tag{24}$$

where $s \ \ s^0$ if there exists $G : X \to PGL_2$ such that

$$G(x) \ \ s(x) = s^0(x);$$

where PGL_2 acts diagonally on $\mathbb{P}^1 \ \ \mathbb{P}^1$.

Let $pt = \mathrm{Spec}(\mathbb{C})$, when we want to show that $(pt; \ )$ for the unique natural transformation $\ : h \to h^{pt}$ satisfies condition a) from Definition 2.16, but not condition b).

a) Assume we have a scheme U and $s \ 2 \ h(U)$ such that $s : U \to \mathbb{P}^1 \ \ \mathbb{P}^1$ factors through the complement $\mathbb{P}^1 \ \ \mathbb{P}^1 \ n \ \ $ of the diagonal $\ $. Show that then there exists an open cover $U = \ \bigcup_i U_i$ of U such that the restrictions $s j_{U_i}$ are equivalent under $\ $ to the constant section

$$s_0 = ([1:0]; [0:1]) : U_i \to \mathbb{P}^1 \ \ \mathbb{P}^1.$$

b) Let $(M^0, \ ^0)$ be a pair of a scheme and a natural transformation $\ ^0 : h \to h^{M^0}$. Let $\ : pt \to M^0$ be the morphism associated via $\ ^0$ to

$$(i : pt \xrightarrow{([1:0];[0:1])} \mathbb{P}^1 \ \ \mathbb{P}^2) \ 2 \ h(pt).$$

Consider the family

$$(s_{\mathbb{P}^1 \ \ \mathbb{P}^1} = \mathrm{id}_{\mathbb{P}^1 \ \ \mathbb{P}^1} : \mathbb{P}^1 \ \ \mathbb{P}^1 \to \mathbb{P}^1 \ \ \mathbb{P}^1) \ 2 \ h(\mathbb{P}^1 \ \ \mathbb{P}^1)$$

and use the previous exercise part to show that the associated morphism $\ ^0(s_{\mathbb{P}^1 \ \ \mathbb{P}^1}) : \mathbb{P}^1 \ \ \mathbb{P}^1 \to M^0$ factors through $\ : pt \to M^0$. (Hint: Use the continuity of $\ ^0(s_{\mathbb{P}^1 \ \ \mathbb{P}^1})$.)

c) Show that given any scheme X and $s \ 2 \ h(X)$ there exists a morphism $f : X \to \mathbb{P}^1 \ \ \mathbb{P}^1$ such that $f \ (s_{\mathbb{P}^1 \ \ \mathbb{P}^1}) = s$. Use this to finish the proof that $(pt; \ )$ satisfies condition a) from Definition 2.16.

d) Why does it not satisfy condition b)?

To summarize, we learned that the functor h embeds the category of schemes into the category of moduli functors. Functors in the image are called representable and have a ne moduli space. However, there are bigger classes of moduli functors, having only a coarse moduli space or even just a pair $(M,)$ satisfying part a) of De nition 2.16. We illustrate this in Figure 7.

Figure 7: The Yoneda embedding of the category of schemes to the category of moduli functors

Exercises

Exercise 2.21. As in Example 2.4, de ne a moduli functor h for "families of k-dimensional subspaces of $\mathbb{C}^n$" and show that h is representable by the Grassmannian $G_k(n)$.

References and further reading

A nice introduction to moduli spaces in general, with applications to moduli spaces of genus 0 curves, can be found in [KV07, Section 0.2]. A short introduction can also be found in [HM98, Chapter 1A].

See [Kle05] for further discussions of Picard functors and schemes.

3 Families of curves and their moduli

3.1 Smooth and nodal curves

Now we want to de ne moduli functors and spaces for algebraic curves. We start by recalling some basics for curves over $\mathrm{Spec}(\mathbb{C})$, which will correspond to the $\mathbb{C}$-points of our moduli space.

De nition 3.1. A (complex) curve is a one-dimensional variety[10] $C \to \mathrm{Spec}(\mathbb{C})$. In other words it is a reduced, separated scheme of nite type over $\mathrm{Spec}(\mathbb{C})$ such that all irreducible components have dimension 1.

In the following, we will mostly be concerned with projective curves. For these, the most important invariant is their genus, coming in two avours.

De nition 3.2. Let C be a complex projective curve.

a) If C is smooth, its geometric genus is de ned to be

$$p_g(C) = h^{1,0}(C) = \dim H^0(C; \Omega^1_C);$$

where Ω^1_C is the cotangent line bundle of C. For a singular curve C, its geometric genus is de ned to be the geometric genus of its normalization.

b) The arithmetic genus of C is de ned as

$$p_a(C) = 1 - \chi(C; \mathcal{O}_C) = 1 - \dim H^0(C; \mathcal{O}_C) + \dim H^1(C; \mathcal{O}_C):$$

Proposition 3.3. For a smooth, irreducible projective curve we have $p_g(C) = p_a(C)$.

Proof. We have $H^0(C; \mathcal{O}_C) = \mathbb{C}$ since C is irreducible projective and $H^1(C; \mathcal{O}_C) = H^0(C; \Omega^1_C)^\vee$ by Serre duality. This implies

$$p_a(C) = 1 - \chi(C; \mathcal{O}_C) = 1 + \dim H^0(C; \Omega^1_C)^\vee = p_g(C): \qquad \square$$

Exercise 3.4. Let $C \subset \mathbb{P}^2$ be a nodal cubic curve, e.g.

$$C = \{[X : Y : Z] \in \mathbb{P}^2 : ZY^2 + X^3 - ZX^2 = 0\}: \tag{25}$$

Show that $p_g(C) = 0$ and $p_a(C) = 1$.

Digression 3.5 (Riemann surfaces) There is a second approach for studying algebraic curves, going via Riemann surfaces. While this is not necessary to build the theory, it is often useful to have in mind.

To start, let C be a smooth, projective, irreducible curve. Consider its set $S = C(\mathbb{C})$ of complex points with the complex topology[11]. Then it turns out that S is a compact, connected complex manifold of complex dimension 1. Thus, seen as a real manifold, it is a compact, connected oriented real surface without boundary.

To give an idea why this is true:

[10] We take the convention that varieties are not necessarily irreducible.

[11] For any scheme X of nite type over $\mathbb{C}$, there is a natural complex topology (or even the structure of a complex analytic space) on its set $X(\mathbb{C})$ of $\mathbb{C}$-points. For an a ne scheme $X = \mathrm{Spec}\,\mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ it is given by the topology on the common zero set $V(f_1, \ldots, f_m) \subset \mathbb{C}^n$ induced from the complex topology on $\mathbb{C}^n$. The construction for general X goes via gluing these topological spaces for an a ne cover of X. See [Har77, Appendix B.1] or [Vak17, Exercise 5.3.G] for more details.

Figure 8: The complex points a curve C of genus g give a compact, connected oriented real surface S without boundary with " g holes"

S is compact since C was assumed to be projective.

The curve C is 1-dimensional over $\mathbb{C}$ and hence $S = C(\mathbb{C})$ is 2-dimensional over $\mathbb{R}$ ($\dim_{\mathbb{R}} C = 2$).

S is a smooth manifold since C is smooth as an algebraic variety.

S is oriented since in fact it is a complex manifold.

There is a complete classi cation of compact, connected oriented real surfaces[12] without boundary. They are all of the form "a donut with g holes" as in Figure 8 and the number $g \quad 0$ is called the *topological genus* of the surface. For $S = C(\mathbb{C})$ it turns out that the topological genus of S is equal to the geometric (or arithmetic) genus of the algebraic curve C.

As mentioned in the introduction, it will turn out that moduli spaces of smooth curves are not compact. To nd a larger moduli space which will be compact, we allow curves to have nodal singularities.

De nition 3.6. Let C be a complex curve.

a) A closed point $q \ 2 \ C$ is a *node*[13] if it satis es one of the following two equivalent conditions:

   There exists a neighbourhood of $q \ 2 \ C(\mathbb{C})$ which is complex-analytically isomorphic to a neighbourhood of the origin in the locus $\{(x; y) : x \ y = 0\} \quad \mathbb{C}^2$.

   The completion $\hat{\mathcal{O}}_{C;q}$ of the local ring of C at q is isomorphic to $\mathbb{C}[[x; y]]=(x \ y)$.

b) The curve C is called *nodal* if every closed point $q \ 2 \ C$ is either a smooth point or a node.

Exercise 3.7. Show that the "nodal cubic curve" (25) from Exercise 3.4 is indeed a nodal curve.

Let C be a complex, projective and nodal curve. Then its normalization $: \tilde{C} \ ! \ C$ is a complex, projective curve which is smooth (but possibly disconnected). The morphism

[12] In this context, surface means a 2-dimensional topological manifold.
[13] Alternative names are *nodal singularity* or *ordinary double point*.

is an isomorphism over the smooth locus of C and every node $q \in C$ has exactly two preimages $q', q'' \in \tilde{C}$. We have the normalization exact sequence[14]

$$0 \to \mathcal{O}_C \to \mathcal{O}_{\tilde{C}} \to \bigoplus_{q \text{ node of } C} \mathcal{O}_q \to 0, \tag{26}$$

where the first map sends a (local) function f on C to f on $\tilde{C}$ and the second map sends a (local) function g on $\tilde{C}$ to $(g(q') - g(q''))_q$ where q runs through nodes of C and q', q'' are the preimages[15] of q under .

Easy exercise 3.8. Use the sequence (26) to show that for a complex, projective, nodal curve C with normalization $\tilde{C} \to C$ we have

$$p_a(C) = p_a(\tilde{C}) + \#\{\text{nodes of } C\}, \tag{27}$$

or equivalently

$$p_a(C) = p_g(\tilde{C}) + 1 - \#\{\text{components of } \tilde{C}\} + \#\{\text{nodes of } C\}. \tag{28}$$

Figure 9: A nodal curve C and its normalization $\tilde{C}$, with $p_g(C) = p_g(\tilde{C}) = 5$ and $p_a(C) = 8$

Fact 3.9. The data of a complex, projective, nodal curve C together with the tuple $(q_i)_{i=1}^{\ell}$ of its nodes, is equivalent to the data of its normalization $\tilde{C}$ together with the sets $(\{q_i', q_i''\})_{i=1}^{\ell}$. In other words there is a unique way to "glue" the components of $\tilde{C}$ together by identifying the pairs q_i', q_i'' to form nodes.

$$C, (q_i)_{i=1}^{\ell} \underset{\text{gluing}}{\overset{\text{normalization}}{\rightleftarrows}} \tilde{C}, (\{q_i', q_i''\})_{i=1}^{\ell} \tag{29}$$

In particular, the data of a morphism $\varphi : C \to X$ to some scheme X is equivalent to the data of $\tilde{\varphi} : \tilde{C} \to X$ such that $\tilde{\varphi}(q_i') = \tilde{\varphi}(q_i'')$ for all i.

$$\varphi : C \to X \underset{\text{gluing}}{\overset{\text{normalization}}{\rightleftarrows}} \tilde{\varphi} : \tilde{C} \to X \text{ s.t. } \tilde{\varphi}(q_i') = \tilde{\varphi}(q_i'') \ \forall i \tag{30}$$

[14] To show exactness of this sequence, one uses that a) it suffices to check it stalkwise, i.e. after tensoring with $\mathcal{O}_{C,p}$ for $p \in C$ and b) that the map $\mathcal{O}_{C,p} \to \hat{\mathcal{O}}_{C,p}$ is faithfully flat. This last point is how the definition of a node comes into play.

[15] For this we need to choose an order on the preimages, alternatively we need to replace $\mathcal{O}_q$ by $(\mathcal{O}_{q'} \times \mathcal{O}_{q''})/\mathcal{O}_q$.

In the next subsection, we will de ne families and then moduli spaces of nodal curves. It turns out (see Remark 3.16) that in a family of nodal curves, the arithmetic genus of the curves stays constant, while the geometric genus can change. For instance, in the family E_t of smooth plane cubic curves from Example 1.2, the general E_t was smooth of (arithmetic and geometric) genus 1, and the nodal cubic E_0 still has arithmetic genus 1 but geometric genus 0 (by Exercise 3.4).

So we see that smooth curves of genus g degenerate to nodal curves of arithmetic genus g. However, it turns out that there are too many nodal curves of arithmetic genus g to obtain a compact moduli space! More precisely, consider the following sequence of nodal curves of genus 2.

Figure 10: An in nite sequence of nodal curves of arithmetic genus 2

If we had a compact moduli space of such curves, this would be a sequence of closed points in this space. After taking a subsequence, this would need to converge (in the complex topology) to some point of the moduli space, but intuitively the above sequence of curves cannot possibly converge to a (nite type) curve! So allowing all nodal curves is too much, but it turns out that things work out very well if we restrict ourself to so-called *stable* curves. Here is the de nition, including a variant for curves together with (marked) points.

De nition 3.10. A connected, nodal, complex projective curve C is called *stable* if the group

$$\mathrm{Aut}(C) = \{\varphi : C \to C : \varphi \text{ isomorphism}\} \tag{31}$$

of its automorphisms is nite. Moreover, given $p_1, \ldots, p_n \in C$ distinct smooth points of C, we say that $(C; p_1, \ldots, p_n)$ is stable if the group

$$\mathrm{Aut}(C; p_1, \ldots, p_n) = \{\varphi : C \to C : \varphi \text{ isomorphism with } \varphi(p_i) = p_i\} \tag{32}$$

of automorphisms of C xing all p_i is nite.

This de nition makes some sense: we have seen before that automorphisms create trouble when trying to nd moduli spaces, so in nite automorphisms can create in nite trouble! More seriously, let's see that this de nition prevents the counterexample from Figure 10. For this we need some facts about automorphisms of curves, starting with the case of smooth curves.

Fact 3.11. Let C be a smooth, complex, irreducible projective curve of genus g.

a) For $g = 0$ we have $C = \mathbb{P}^1$ and $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C})$, where

$$\mathrm{PGL}_2 = \begin{matrix} a & b \\ c & d \end{matrix} \in \mathbb{P}(\mathrm{Mat}_{2\times 2;\mathbb{C}}) = \mathbb{P}^3 : ad - bc \neq 0$$

is the projective linear group. The action of $PGL_2(\mathbb{C})$ on $\mathbb{P}^1$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : [X : Y] = [aX + bY : cX + dY]:$$

Moreover, this action is 3-transitive, i.e. for $p_1, p_2, p_3 \in \mathbb{P}^1$ pairwise distinct closed points there exists a unique element of $PGL_2(\mathbb{C})$ sending them to $0, 1, \infty \in \mathbb{P}^1$, respectively. Or, formulated differently, the morphism

$$\begin{aligned} PGL_2 &\to (\mathbb{P}^1)^3 \setminus \Delta; \\ A &\mapsto (A:[0:1], A:[1:1], A:[1:0]) \end{aligned}$$

to the complement of the big diagonal $\Delta \subset (\mathbb{P}^1)^3$ is an isomorphism.

b) For $g = 1$ the curve $C = E$ has automorphism group is isomorphic to $\mathrm{Aut}(E) = E(\mathbb{C}) \rtimes G$ for G one of the finite groups $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/6\mathbb{Z}$. The normal subgroup $E(\mathbb{C}) \subset \mathrm{Aut}(E)$ acts simply transitively, i.e. for $p, p' \in E$ closed points there exists a unique element of $E(\mathbb{C})$ sending p to p'.

c) For $g \geq 2$ the automorphism group $\mathrm{Aut}(C)$ is finite, of order at most $84(g-1)$.

Easy exercise 3.12. Let C be a smooth, complex, irreducible projective curve of genus g and $p_1, \ldots, p_n \in C$ be distinct points. Show that $\mathrm{Aut}(C; p_1, \ldots, p_n)$ is finite if and only if $2g - 2 + n > 0$.

From this we can get a very explicit criterion for a nodal and pointed curve to be stable.

Proposition 3.13. Let C be a connected, nodal, complex projective curve and let $p_1, \ldots, p_n \in C$ be distinct smooth points. Then $(C; p_1, \ldots, p_n)$ is stable if and only if every irreducible component $\tilde{C}_v \subset \tilde{C}$ of the normalization of C satisfies

$\tilde{C}_v$ has genus 0 and contains at least 3 special points, i.e. preimages of nodes of C or markings p_i, or

$\tilde{C}_v$ has genus 1 and contains at least 1 special point, or

$\tilde{C}_v$ has genus at least 2.

Proof. By Fact 3.9 together with the universal property of the normalization, an automorphism $\varphi : C \to C$ of C is equivalent to an automorphism $\tilde{\varphi} : \tilde{C} \to \tilde{C}$ mapping each pair $q_j', q_j'' \in \tilde{C}$ of preimages of nodes to some other such pair. In particular, we get a group morphism

$$\mathrm{Aut}(C; p_1, \ldots, p_n) \to \mathrm{Sym}(\{\text{components } \tilde{C}_v \text{ of } \tilde{C}\}) \times \mathrm{Sym}(\{q_j', q_j'' : j = 1, \ldots, \ell\}) \quad (33)$$

sending an automorphism to the permutation on the set of components of $\tilde{C}$ and the set of preimages of nodes. Since this permutation group is finite, the group $\mathrm{Aut}(C; p_1, \ldots, p_n)$ is finite if and only if the kernel K of the above map is finite.

But an element of the kernel is precisely a collection of automorphisms $\tilde{\varphi}_v : \tilde{C}_v \to \tilde{C}_v$ of the components of $\tilde{C}$ which fix all special points (i.e. the points q_j', q_j'' and the points p_i). Comparing with Fact 3.11 (or with Easy Exercise 3.12), we see that the group of automorphisms of $\tilde{C}_v$ fixing a number m of distinct points of $\tilde{C}_v$ is finite if and only if $\tilde{C}_v$ is of genus 0 with $m \geq 3$, of genus 1 with $m \geq 1$ or of genus at least 2. □

Example 3.14. For the nodal curve in Figure 11, we see that its normalization has three components of genus $0, 1, 2$ respectively. They have $2, 1$ and 3 preimages of nodes, respectively, and the genus 0 component has also a preimage of the marked point $p_1 \in C$. Checking with the criterion above, we see that C itself is not stable (the genus 0 component only has 2 special points), but (C, p_1) is stable (together with the preimage of p_1 it now has 3 special points).

Figure 11: The nodal curve C and its normalization $\tilde{C}$ together with the special points on $\tilde{C}$

Looking back at Figure 10 we see that only the first curve in the sequence is stable: all others have components of genus 0 with only one special point on them. Thus this sequence will not give a counterexample to the compactness of the moduli space of stable curves!

3.2 Families and moduli spaces of smooth and stable curves

Now we define the notion of a family of curves. As we will see later, it is very natural to not just consider the the curve C itself, but the additional data of $p_1, \ldots, p_n \in C$ of n distinct points. Moreover, we will treat families of smooth and stable curves simultaneously.

Definition 3.15. Given $g, n \geq 0$, an n-pointed family of (smooth/stable) genus g curves over a scheme S is a tuple

$$(: C \to S; p_1, \ldots, p_n : S \to C) \tag{34}$$

where

- is a (smooth/flat), proper, surjective, finitely presented morphism of schemes such that the fibre C_s over any geometric point $s \in S$ is a (smooth/stable), projective, connected curve of arithmetic genus g,
- the morphisms $p_1, \ldots, p_n$ are pairwise disjoint sections of , with image in the smooth locus of .

We say that a second family $(C^0 \to S; p_1^0, \ldots, p_n^0)$ is isomorphic to $(C \to S; p_1, \ldots, p_n)$ if there exists an isomorphism $' : C \to C^0$ over S such that $' \circ p_i = p_i^0$, i.e. such that the

Figure 12: A family of stable curves

diagram

$$\begin{tikzcd} C \arrow[rr, "'"] \arrow[dr] & & C^0 \arrow[dl, "0"'] \\ & S \arrow[ul, bend left, "p_i"] \arrow[ur, bend right, "p_i^0"'] & \end{tikzcd}$$

commutes. Given a map $f : T \to S$, we define the pullback of $(C/S; p_1; \dots; p_n)$ under f to be the family $(C_T/T; p_{1;T}; \dots; p_{n;T})$ for the fibre product $C_T = C \times_S T$ with induced sections $p_{i;T} = (p_i \circ f) \times \mathrm{id}_T$.

$$\begin{tikzcd} C_T \arrow[r] \arrow[d] & C \arrow[d] \\ T \arrow[r, "f"] \arrow[u, bend left, "p_{i;T}"] & S \arrow[u, bend right, "p_i"'] \end{tikzcd}$$

Remark 3.16. Most of the above definition should be very reasonable, but I want to comment on two small points.

a) Over reasonable bases (i.e. locally Noetherian), the assumption of f being finitely presented already follows from the map being proper. But we need it for technical reasons (see point c) below).

b) Given that we ask all fibres C_s of the morphism to be projective, it is tempting to just require being projective instead. However, this would lead to some technical difficulties (e.g. being projective cannot be checked on a Zariski open cover of S). However, it turns out to be true that for a stable family of curves, the morphism is locally projective, i.e. there exists a Zariski open cover of S such that is projective restricted to the open sets in this cover[16].

[16] Idea of proof: we will later see that there is a so-called canonical line bundle of and that the family being stable guarantees that $(\omega_{C/S} \otimes \sum_i p_i)$ is ample on the fibres C_s. By EGA IV$_3$, 9.6.4 this implies that this line bundle is -relatively ample. Then this line bundle makes projective locally on S.

c) For the family of stable curves, we require the map to be at. In the case of smooth curves we did not have to add this extra assumption since every smooth morphism is automatically at. One nice consequence is that for a at, proper, locally nitely presented morphism the Euler characteristic of the bres is locally constant ([Vak17, ,Exercise 28.2.M]). This means that even if we did not ask all bres to have arithmetic genus g, we would at least know that their arithmetic genus is constant on connected components of the base.

De nition 3.17. Let $M_{g;n}$ and $\overline{M}_{g;n}$ be the moduli functors sending a scheme $S \in Sch_C$ to the sets

$$M_{g;n}(S) = \{(\ : C \to S; p_1; \dots; p_n : S \to C) : \text{ smooth curve over } S\}/\text{iso};$$
$$\overline{M}_{g;n}(S) = \{(\ : C \to S; p_1; \dots; p_n : S \to C) : \text{ stable curve over } S\}/\text{iso}:$$

of n-pointed families of smooth (or stable) genus g curves over S, up to isomorphism. Given a morphism $f : T \to S$, the induced maps

$$M_{g;n}(S) \to M_{g;n}(T); \quad \overline{M}_{g;n}(S) \to \overline{M}_{g;n}(T)$$

are de ned by the pullback of the families of curves over S to T.

Easy exercise 3.18. Convince yourself that this de nes a moduli functor. In particular, check that the pullback of a family of (smooth/stable) curves is again (smooth/stable).

Theorem 3.19 (see [DM69, Knu83b]). Let $g; n \geq 0$ with $2g - 2 + n > 0$.

a) There exist coarse moduli spaces $M_{g;n}$ and $\overline{M}_{g;n}$ of $M_{g;n}$ and $\overline{M}_{g;n}$.

b) They are normal algebraic varieties of dimension $3g - 3 + n$ and there is a natural inclusion $M_{g;n} \hookrightarrow \overline{M}_{g;n}$ as a nonempty, open and dense subvariety.

c) The variety $\overline{M}_{g;n}$ is irreducible, projective and has quotient singularities[17].

d) The complement $\partial\overline{M}_{g;n} = \overline{M}_{g;n} \setminus M_{g;n}$ of the locus of smooth curves, called the boundary of $\overline{M}_{g;n}$, is a (Weil) divisor[18].

e) The locus $\overline{M}^0_{g;n} \hookrightarrow \overline{M}_{g;n}$ of $[(C; p_1; \dots; p_n)]$ with trivial automorphism group

$$\mathrm{Aut}(C; p_1; \dots; p_n) = \{id_C\}$$

is an open and smooth subvariety. It is a ne moduli space for the moduli functor $\overline{M}^0_{g;n}$ of stable curves with trivial automorphism group and thus has a universal family

$$\begin{tikzcd} \overline{C}^0_{g;n} \arrow[d] \arrow[d, bend left, leftarrow, "p_i"] & \\ \overline{M}^0_{g;n} \arrow[r, hook] & \overline{M}_{g;n} \end{tikzcd} \tag{35}$$

[17] That means it has a Zariski open cover by varieties V/G for V smooth a ne and G a nite group acting on V. Here, if V is the spectrum of a ring R, the action of G on V induces an action of G on R and the variety V/G is the spectrum of the ring R^G of G-invariant elements in R.

[18] In general, the divisor $\overline{M}_{g;n} \setminus M_{g;n}$ is not Cartier (see here). However, since $\overline{M}_{g;n}$ has quotient singularities it is $\mathbb{Q}$-factorial so a multiple of the divisor is Cartier.

Figure 13: The moduli space of stable curves $\overline{M}_{g;n}$ with the open set $M_{g;n}$ of smooth curves and the boundary $\partial\overline{M}_{g;n}$ illustrated

Remark 3.20. Using Easy exercise 3.12 we see that the reason for the requirement $2g - 2 + n > 0$ of the theorem is precisely that it ensures that any smooth curve $(C; p_1; \ldots; p_n)$ has finite automorphism group (so that indeed $\emptyset \neq M_{g;n} \subset \overline{M}_{g;n}$). In the end, it is just a concise way to exclude the finite list of cases

$$(g; n) = (0; 0); (0; 1); (0; 2); (1; 0):$$

Concerning part e) of the Theorem, one can check that for $g \geq 2$ $2 + n > 0$ the set $\overline{M}^0_{g;n}$ of curves with trivial automorphism group is nonempty if and only if $(g; n) \notin (1; 1); (2; 0)$.

In order to help us digest this big theorem, we will start by looking at some concrete examples of moduli spaces $\overline{M}_{g;n}$ in low genus in the next section. Along the way we will also introduce some general concepts, like the dual graph of a stable curve. Later we will discuss the main ideas of how to prove the various properties of $\overline{M}_{g;n}$ stated above.

References and further reading

A great introduction to (smooth) algebraic curves with an overview of the behaviour for small genus is given in [Vak17, Chapter 19]. See also [Vak17, Exercise 16.4.C] for the automorphism group of $\mathbb{P}^1$ and [Vak17, Section 21.7.8] for the finiteness of automorphisms for $g \geq 2$ from Fact 3.11. See this great poster about automorphism groups of smooth, complete curves over more general algebraically closed fields.

For an introduction to nodal curves see [ACG11, Chapter X, §2]. For some more technical results about (nodal) curves, see [Sta13, Tag 0BRV], in particular subsections 14, 15 and 19.

4 Examples of moduli of curves and basic constructions

4.1 Smooth curves of genus 0

Let's start with the moduli spaces $\mathcal{M}_{0;n}$ of smooth curves, which is defined for

$$2g - 2 + n = -2 + n > 0 \iff n \geq 3.$$

From Fact 3.11 a) we know that every smooth genus 0 curve C is isomorphic to $\mathbb{P}^1$. Thus, beginning with the simplest case $n = 3$, every curve $(C; p_1; p_2; p_3) \in \mathcal{M}_{0;3}(\mathrm{Spec}(\mathbb{C}))$ is isomorphic to $(\mathbb{P}^1; p_1; p_2; p_3) \in \mathcal{M}_{0;3}(\mathrm{Spec}(\mathbb{C}))$. Again by Fact 3.11 a) there exists an automorphism of $\mathbb{P}^1$, an element of $PGL_2(\mathbb{C})$, sending $p_1; p_2; p_3$ to $0; 1; \infty$. Thus we have

$$(C; p_1; p_2; p_3) \cong (\mathbb{P}^1; p_1; p_2; p_3) \cong (\mathbb{P}^1; 0; 1; \infty).$$

Thus, up to isomorphism, there exists a unique smooth genus 0 curve with three distinct marked points! Therefore, we expect that the moduli space $\mathcal{M}_{0;3}$ is a point. This turns out to be true, but the proof (without using Theorem 3.19) is actually quite involved!

Proposition 4.1. The variety $M_{0;3} = pt = \mathrm{Spec}(\mathbb{C})$ is a fine moduli space for the functor $\mathcal{M}_{0;3}$ and the universal family is given by

$$(\pi : \mathbb{P}^1 \times \mathrm{Spec}(\mathbb{C}); p_1 = 0; p_2 = 1; p_3 = \infty : \mathrm{Spec}(\mathbb{C}) \to \mathbb{P}^1) \tag{36}$$

For the proof we are going to use the following result. Proving it is quite technical, so we only give the proof in the appendix in the optional Section 4.6.

Proposition 4.2. Let $\pi : C \to B$ be a smooth, proper, surjective, locally finitely presented morphism of relative dimension 1 with geometric fibres isomorphic to $\mathbb{P}^1$.

a) If π admits a section $p_1 : B \to C$, then there exists a rank 2 vector bundle E on B such that C is isomorphic to the projective bundle $C = \mathbb{P}(E)$ over B.

b) If π admits two disjoint sections $p_1; p_2 : B \to C$, then the bundle E splits as a direct sum $E = L_1 \oplus L_2$ of line bundles.

c) If π admits three disjoint sections $p_1; p_2; p_3 : B \to C$, then we can take $E = \mathcal{O}_B \oplus \mathcal{O}_B$ above, so that C is isomorphic to the trivial projective bundle $C = B \times \mathbb{P}^1$.

Proof of Proposition 4.1. By definition, we need to show that the functor $\mathcal{M}_{0;3}$ is isomorphic to h^{pt}. Since for any scheme S the set $h^{pt}(S) = \{S \to pt\}$ has a unique element, this means we need to show that any family of smooth genus 0 curves

$$(\pi : C \to S; p_1; p_2; p_3 : S \to C) \in \mathcal{M}_{0;3}(S) \tag{37}$$

is isomorphic to the trivial family

$$(S \times \mathbb{P}^1 \to S; 0; 1; \infty : S \to S \times \mathbb{P}^1); \tag{38}$$

so that really there exists a unique element of $\mathcal{M}_{0;3}(S)$. By Proposition 4.2 we see that indeed $C \cong S \times \mathbb{P}^1$ over S. Then since the three sections $p_1; p_2; p_3 : S \to S \times \mathbb{P}^1$ are assumed to be disjoint, they induce a map

$$A = (p_1; p_2; p_3) : S \to (\mathbb{P}^1)^3 \setminus \Delta = PGL_2 \tag{39}$$

to the complement of the big diagonal $(P^1)^3$, which by Fact 3.11 a) is isomorphic to PGL_2. Then an isomorphism $S \times P^1 \to S \times P^1$ sending the sections p_1, p_2, p_3 to $0, 1, 1$ is given by

$$S \times P^1 \to S \times P^1, (s, p) \mapsto (s, A(s)^{-1} \cdot p).$$

This gives the desired isomorphism of the families (37) and (38). □

It turns out that once we understand the case $n = 3$, the case of general $n \geq 4$ is actually not much more difficult. Indeed, given any curve $(C, p_1, \ldots, p_n) \in M_{0,n}(C)$ we still have $C = P^1$ and there exists a unique element $B \in PGL_2(C)$ of the automorphism group of P^1 sending p_1, p_2, p_3 to $0, 1, 1$. Let $p_j^0 = B \cdot p_j$, then we have

$$(C, p_1, \ldots, p_n) = (P^1, 0, 1, 1, p_4^0, \ldots, p_n^0).$$

The elements $p_4^0, \ldots, p_n^0 \in P^1$ are pairwise distinct and also distinct from $0, 1, 1$ and uniquely determine the isomorphism class of $(C, p_1, \ldots, p_n)$[19]. This gives the following result.

Proposition 4.3. For $n \geq 3$ the moduli functor $M_{0,n}$ is representable by the variety

$$M_{0,n} = (P^1 \setminus \{0, 1, 1\})^{n-3} \setminus \Delta, \tag{40}$$

where $\Delta = \{(q_i)_i : \exists i \neq j \text{ with } q_i = q_j\}$ is the big diagonal.

Exercise 4.4. Give a proof of Proposition 4.3, following the proof of Proposition 4.1. What is the universal family over $M_{0,n}$?

From the concrete description above we can actually now verify several of the statements from Theorem 3.19:

- The space $M_{0,n}$ exists as a coarse (even fine!) moduli space.
- It is normal (even smooth!) and irreducible of dimension $3 - 3 + n = n - 3$.
- In fact one easily checks that every smooth genus 0 curve $(C, p_1, \ldots, p_n)$ has trivial automorphism group, so that $M_{0,n} = \overline{M}^0_{0,n}$. This explains the fact that it is a fine moduli space and smooth instead of just normal!

Next we want to look at moduli spaces of stable curves in genus 0, to see if the nice properties above (being smooth and a fine moduli space) extend. For this, we will now define a very useful tool (working for any genus g), the dual/stable graph of a stable curve.

4.2 Stable graphs and gluing morphisms

Given a stable curve $(C, p_1, \ldots, p_n)$, we want to define a combinatorial object Γ_C that allows us to describe its shape, i.e. how many components it has, of which genus they are and how they intersect among themselves. Consider Figure 14 for illustration.

We see there that the combinatorial object is a graph together with some decorations. In the graph

- the vertices v correspond to irreducible components C_v of C and are decorated with the geometric genus of the component,

[19] For $n = 4$, the value p_4^0 is called the cross ratio of the points $p_1, \ldots, p_4$.

Figure 14: A stable curve and the associated dual graph

the edges correspond to nodes of the curve, where a node connecting two components $C_v; C_w$ (or C_v with itself) gives an edge between v and w (or from v to itself),

there are legs attached to the vertices v, numbered $1; \ldots; n$, describing in which components C_v the markings $p_1; \ldots; p_n \in C$ are contained.

Now we set up the notation how to formally encode all this data. This involves a bunch of sets and maps satisfying properties and as you go through the de nition you should check back to Figure 14 to match the parts of the de nition to the picture.

De nition 4.5. A stable graph is a tuple

$$= (V; H; L; g : V \to Z_0; v : H \to V; : H \to H; ` : L \to f 1; \ldots; ng) \qquad (41)$$

where

a) $V = V( )$ is a nite set (the vertices of) and $g : V \to Z_0$ is a map associating a genus $g(v)$ to each vertex v,

b) $H = H( )$ is a nite set (the half-edges of). The map $v : H \to V$ associates to each half-edge h a vertex $v(h)$ (the vertex incident to h). We denote by

$$H(v) = f h \in H : v(h) = vg$$

the half-edges incident at v and by $n(v) = \# H(v)$ the number of these half-edges. The map $: H \to H$ is an involution (i.e. $= id_H$). Thus H decomposes into pairs of half-edges switched by and xed points of .

c) The pairs $e = f h; h^0 g$ of distinct half-edges exchanged by, i.e. $(h) = h^0$, are called the edges $E = E( )$ of .

d) The set $L = L( ) \quad H$ is the set of half-edges xed by (the legs of) and $` : L \to f 1; \ldots; ng$ is a bijective map.

e) The graph is connected, i.e. any two vertices can be connected by a path consisting of edges[20].

f) We require the stability condition that for each vertex $v \in V$ we have

$$2g(v) \quad 2 + n(v) > 0:$$

An isomorphism $: \quad ! \quad {}^0$ of stable graphs is a collection of bijective maps

$$'_V : V ! \quad V^0; '_H : H ! \quad H^0;$$

of their sets of vertices and half-edges which are compatible with the functions $g, v, \ ,$ i.e.

$$g^0('_V(v)) = g(v); v^0('_H(h)) = '_V(v(h)); \ ^0(\ _H(h)) = \ _H(\ (h)); `^0('_H(h)) = `(h):$$

Denote by $\mathrm{Aut}(\ )$ the set of isomorphisms $\quad ! \quad$. This is a group, with group law given by composition. De ne the genus $g(\ )$ of as the number

$$g(\ ) = \sum_{v \in V(\ )} g(v) + 1 + \# \ E(\ ) \quad \# V(\ ) \ ; \tag{42}$$

and the number of legs/markings $n(\ )$ as $n(\ ) = \ n = \# \ L(\ )$.

Example 4.6. Let's work out all these things in an example. Consider the stable graph in Figure 15. Its data is given by

$$\begin{aligned}
V(\ ) &= \{v_0; v_1; v_2\} \text{ and } g(v_0) = 1 ; g(v_1) = 1 ; g(v_2) = 2 ; \\
H(\ ) &= \{h_1; : : : ; h_7\} \text{ and } v : H(\ ) \ ! \ V(\ ) \text{ with} \\
&\quad v^{\ 1}(v_0) = \{h_1\}; v^{\ 1}(v_1) = \{h_3\}; v^{\ 1}(v_2) = \{h_2; h_4; h_5; h_6; h_7\}; \\
E(\ ) &= \{\{h_1; h_2\}; \{h_3; h_4\}; \{h_6; h_7\}\}; \text{ in particular} \\
&\quad (h_1) = h_2; \ (h_2) = h_1; : : : ; \ (h_7) = h_6; \ (h_5) = h_5; \\
L(\ ) &= \{h_5\} \text{ and } `(h_5) = 1 :
\end{aligned}$$

Concerning the automorphism group of , there is an automorphism $= (\ _V; \ _H) : \quad !$ with

$$_V(v_0) = v_1; \ _V(v_1) = v_0; \ _V(v_2) = v_2; \ _H(h_1) = h_3; \ _H(h_3) = h_1; \ _H(h_2) = h_4; \ _H(h_4) = h_2:$$

Similarly, there is an automorphism , where $_V; \ _H$ x all vertices and half-edges except for $_H(h_6) = h_7; \ _H(h_7) = h_6$. You can check that these two commute and generate the automorphism group

$$\mathrm{Aut}(\ ) = \ \langle \ ; \ \rangle = \mathbb{Z}/2\mathbb{Z} \quad \mathbb{Z}/2\mathbb{Z}:$$

Finally we see that

$$g(\ ) = (1 + 1 + 2) + 1 + 3 \quad 3 = 5 \text{ and } n(\ ) = 1 \ :$$

Exercise 4.7. De ne the stable graph associated to a stable curve $C(p_1; : : : ; p_n)$. Check your de nition against the picture in Figure 14 and convince yourself that conditions e) and f) of De nition 4.5 are satis ed. We give the de nition below for completeness, but it will be less confusing if you rst try writing it down yourself.

[20] Mini-Exercise: make precise what this means.

Figure 15: A stable graph with labeled vertices and half-edges (in red) and automorphisms ; generating its automorphism group Aut( ) = h ; i = Z=2Z Z=2Z

De nition 4.8. Given a stable curve $(C; p_1; : : : ; p_n)$ its associated dual graph = $_C$ is the stable graph de ned as follows:

The vertices v 2 V of are in one-to-one correspondence to the irreducible components C_v of C (which canonically correspond to the components $\tilde{C}_v$ of the normalization $\tilde{C}$).

$$V = f C_v : \text{component of } C g = f \tilde{C}_v : \text{component of } \tilde{C} g$$

The map $g : V \to Z_0$ sends a vertex v to the genus $g(\tilde{C}_v)$ of the component in the normalization.

The half-edges h 2 H of are in one-to-one correspondence to the union of the preimages $q^0; q^{00}$ 2 $\tilde{C}$ of nodes q 2 C under the normalization map : $\tilde{C} \to C$ and the marked points $p_1; : : : ; p_n$ 2 C.

$$H = \left(\bigsqcup_{\substack{q \text{ node in } C \\ {}^{-1}(q) = f q^0; q^{00} g}} f q^0; q^{00} g \right) \sqcup f p_1; : : : ; p_n g$$

The map $v : H \to V$ sends half-edges of the form $q^0; q^{00}$ to the vertex v for the component $\tilde{C}_v$ of the normalization containing them, and the half-edges of the form p_i to the vertex v for the component C_v of C containing them. The involution exchanges the preimages of nodes $(q^0) = q^{00}$, $(q^{00}) = q^0$) and xes the marked points $((p_i) = p_i)$.

The legs L H are precisely the marked points

$$L = f p_1; : : : ; p_n g$$

and the map $` : L \to f 1; : : : ; n g$ sends p_i to i.

Easy exercise 4.9. Check that the genus $g(_C)$ of the dual graph of a curve $(C; p_1; : : : ; p_n)$ equals the arithmetic genus of C.

Exercise 4.10. Let $(C; p_1; \dots; p_n)$ be a stable curve with dual graph Γ and let $(\tilde{C}_v; (q_h)_{h \in H(v)})_{v \in V(\Gamma)}$ be the components of the normalization of C with marked preimages of nodes and markings. Then there is an exact sequence of groups

$$0 \to \prod_{v \in V(\Gamma)} \mathrm{Aut}(\tilde{C}_v; (q_h)_{h \in H(v)}) \to \mathrm{Aut}(C; p_1; \dots; p_n) \to \mathrm{Aut}(\Gamma) \quad : \tag{43}$$

Can this sequence in general be extended on the right by 0, i.e. is the map $\mathrm{Aut}(\Gamma)$ surjective?

Exercise 4.11. a) Show that a stable graph of genus g with n legs has at most $3g - 3 + n$ edges. (Note: This follows from Proposition 4.14 below, and you might look at the proof for inspiration, but there is a purely combinatorial argument for this statement, just using the definition above).

b) Show that given $g; n$, there are only finitely many stable graphs of genus g with n legs, up to isomorphism.

c) For fixed genus g and number n of legs, convince yourself that there is precisely one stable graph with no edge at all (the trivial stable graph). Compute a formula, depending on $g; n$, for the number of isomorphism classes of stable graphs with exactly one edge.

Remark 4.12. It turns out that stable graphs are incredibly useful when studying problems related to stable curves and their moduli spaces, since they can describe important information about the curves in a purely combinatorial fashion.

However, one issue one encounters is that the number of isomorphism classes of stable graphs grows drastically with $g; n$. For instance, even for $g = 1; n = 5$ there are already 1576 isomorphism classes of stable graphs. How did I come up with this number? Have I locked myself in my basement for a week, scribbling pages upon pages of stable graphs? No, in fact a few years ago, I locked myself in my basement for 5 months and wrote a computer program to count the graphs for me[21]!

With the help of Jason van Zelm and Vincent Delecroix, this has by now grown into the software package admcycles [DSv20] for the open source mathematical software SageMath [S+20]. This package can perform intersection theory on the spaces $\overline{\mathcal{M}}_{g;n}$ and as part of this, it can enumerate stable graphs. It can be used online without installation, and you can click on this link to see some example computations, enumerating stable graphs up to isomorphism.

I will occasionally show some examples using this program, but this will be an entirely optional part of the course, and in particular I will not ask anything related to this in the exam.

*Exercise 4.13. Check your answer to Exercise 4.11 c) for small values of $g; n$ using the software described above. You can also verify that the graph in Figure 15 has precisely 4 automorphisms.

One reason why stable graphs are useful is that the moduli space $\overline{\mathcal{M}}_{g;n}$ decomposes as a disjoint union according to possible stable graphs.

[21] If I had just been interested in this particular number of stable graphs, I could also have used an existing computer program written by Aaron Pixton, but I wanted to perform some computations that Aaron's program could not do.

Proposition 4.14. Let $g; n \quad 0$ with $2g \quad 2 + n > 0$, then for any stable graph of genus g with n legs, the set

$$M \quad = f(C; p_1; : : : ; p_n) : \quad {}_C = \quad g \quad \overline{M}_{g;n}$$

of curves with stable graph isomorphic to is a nonempty, irreducible, locally closed subset of $\overline{M}_{g;n}$. In particular, the space $\overline{M}_{g;n}$ is the disjoint union

$$\overline{M}_{g;n} = \overset{a}{\quad} M \quad ;$$

where runs through isomorphism classes of stable graphs. We have

$$\dim M \quad = \sum_{v2V()} 3g(v) \quad 3 + n(v) = \dim \overline{M}_{g;n} \quad \# E() : \tag{44}$$

The sets M are called the *strata* of $\overline{M}_{g;n}$ and the decomposition is called the *strati - cation according to dual graph*

To prove the Proposition, let's recall from Fact 3.9 that a nodal curve C is uniquely determined by its normalization together with the data of the pairs of preimages of nodes under the normalization map. Thus the curves $(C; p_1; : : : ; p_n) \ 2 \ M$ can be uniquely described by specifying the components $\tilde{C}_v$ of their normalization ($v \ 2 \ V()$) together with the preimages of nodes and markings p_i under the normalization map $: \tilde{C} \ ! \ C$. This leads to the idea of *gluing morphisms*

Proposition 4.15. Let be a stable graph of genus g with n legs, then there exists a morphism

$$: \overline{M} \quad = \prod_{v2V()} \overline{M}_{g(v);n(v)} \ ! \ \overline{M}_{g;n} \tag{45}$$

sending a tuple $(C_v; (q_h)_{h2H(v)})_{v2V()}$ to the curve $(C; p_1; : : : ; p_n)$ obtained by gluing all pairs $q_h; q_{h^0}$ of points corresponding to pairs $\{h; h^0\}$ forming edges of and setting $p_i \ 2 \ C$ to be the image of the marking $q_{\ 1(i)}$ belonging to the half-edge $\ 1(i) \ 2 \ H()$. The morphism is *nite* and its image is the closure $\overline{M}$ of M .

Remark 4.16. The fact that the strata of $\overline{M}_{g;n}$ are parametrized under the maps by products of smaller-dimensional spaces $\overline{M}_{g(v);n(v)}$ is sometimes called the *recursive boundary structure* of $\overline{M}_{g;n}$. It is one of the most important features of the moduli space of stable curves and the proofs of many results about $\overline{M}_{g;n}$ use it in a very essential way.

Example 4.17. In Figure 16 we see how the gluing map for a particular stable graph works. It identi es the markings $q_h; q_{h^0}$ belonging to half-edges $\{h; h^0\}$ of forming an edge.

Idea of proof. For the domain of the map , one can check that $\overline{M}$ is a coarse moduli space for the moduli functor $\overline{M} \quad : Sch^{op} \ ! \ Sets$ de ned by

$$\overline{M} \quad (S) = \prod_{v2V()} \overline{M}_{g(v);n(v)}(S): \tag{46}$$

Then we can obtain the map above by constructing a natural transformation

$$b \ : \overline{M} \quad ! \quad \overline{M}_{g;n}$$

between moduli functors and then using the properties of coarse moduli spaces.

Figure 16: The gluing map associated to a stable graph

Given a scheme S, the natural transformation $\flat$ takes an element of $\overline{M}\ (S)$, i.e. a tuple

$$({}_v : C_v \to S; (q_h : S \to C_v)_{h \in H(\ )})_{v \in V(\ )}$$

of stable curves over S and glues them to a stable curve by identifying the sections $q_h; q_{h^0}$ corresponding to pairs $\{h; h^0\}$ forming edges of . The fact that this gluing can be performed in families requires an argument (see e.g. [ACG11, Chapter X, Section 7]). As an illustration, in the simplest case of having two curves $_1 : C_1 \to S$, $_2 : C_2 \to S$ glued along sections $q_1 : S \to C_1$, $q_2 : S \to C_2$, the glued family can be obtained as the union of the images of C_1, C_2 inside the fibre product $C_1 \times_S C_2$ under the maps

$$\begin{tikzcd}
C_1 \arrow[dr, "\mathrm{id}_{C_1} \times (q_2 \circ {}_1)"] & \\
& C_1 \times_S C_2 \\
C_2 \arrow[ur, "(q_1 \circ {}_2) \times \mathrm{id}_{C_2}"'] &
\end{tikzcd}$$

Once we construct the natural transformation $\flat$, we obtain the map of coarse moduli spaces by considering the diagram

$$\begin{tikzcd}
\overline{M} \arrow[r] \arrow[d, "\flat"] & h^{\overline{M}} \arrow[d, dashed] \\
\overline{M}_{g;n} \arrow[r] & h^{\overline{M}_{g;n}} :
\end{tikzcd} \tag{47}$$

Here the horizontal arrows come from the fact that $\overline{M}$ and $\overline{M}_{g;n}$ are coarse moduli spaces of the functors on the right. By definition of a coarse moduli space, the composition

$\overline{M}$! $h^{\overline{M}_{g;n}}$ of the morphisms on the left and bottom must factor through the morphism at the top, giving the map on the right as desired. From the de nition of b we see that does what we want on C-points of $\overline{M}$.

As for the properties of , note that it is proper since its domain is proper and its target is separated ([Vak17, Proposition 10.3.4]). Thus to show that it is nite, it su ces to show that it is quasi nite ([Vak17, Theorem 29.6.2]). Given $(C; p_1; : : : ; p_n)$ in the image of , what is the preimage? It is the set of tuples of curves $(C_v; (q_h)_{h2H(v)})_{v2V()}$ which can be glued together to form $(C; p_1; : : : ; p_n)$.

Figure 17: Di erent choices to obtain a stable curve C by gluing according to the graph , as $C = C_1 t C_2$ or $C = C_1^0 t C_2^0$

You can check that such a tuple can be speci ed by making two nite lists of choices: rst you specify the subset Q of the nodes of C which are obtained by gluing markings in the preimage. Given this, you get a (possibly disconnected) curve $\hat{C}$ by normalizing the nodes in Q[22]. Then you obtain the tuple $(C_v; (q_h)_{h2H(v)})_{v2V()}$ of curves by identifying the connected components C_v of $\hat{C}$ with the vertices $v 2 V( )$ and the preimages q_h of nodes and markings with the half-edges $h 2 H( )$. Both the choice of nodes Q and the identi cation of components and preimages are nite choices, so there are only nitely many possibilities[23].

To show that the image of is the closure of M , rst note that

$$M = (M); \text{ where } M = \prod_{v2V()} M_{g(v);n(v)} \quad \overline{M} :$$

Indeed, given a curve in M we can certainly obtain it by gluing a tuple of smooth curves C_v under the map (C_v are the components of the normalization of C). Conversely, any

[22] Normally, you cannot choose to "partially normalize" a variety at some points. If you want, you can obtain the normalization $\hat{C}$ by rst normalizing all of C and then gluing back together all nodes not in Q.

[23] Note: I am not saying that any choice of Q is allowed: if you choose the wrong set of nodes, you won't get the right number of connected components of $\hat{C}$ or they won't have the right genus or markings p_i. You can check that if the stable graph of C is isomorphic to , then Q must be the set of all nodes of C and the number of choices we have in the second step is exactly the size of the automorphism group Aut( ). Thus we expect that the degree of to its image is generically #Aut( ).

such gluing of smooth curves has stable graph . Since is proper, its image is closed, hence $\overline{M}$ $(\overline{M}\ )$. On the other hand, since the moduli spaces of curves are irreducible, so is the product $\overline{M}$ and thus the nonempty open subset M is dense in $\overline{M}$. Finally, this implies

$$(\overline{M}\ ) \quad \overline{(M\ )} = \overline{M}\ :$$

□

Exercise 4.18. a) Show (without using Proposition 4.14) that the variety $\overline{M}$ has dimension

$$\dim \overline{M}\ = \sum_{v \in V(\ )} 3g(v) \quad 3 + n(v) = \dim \overline{M}_{g;n} \quad \# E(\ )\ :$$

b) Show that the complement $\overline{M}_{g;n}$ of the locus $M_{g;n}$ $\overline{M}_{g;n}$ of smooth curves is given by the union of the sets $\overline{M}$ for a stable graph with exactly one edge

$$\overline{M}_{g;n} \cap M_{g;n} = \bigcup_{:\# E(\ )=1} \overline{M}\ :$$

*c) Show that for any stable graph , the set $\overline{M}$ is a union of strata $M^{\ 0}$. Give a purely combinatorial description of the $^{\ 0}$ which appear. (Hint: A particularly nice way to put the answer to the second part of the question starts with "Consider the category whose objects are stable graphs of genus g with n legs and whose morphisms $^{\ 0}!$ are given by $:\ :\ :$". We are going to see this appear later when discussing the intersection theory of $\overline{M}_{g;n}$.)

By Proposition 4.14, the $\overline{M}$ with $\# E(\ ) = 1$ have codimension 1 in $\overline{M}_{g;n}$ and they are called the *boundary divisors* of $\overline{M}_{g;n}$. The stable graphs with $\# E(\ ) = 1$ are of one of the two forms in Figure 18 (this is part of the solution of Exercise 4.11 c)).

Figure 18: List of stable graphs with precisely one edge; if $g_1 = 0$ we require $n_1 = \# N_1 \quad 2$ because of the stability condition and similarly for $g_2 = 0$ we ask $n_2 = \# N_2 \quad 2$

The corresponding gluing maps take the form

$$\overline{M}_{g_1;n_1+1} \quad \overline{M}_{g_2;n_2+1} \ ! \ \overline{M}_{g;n};$$
$$\overline{M}_{g\ \ 1;n+2} \ ! \ \overline{M}_{g;n};$$

and we denote by $_{g_1;N_1} = \quad _{g_2;N_2}$ and $\quad _0$ the images of the respective map. These are precisely the irreducible components of the boundary of $\overline{M}_{g;n}$.

Thus once we prove Proposition 4.14, the exercise above completes the proof of Theorem 3.19 d), stating that the boundary is a Weil divisor (where of course we assumed all the other parts of the Theorem).

Proof of Proposition 4.14. We saw above that $M \quad$ is the image of $M \quad = \prod_v M_{g(v);n(v)}$ under $\quad$. Since $M \quad$ is nonempty and irreducible, so is $M \quad$. Also, the closure $\overline{M} \quad$ is the image of $\overline{M} \quad$ and by a slight extension of the argument above, one can show

$$(\overline{M} \quad \cap M \quad) = \overline{M} \quad \cap M \quad :$$

Indeed, a tuple $(C_v;(q_h)_h)_v \ 2 \ \overline{M} \quad \cap M \quad$ satis es that one of the curves C_v has a node, and then the curve $((C_v;(q_h)_h)_v) \ 2 \ \overline{M} \quad$ has at least $\#E(\ ) + 1$ nodes, and thus cannot have dual graph isomorphic to $\quad$. Since $\overline{M} \quad \cap M \quad$ is closed and $\quad$ is proper, the set $\overline{M} \quad \cap M \quad$ is closed in $\overline{M} \quad$ and thus $M \quad$ is locally closed inside $\overline{M}_{g;n}$ since it is open in the closed set $\overline{M} \quad$. Finally, we have $\dim \overline{M} \quad = \dim \overline{M} \quad$ since $\quad$ is nite and thus the formula for $\dim \overline{M} \quad$ follows from Exercise 4.18. □

4.3 Stable curves of genus 0

General results

Before we start looking at examples, let us use the new tool of stable graphs to prove that automorphism groups of genus 0 stable curves are trivial.

Exercise 4.19. Let $\quad$ be a stable graph of genus 0.

a) Show that the undirected graph $\quad$ with vertex set $V(\ )$ and edges $\{v(h);v(h^0)\}$ for $\{h;h^0\} \ 2 \ E(\ )$ is a tree. (Hint: See e.g. here for the de nition of a tree)

b) Show that $\mathrm{Aut}(\ ) = \{id \quad\}$ is trivial.

c) Show that any stable curve $(C;p_1;\dots;p_n)$ of genus 0 has trivial automorphism group $\mathrm{Aut}(C;p_1;\dots;p_n) = \{id_C\}$. (Hint: Exercise 4.10)

Corollary 4.20. For $n \quad 3$, the space $\overline{M}_{0;n}$ is a ne moduli space for the functor $\overline{M}_{0;n}$ and a smooth, irreducible projective variety of dimension $n \quad 3$.

Proof. By Exercise 4.19 we have $\overline{M}^0_{0;n} = \overline{M}_{0;n}$, so the statement follows from Theorem 3.19, in particular part e). □

Example: $n = 3$

Let's turn to some examples. From Exercise 4.11 we know that a stable graph in genus 0 with n legs has at most $3g \quad 3 + n = n \quad 3$ edges. Thus for $n = 3$ the only stable graph is the trivial one, which shows $\overline{M}_{0;3} = M_{0;3} = pt$ is a point.

Example: $n = 4$

For $n = 4$, every nontrivial stable graph has exactly one edge and by Exercise 4.19 above it must have precisely two vertices v_1, v_2 (if it had only one, the edge would be a loop at this vertex, so the graph would not be a tree, if it had more than two vertices, the graph would not be connected). Each vertex is incident to one of the half-edges forming the single edge and by stability we must have $n(v_1), n(v_2) \quad 3$. Since we have precisely four legs to distribute (corresponding to the four marked points), every vertex must get exactly two of them. To make a long story short, here are the possible stable graphs with $(g, n) = (0, 4)$:

Figure 19: The stable graphs of genus 0 with 4 legs

By Proposition 4.14 we know that

$$\overline{M}_{0,4} = M^{0} \sqcup M^{1} \sqcup M^{2} \sqcup M^{3}, \tag{48}$$

where $M^{0} = M_{0,4} = \mathbb{A}^1 \setminus \{0, 1\}$ and the M^{i} $(i = 1, 2, 3)$ are irreducible, locally closed subset of $\overline{M}_{0,4}$ of dimension 0, i.e. points. Maybe by now you have (correctly) guessed that $\overline{M}_{0,4} = \mathbb{P}^1$. Indeed, this follows from Corollary 4.20 and (48) since $\mathbb{P}^1$ is the only smooth, irreducible, projective variety of dimension 1 which contains $M_{0,4} = \mathbb{A}^1 \setminus \{0, 1\}$ as an open subvariety[24]. In Figure 20 you see a picture showing which points in $\mathbb{P}^1$ correspond to which stable curves.

In fact, since $\overline{M}_{0,4}$ is a fine moduli space, we know that the individual curves we drew in Figure 20 actually fit together into a universal curve $\pi : \overline{C}_{0,4} \to \overline{M}_{0,4}$, the universal family of curves for the moduli functor $\overline{M}_{0,4}$. If you worked on Exercise 4.4, you know that the universal family over $M_{0,4}$ is the trivial family $M_{0,4} \times \mathbb{P}^1 \to M_{0,4}$. The correct way to fill in the missing fibres over $0, 1, 1 \times 2 \in \overline{M}_{0,4}$ is shown in Figure 21.

Thus the universal curve is given by the composition

$$\pi : \overline{C}_{0,4} = \mathrm{Bl}_{(0,0),(1,1),(1,\lambda)} \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 = \overline{M}_{0,4} \tag{49}$$

of the blow-up map of $\mathbb{P}^1 \times \mathbb{P}^1$ at three points and the projection from $\mathbb{P}^1 \times \mathbb{P}^1$ to the first factor. The sections $p_1, \ldots, p_4 : \overline{M}_{0,4} \to \overline{C}_{0,4}$ are the strict transforms of the four maps

$$\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1, q \mapsto (q, 0), (q, 1), (q, \lambda), (q, \infty). \tag{50}$$

[24] The varieties $\overline{M}_{0,4}$ and $\mathbb{P}^1$ are both smooth, irreducible projective curves, and such curves are isomorphic if and only if they are birational. And indeed, both contain $\mathbb{A}^1 \setminus \{0, 1\}$.

Figure 20: The moduli space $\overline{M}_{0;4} = P^1$ of stable curves

The singular fibres of over $0, 1; 1 \quad 2 \quad P^1$ are the unions of the strict transform of the fibre of the projection $P^1 \quad P^1 \to \quad P^1$ and the exceptional divisor of the blowup, which meet transversally and thus form a nodal curve.

Exercise 4.21. Show that indeed the singular fibres of over $0, 1; 1 \quad 2 \quad P^1$ are nodal curves according to Definition 3.6.

On the exceptional divisor over $(0, 0) \quad 2 \quad P^1 \quad P^1$, you can check that the strict transforms $p_1; p_4$ of $q \mapsto (q; 0)$ and $q \mapsto (q; q)$ go to distinct points, since the maps $q \mapsto (q; 0); (q; q)$ meet at $(0; 0)$ with distinct tangent vectors[25].

The general case $n \quad 5$

In the case $n = 5$ it turns out that

$$\overline{M}_{0;5} = \mathrm{Bl}_{(0;0);(1;1);(1 \quad ;1 \quad)} P^1 \quad P^1:$$

Sounds familiar? Indeed, we have $\overline{M}_{0;5} = \overline{C}_{0;4}$. This is the beginning of a more general story, which we summarize in the following theorem.

Theorem 4.22 ([Knu83a, Kee92]). Let $\overline{M}_{0;n}$ be the moduli space of stable curves of genus 0 with n marked points and let $: \overline{C}_{0;n} \to \overline{M}_{0;n}$ be its universal curve.

a) For $n \quad 3$ we have $\overline{M}_{0;n+1} = \overline{C}_{0;n}$ and under this identification, the map

$$: \overline{M}_{0;n+1} \to \overline{M}_{0;n}$$

is the so-called *forgetful morphism*[26] of the marking $n + 1$.

b) The universal curve $\overline{C}_{0;n} \to \overline{M}_{0;n}$ can be obtained from the projection

$$\overline{M}_{0;n} \quad P^1 \to \overline{M}_{0;n}$$

by an iterated blowup along smooth codimension 2 subvarieties (see [Kee92, Section 1] for a precise description).

[25] This uses the identification of the exceptional divisor with $P(T_{(0;0)} P^1 \quad P^1)$, see [Vak17, Section 22.3] for details.

[26] See the next section for the general definition of such forgetful morphisms.

Figure 21: The universal family of curves over the moduli space $\overline{M}_{0;4}$

Remark 4.23. a) Combining the two parts of the above result, we can construct $\overline{M}_{0;n+1}$ from $\overline{M}_{0;n}$ so starting with $\overline{M}_{0;3} = \mathrm{Spec}(\mathbb{C})$ we can find all moduli spaces of stable curves in genus 0 recursively. Alternatively, since we get a factor $\mathbb{P}^1$ every time n increases starting from $n = 3$, we can see that the above procedure will produce $\overline{M}_{0;n}$ as an iterated blowup of the variety $(\mathbb{P}^1)^{n-3}$.

b) If you completed Exercise 4.4, you saw already that the universal curve over $M_{0;n}$ is isomorphic to $M_{0;n+1} \times \mathbb{P}^1$. This is naturally contained as an open subvariety of the blowup $\overline{C}_{0;n}$ of $\overline{M}_{0;n} \times \mathbb{P}^1$ from above. The universal sections $p_1; \ldots; p_n : \overline{M}_{0;n} \to \overline{C}_{0;n}$ are then the unique extensions of the universal sections $M_{0;n} \to M_{0;n+1} \times \mathbb{P}^1$ from Exercise 4.4.

The next section will introduce the forgetful morphisms $: \overline{M}_{g;n+1} \to \overline{M}_{g;n}$ mentioned above for arbitrary $g; n$. We will see that they almost, but not quite, identify $\overline{M}_{g;n+1}$ as the universal curve over $\overline{M}_{g;n}$. Again, our old enemies the automorphisms will ruin everything.

4.4 The forgetful morphism and the universal curve

Since $\overline{M}_{g;n}$ parametrizes curves C together with marked points $p_1; \ldots; p_n$, we might expect that there exist morphisms $: \overline{M}_{g;n+1} \to \overline{M}_{g;n}$ sending $(C; p_1; \ldots; p_n; p_{n+1})$ to the curve $(C; p_1; \ldots; p_n)$ obtained by forgetting the last marking. Below we will see that this certainly works for smooth curves C, but that the condition that the resulting n-pointed curve is stable requires some slight adjustment in our de nition.

Easy exercise 4.24. For $2g - 2 + n > 0$, show that there exists a morphism $M_{g;n+1} \to M_{g;n}$ of the moduli spaces of smooth curves, which on complex points is given by

$$M_{g;n+1}(C) \to M_{g;n}(C); (C; p_1; \ldots; p_n; p_{n+1}) \mapsto (C; p_1; \ldots; p_n):$$

What goes wrong when we try to write down the same map for arbitrary stable curves $(C; p_1; \ldots; p_n; p_{n+1})$? Well, it can happen that $(C; p_1; \ldots; p_n)$ is no longer stable. Indeed, the component C_v of C containing the marked point p_{n+1} has one special point less than

before in $(C; p_1; \dots; p_n)$. Thus by Proposition 3.13 it can become unstable if $g(C_v) = 0$ and C_v had exactly 3 special points[27].

The left side of Figure 22 shows the various ways in which this situation can happen.

Figure 22: The map $'$ contracting the component C_v that becomes unstable by forgetting the marking p_{n+1} of C to a point $q \in C^0$; the map in the opposite direction (taking (C^0, q) and inserting a component C_v isomorphic to P^1) is called stabilization and will appear in the proof of Proposition 4.25

Indeed, the other two special points on C_v besides p_{n+1} can be

a) two preimages of nodes, or

b) one preimage of a node and one other marked point p_i.

The right side of Figure 22 shows the solution to our problem of de ning the forgetful morphism - we need to construct a morphism $: C \to C^0$ contracting the component C_v to a point $q \in C^0$. Then our forgetful morphism $: \overline{M}_{g;n+1} \to \overline{M}_{g;n}$ is given by

$$((C; p_1; \dots; p_n; p_{n+1})) = \begin{cases} (C; p_1; \dots; p_n) & \text{if } (C; p_1; \dots; p_n) \text{ is stable,} \\ (C^0; '(p_1); \dots; '(p_n)) & \text{otherwise.} \end{cases} \tag{51}$$

[27] The case of $g(C_v) = 1$ does not cause problems: it would require that p_{n+1} was the only special point of C_v. If $C = C_v$ was smooth, we would be in the case $(g; n+1) = (1; 1)$ which violates $2g - 2 + n > 0$, if C was singular then C_v would have to contain at least one node (otherwise it would be an isolated component of C, so C would not be connected).

To de ne the map ' above, we can use Fact 3.9: the curve C^0 is obtained by taking the normalization $\tilde{C}$ of C, removing the normalization of the component C_v and gluing the remaining components back together along preimages of nodes according to Figure 22. In case a) above, the two preimages of nodes on C_v are identi ed, in case b) we simply remove the component C_v. The morphism ' : C ! C^0 is then de ned via the second part of Fact 3.9 from the natural map $\tilde{C}$! C^0, which contracts the normalization of C_v to the point where it was previously attached.

It is not a priori obvious that (51) is the right thing to do, or even that it gives a continuous map. However, the next Proposition tells us that both are the case.

Proposition 4.25 ([Knu83a]). There exists a morphism : $\overline{M}_{g;n+1}$! $\overline{M}_{g;n}$ de ned on C-points by (51). Over the locus $\overline{M}^0_{g;n}$ $\overline{M}_{g;n}$ of curves without automorphisms, this map is isomorphic to the universal curve of $\overline{M}^0_{g;n}$. The universal sections $p_1; : : : ; p_n$: $\overline{M}^0_{g;n}$! $^1(\overline{M}^0_{g;n})$ are given by the restriction of the gluing morphisms

$$p_i = {}_i : \overline{M}_{g;n} = \overline{M}_{0;3} \quad \overline{M}_{g;n} ! \quad \overline{M}_{g;n+1}$$

associated to the stable graphs

Before saying something about the proof, note that the morphism : $\overline{M}_{g;n+1}$! $\overline{M}_{g;n}$ above is the unique extension of the morphism $M_{g;n+1}$! $M_{g;n}$ from Easy exercise 4.24 which really just forgets the marking p_{n+1}. The uniqueness follows since the domain $\overline{M}_{g;n+1}$ of is reduced, the target $\overline{M}_{g;n}$ is separated and since is determined on the open dense subset $M_{g;n+1}$ $\overline{M}_{g;n+1}$ (see the "Reduced-to-Separated Theorem" [Vak17, Theorem 10.2.2]). Thus you can see the Proposition above as saying that the extension exists and that (51) gives us a modular interpretation, i.e. an interpretation what this extension does on the geometric objects (stable curves) that our moduli spaces parametrize.

For the second part of Proposition 4.25 which allows us to interpret the forgetful map as the universal curve over part of $\overline{M}_{g;n}$ you can have a look at Figure 23, where for chosen points of the bre of the universal curve we illustrate which $n(+1)$-pointed curves they correspond to.

Below we sketch the formal proof of Proposition 4.25, but even this sketch is rather technical. So if you are happy with the picture and explanation above, feel free to skip it for now.

*Sketch of proof. As in the proof of Proposition 4.15 we can construct the morphism by de ning a corresponding natural transformation $\overline{M}_{g;n+1}$! $\overline{M}_{g;n}$ and using that $\overline{M}_{g;n+1}; \overline{M}_{g;n}$ are coarse moduli spaces of the corresponding functors.

In fact, we can do this in two steps: we de ne a moduli functor $\overline{C}_{g;n}$ sending a scheme S to the tuples

$$({}^0 : C^0 ! \quad S; p^0_1; : : : ; p^0_n; q : S ! \quad C^0); \tag{52}$$

such that

Figure 23: Correspondence of points in the bre of the universal curve over $\overline{M}^0_{g;n}$ with $(n+1)$-pointed stable curves

$({}^0 : C^0 \to S; p^0_1; \dots; p^0_n : S \to C)$ is a family of stable curves of genus g with n marked points,

$q : S \to C^0$ is any section of .

There is a natural transformation $\overline{C}_{g;n} \to \overline{M}_{g;n}$ which simply forgets the marking q (and doesnot change the curve C). On the other hand, Knudsen [Knu83a] constructs isomorphism of functors

$$\overline{M}_{g;n+1} \underset{\text{stabilization}}{\overset{\text{contraction}}{\rightleftarrows}} \overline{C}_{g;n}; \tag{53}$$

and the natural transformation $\overline{M}_{g;n+1} = \overline{C}_{g;n} \to \overline{M}_{g;n}$ de nes us the morphism : $\overline{M}_{g;n+1} \to \overline{M}_{g;n}$. For an excellent explanation of the details of the equivalence (53) see [KV07, Section 1.3] (or look at [Knu83a, Proposition 2.1, Theorem 2.4] for the original formulation and proof).

To summarize the sources above, the contraction map in (53) takes an $(n+1)$-pointed family of curves

$$(: C \to S; p_1; \dots; p_n; p_{n+1} : S \to C); \tag{54}$$

constructs a morphism $' : C \to C^0$ which contracts the unstable components of bres of $C \to S$ as in (51). Then (54) is sent to (52) setting $p^0_i = ' \quad p_i$ and $q = ' \quad p_{n+1}$.

The stabilization functor in (53) does the opposite: starting with the family (52) it modi es the bres C^0_s of $C^0 \to S$ in which the section q collides with one of the nodes or one of the markings p^0_i by inserting an extra component isomorphic to $\mathbb{P}^1$ at the point of intersection. This creates a new family of curves $: C \to S$ and the sections

$p_1, \ldots, p_{n+1} : S \to C$ are obtained from $p_1^0, \ldots, p_n^0, q$ as "strict transforms" under the map $C \to C^0$.

Finally, we know that the locus $\overline{M}^0_{g,n}$ is a fine moduli space for the functor $\overline{M}^0_{g,n}$. It is easy to see that the preimage $^{-1}(\overline{M}^0_{g,n}) \subset \overline{M}^0_{g,n+1}$ is also contained in the set $\overline{M}^0_{g,n+1}$ which is a fine moduli space. Defining the functor $\overline{C}^0_{g,n}$ as the families (52) such that $(C, p_1, \ldots, p_n)$ has only trivial automorphisms, this easily shows that $^{-1}(\overline{M}^0_{g,n})$ is a fine moduli space for $\overline{C}^0_{g,n}$. On the other hand, one can write down a natural transformation $\overline{C}^0_{g,n} \to h_{\overline{C}^0_{g,n}}$ making the universal curve $\overline{C}^0_{g,n}$ a fine moduli space of $\overline{C}^0_{g,n}$. So both $^{-1}(\overline{M}^0_{g,n})$ and $\overline{C}^0_{g,n}$ are fine moduli spaces of the same functor $\overline{C}^0_{g,n}$, and thus they are isomorphic. From the definition of the stabilization in (53) it follows that the gluing morphisms $_i$ are obtained from the stabilization of the family

$$(: \overline{C}^0_{g,n} \to \overline{M}^0_{g,n}, p_1, \ldots, p_n, q = p_i : \overline{M}^0_{g,n} \to \overline{C}^0_{g,n}) \in \overline{C}_{g,n}(\overline{M}^0_{g,n}).$$

But setting $q = p_i$ exactly corresponds to the i-th section of the universal curve, and this completes the proof that $p_i = \ _i$ over $\overline{M}^0_{g,n}$. □

A final question remains: if $: \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ is not the universal curve outside the locus of curves $(C, p_1, \ldots, p_n) \in \overline{M}_{g,n}$ which have only trivial automorphisms, then what are the fibres over curves? For this let $: C \to C$ be such an automorphism fixing the points $p_1, \ldots, p_n$, then we have for any point $q \in C$ which is smooth and does not coincide with one of the markings $p_1, \ldots, p_n$ that

$$(C, p_1, \ldots, p_n, q) \quad (C, \ (p_1), \ldots, \ (p_n), \ (q)) = (C, p_1, \ldots, p_n, \ (q)) \in \overline{M}_{g,n+1}. \tag{55}$$

From this one can check that the closed points of the fibre $^{-1}((C, p_1, \ldots, p_n))$ are in one-to-one correspondence with $\mathrm{Aut}(C, p_1, \ldots, p_n)$-orbits of points in C, and so the fibre is isomorphic to the quotient

$$^{-1}((C, p_1, \ldots, p_n)) = C/\mathrm{Aut}(C, p_1, \ldots, p_n). \tag{56}$$

4.5 Genus 1

In comparison to the genus 0 case, the case of genus 1 is much harder: nice properties, such as having a fine moduli space, are missing and there is - to my knowledge - no nice recursive construction as for $\overline{M}_{0,n}$. We content ourselves in looking at one nontrivial example - the case of precisely one marking[28] - and in pointing out the new phenomena that arise from the fact that we can have automorphisms.

The case $n = 1$

The following result makes precise some of the discussion in Section 1.

Proposition 4.26 (see Proposition 4.18 in [Ber13]). The moduli functor $M_{1,1}$ has as coarse moduli space the affine line $M_{1,1} = \mathbb{A}^1$.

*Sketch of proof. We must show that for any family of elliptic curves over a scheme S we can construct a natural morphism $S \to \mathbb{A}^1$. In the introduction we saw that this should be related to the j-invariant of the elliptic curves. But a priori, this j-invariant only makes sense for elliptic curves given as cubic curves in $\mathbb{P}^1$.

[28] Note that the condition $2g \quad 2 + n > 0$ implies that $n = 1$ is the simplest possible case for $g = 1$.

To approach this situation, note that given a (smooth) elliptic curve $(E; p)$ one can use Riemann-Roch and Serre Duality to show

$$h^0(E; \mathcal{O}_E(3p)) = 3\,; \quad h^1(E; \mathcal{O}_E(3p)) = 0 \tag{57}$$

and for any basis $s_1; s_2; s_3$ of sections of $\mathcal{O}_E(3p)$ we obtain an embedding[29]

$$E \hookrightarrow \mathbb{P}^2; \; q \mapsto [s_1(q) : s_2(q) : s_3(q)]$$

of E as a smooth cubic curve in $\mathbb{P}^2$ (see [Vak17, Section 19.9] for details).

Now we need to do this for families of elliptic curves. Given such a family

$$(\pi : E \to S; p : S \to E) \in \mathcal{M}_{1;1}(S) \tag{58}$$

over some scheme S, we see as in the proof of Proposition 4.2 that (57) implies that the pushforward $\mathcal{V} = \pi_* \mathcal{O}_E(3p)$ is a rank 3 vector bundle over S. Let V be the total space of this vector bundle, then we have an open dense subset

$$U = \{(s_1; s_2; s_3) : s_1; s_2; s_3 \in \mathcal{V}_s \text{ form a basis of } \mathcal{V}_s = H^0(E_s; \mathcal{O}_{E_s}(3p(s)))\} \subset V \times_S V \times_S V.$$

Let $E_U = E \times_S U$ be the pullback of $E \to S$ under the natural map $U \to S$, then the pullback of V to E_U naturally has three sections $s_1; s_2; s_3$ (given by the coordinates on U), and they define an embedding $E_U \hookrightarrow \mathbb{P}^2_U = U \times \mathbb{P}^2$.

For this embedding, we want to say that on each fibre there is a cubic equation on $\mathbb{P}^2$, unique up to scaling, that cuts out the image of the elliptic curve in $\mathbb{P}^2$. And indeed, denoting $\mathcal{F}$ the pushforward of $\mathcal{O}_{\mathbb{P}^2_U}(3)$ under $\mathbb{P}^2_U \to U$, there exists a unique $F \in \mathbb{P}(\mathcal{F})$ such that E_U is the vanishing locus of F in $\mathbb{P}^2_U$. Now what data is $\mathcal{F}(u)$ at a point $u \in U$? It's just the coefficients of a cubic curve in the corresponding projective space $\mathbb{P}^2_u$. But from these coefficients you can compute the j-invariant of this cubic curve (and the formula is invariant under scaling). This means that we obtain a map $\tilde{j} : U \to \mathbb{A}^1$. But the value of $\tilde{j}$ is constant on the fibres of the morphism $U \to S$ - different points in the fibre just correspond to different ways to embed the same elliptic curve in $\mathbb{P}^2$ and the j-invariant is, well, invariant under this. Using fpqc descent (see [Sta13, Tag 023Q]), we then obtain a morphism $j : S \to \mathbb{A}^1$ as desired.

$$\begin{tikzcd} U \arrow[d] \arrow[dr, "\tilde{j}"] & \\ S \arrow[r, "j"'] & \mathbb{A}^1 \end{tikzcd}$$

We'll not go into the details in how to verify that this natural transformation $\mathcal{M}_{1;1} \to h^{\mathbb{A}^1}$ satisfies property a) of a coarse moduli space (being initial among natural transformations $\mathcal{M}_{1;1} \to h^M$ for M a scheme). But concerning part b) of the definition, the above proof showed that every elliptic curve $(E; p)$ can be embedded as a smooth cubic in $\mathbb{P}^2$, so if we accept that the j-invariant classifies those up to isomorphism, it implies that $\mathcal{M}_{1;1} \to h^{\mathbb{A}^1}$ is a bijection on geometric points. □

Corollary 4.27. The moduli space $\overline{\mathcal{M}}_{1;1}$ is isomorphic to $\mathbb{P}^1$.

Proof. By Theorem 3.19, the space $\overline{\mathcal{M}}_{1;1}$ is a normal, projective variety of dimension 1. Being normal in dimension 1 means it is actually smooth, and as seen in Section 4.3, the only such curve containing an open subset of $\mathbb{A}^1$ is $\mathbb{P}^1$. □

The point $1 \in \mathbb{P}^1 = \overline{\mathcal{M}}_{1;1}$ corresponds to the stable curve $(E_0; p_0)$ obtained by starting with $p_0 \in \mathbb{P}^1$ and identifying two points (not equal to p_0) to a node.

[29] Here we use that $\mathbb{P}^2$ is a moduli space of line bundles together with 3 sections not vanishing simultaneously, just what we saw in the proof of Example 2.4!

The case $n = 2$

Exercise 4.28. Figure 24 illustrates the forgetful morphism $: \overline{M}_{1;2} \to \overline{M}_{1;1}$ with the boundary of both spaces marked in red. For each of the points marked in blue, draw their corresponding curves and their dual graphs.

Figure 24: The forgetful morphism $: \overline{M}_{1;2} \to \overline{M}_{1;1}$

A fun construction in $n = 9$

Exercise 4.29. Show that for $Q_1; \dots; Q_9 \in \mathbb{P}^2$ general points, there exists a unique cubic curve E_Q going through $Q_1; \dots; Q_9$. Show that this gives rise to a rational map

$$(\mathbb{P}^2)^9 \dashrightarrow \overline{M}_{1;9}; (Q_1; \dots; Q_9) \mapsto (E_Q; Q_1; \dots; Q_9): \tag{59}$$

Show that this map is dominant, i.e. that the generic point of $\overline{M}_{1;9}$ is contained in the image. (Hint: This last part will involve showing that for any fixed smooth genus 1 curve E, embedded in some way as a cubic $E \hookrightarrow \mathbb{P}^2$ and $Q_1; \dots; Q_9 \in E$ general points in E, the curve E is the unique cubic through $Q_1; \dots; Q_9$).

Remark 4.30 (*, for people interested in birational geometry) The above exercise shows that $\overline{M}_{1;9}$ is unirational, i.e. that it admits a dominant rational map from a projective space. For $g = 0$ we already saw that all spaces $\overline{M}_{0;n}$ have the stronger property of being rational, i.e. birational to a projective space. This is the start of an interesting story: for many (small) values of $(g; n)$ there have been rational parametrizations of $\overline{M}_{g;n}$ as in (59) (see [Ben14, BV05, CF07, Far09, Log03, Ver05]). However, in general the spaces $\overline{M}_{g;n}$ are neither rational nor unirational:

> It turns out that in genus 1, the Hodge number $h^{11;0}(\overline{M}_{1;11})$ is equal to 1, so there exists a nonzero holomorphic 11-form on $\overline{M}_{1;11}$. This implies that there cannot be a dominant rational map $\mathbb{P}^N \dashrightarrow \overline{M}_{1;11}$.

Classical results by Eisenbud, Harris and Mumford [M82, Har84, EH87] say that $\overline{M}_g$ is of general type for g 24 and has positive Kodaira dimension for $g = 23$. A variant of this result by Logan [Log03] says that for $g > 3$ the space $\overline{M}_{g;n}$ is of general type for all but nitely many pairs $(g; n)$.

The birational geometry and speci cally the Kodaira dimension of the moduli spaces of curves are still an active area of research.

Non-existence of a ne moduli space / universal curve

To show that $M_{1;1}$ cannot have a ne moduli space, we will construct an explicit example $(: E \to S; p : S \to E)$ of a family of smooth genus 1 curves over a base S, such that the family is Zariski-locally trivial (i.e. a constant family) but not globally trivial. This gives a contradiction: if $M_{1;1}$ was a ne moduli space, the family would be the pullback of the universal family over $M_{1;1}$ under a unique map $S \to M_{1;1}$. The fact that is Zariski locally trivial would imply that on a Zariski cover of S this map to $M_{1;1}$ is constant (i.e. factors through a point). Since a map is determined by its restriction to a Zariski cover, the map $S \to M_{1;1}$ itself would be constant. But the pullback of a family of curves under a constant map is a trivial family, a contradiction.

For constructing , since the Zariski topology is very coarse we will need a slightly ugly base S: it is a nodal curve consisting of four rational curves forming a chain, as indicated in Figure 25. To obtain let $(E_0; p_0)$ be a smooth elliptic curve. By Fact 3.11 there exists a nontrivial automorphism $: E_0 \to E_0$ which xes p_0 [30]. We obtain the family $: E \to S$ by gluing the trivial families $U_1 \quad E_0$, $U_2 \quad E_0$ over the indicated Zariski open cover of S. The intersection $U_1 \setminus U_2 \quad S$ has two components, and we glue the families by the identity of E_0 on one component and by on the other. The fact that p_0 is invariant under shows that the sections $p_0 : U_1 \to U_1 \quad E_0$ and $p_0 : U_2 \to U_2 \quad E_0$ glue together over the overlaps to a section $p_0 : S \to E$.

Exercise 4.31. Show that the sections $s : S \to E$ of the morphism $: E \to S$ are in bijection with the xed points of the automorphism . Conclude that $: E \to S$ is not isomorphic to the trivial family $S \quad E_0 \to S$. (Hint: Use that every morphism $\mathbb{P}^1 \to E_0$ is constant. This follows from the more general fact that every morphism $C \to D$ of smooth, projective, irreducible curves with $g(C) < g(D)$ is constant.)

4.6 *Proof of Proposition 4.2

Parts of the script indexed by a * are facultative, so they can be skipped on a rst reading and are not part of the exam material. Concerning Proposition 4.2, it is mentioned in [KV07, Section 1.1.1], but proving it in the stated generality is actually quite nontrivial! If you spot gaps or mistakes in the proof below or nd better references for the statements I cite, I would be happy if you write me an email.

* Proof of Proposition 4.2. For part a) note that since $: C \to B$ is proper and the composition $p_1 = \mathrm{id}_B$ is a closed embedding, by the cancellation theorem for properties of morphisms ([Vak17, Theorem 10.1.19]) the map p_1 is a closed embedding. Since the image of p_1 is an e ective Cartier divisor when restricted to each bre, by [Sta13, Tag 062Y] it de nes a relative e ective Cartier divisor. Let $L = \mathcal{O}_C(p_1)$ be the associated line bundle on C. We claim that $E = \quad L$ is a locally free sheaf on B of rank 2. Indeed, for any

[30] If we identify E_0 as the quotient $E_0 = C/ \quad$ of a C by a lattice $\quad C$ and choose $p_0 = 0$, one such automorphism is induced by the map $C \to C; z \mapsto z$.

Figure 25: Constructing a nontrivial family over the base S by gluing trivial families on an open cover $S = U_1 \cup U_2$ along a nontrivial automorphism

point $s \in B$ the cohomology group of the fibre $H^1(C_s; L_s)$ vanishes, since after passing to the algebraic closure $\overline{k(s)}$ of the residue field $k(s)$ (which is flat and thus commutes with formation of H^1) it is isomorphic to

$$H^1(C_{\overline{k(s)}}; L_{\overline{k(s)}}) = H^1(\mathbb{P}^1_{\overline{k(s)}}; \mathcal{O}(1)) = 0 :$$

By the Cohomology and Base Change Theorem (see [Vak17, Theorem 28.1.6., Exercise 28.1.D.] and note that the hypothesis of the base B being locally Noetherian can be removed since f is assumed locally of finite presentation, see [Vak17, Exercise 28.2.M.]), the sheaf $E = f_* L$ is indeed locally free. By going to a geometric fibre, we see that its rank is

$$h^0(C_{\overline{k(s)}}; L_{\overline{k(s)}}) = h^0(\mathbb{P}^1_{\overline{k(s)}}; \mathcal{O}(1)) = 2 :$$

As in the proof of [Har77, V, Proposition 2.2.] one then shows that $C = \mathbb{P}(E)$, completing the proof of a).

Now in case b) assume we have an additional section $p_2 : B \to C$. As in part a) we obtain a line bundle $L^0 = \mathcal{O}_C(p_2)$ on C and we claim that the line bundle $L - L^0$ is a pullback from the base, i.e. there exists M a line bundle on B with $L - L^0 = f^* M$. This follows e.g. by [Vak17, Proposition 28.1.11.] (note that we can remove the assumptions on the base being reduced and locally Noetherian with the same arguments used in the previous part). Let $s_0 \in H^0(C; L)$ be the section vanishing along p_1 and $s_1 \in H^0(C; L^0)$ the section vanishing along p_2. Then we have a map of locally free sheaves on B

$$: \mathcal{O}_B \oplus M \to E ; (a; b) \mapsto a \cdot s_0 + b \cdot s_1 : \tag{60}$$

Here the section $b \cdot s_1$ makes sense since

$$E = f_* L = f_*(L^0 \otimes f^* M) = M \otimes f_* L^0 .$$

On an open cover of B which trivializes E (so that over the open sets $U \quad B$ the space C_U is isomorphic to $C_U = U \quad P^1$) it is easy to check that is an isomorphism. This open cover also trivializes the line bundle M and then the sections $s_0; s_1$ restrict to a basis of the sections of L on the bres of (since $p_1; p_2$ are disjoint).

Finally, in case c) we have a third section p_3 and since it is disjoint from $p_1; p_2$ we have:

$$M = p_3 \quad M = p_3 L - \quad L \quad {}^0 = p_3 O_C(\quad p_1 + p_2) = O_B :$$

Thus by the proof of part b) we can take $E = O_B \quad O \quad {}_B$. □

Remark 4.32. A morphism as in Proposition 4.2 is a special case of a Brauer-Severi scheme. By [Gro66, Theorème 8.2] such morphisms are always étale locally isomorphic to projective bundles, but not necessarily Zariski locally. An example for a family of smooth genus 0 curves which is not a projective bundle is given by the universal plane conic, de ned over an open subset of $P^5 = P(H^0(P^2; O(2)))$, see [Vak17, Section 18.4.5].

References and further reading

A great introduction to moduli spaces of genus 0 curves is given in [KV07, Chapter 1]. You can also have a look at these lecture notes by Renzo Cavalieri. More material on gluing and forgetful maps (though phrased in the language of stacks that we will see in the next section) can be found in [ACG11, Chapter XII, Section 10].

The moduli problem of elliptic curves $(E; p)$ has a much richer structure than you would expect from the isomorphism $\overline{M}_{1;1} = P^1$ that we give above. You can have a look at the lecture notes [Hai08] for a much more complete picture.

5 Moduli stacks of curves

We have seen in many instances that the existence of automorphisms prevents fine moduli spaces from existing and we mentioned that it causes unpleasant phenomena (e.g. singularities) in the coarse moduli spaces. The solution is that we define a generalization of the notion of a scheme, called an (algebraic) stack, such that

- any scheme S can be interpreted as an algebraic stack (similar to the Yoneda embedding allowing us to see schemes as particular examples of moduli functors),
- tools and results from algebraic geometry can be generalized to stacks (i.e. we can define when an algebraic stack X is smooth, when a morphism $X \to X^0$ is proper, etc),
- there exists an algebraic stack $\overline{M}_{g;n}$ serving as a "moduli stack of stable curves" (i.e. morphisms from a scheme S to $\overline{M}_{g;n}$ are exactly equivalent to families of stable curves over S),
- the algebraic stack $\overline{M}_{g;n}$ has many nice properties (e.g. $\overline{M}_{g;n}$ is smooth, the morphism $\overline{M}_{g;n} \to \operatorname{Spec}(\mathbb{C})$ is proper, etc.).

However, defining stacks and developing the language and results to talk about them requires some serious effort, and goes beyond the scope of this course. So what I will do is to give an outline of the ideas of the definition together with a guide where to learn more. We'll then go on to treat the stacks essentially as a black-box, pretending that they are schemes, with occasional remarks where we need to be more careful.

5.1 An outline of the theory of algebraic stacks

In Lemma 2.1 we saw that we can embed the category of schemes into the category of moduli functors by sending M to the functor $h^M = \operatorname{Mor}(\ ; M)$. But we saw that the moduli functors $\overline{M}_{g;n}$ are not of the form h^M for some M. The main reason is that given a scheme S and an open cover $S = U_1 \cup U_2$, a morphism $S \to M$ to a scheme M is uniquely determined by its restriction to U_1 and U_2, but a family of curves up to isomorphism is not necessarily uniquely determined by its restrictions to $U_1; U_2$. As we saw in Exercise 4.31, we can obtain a nontrivial family by taking two trivial families of curves over $U_1; U_2$ and gluing them along a nontrivial automorphism on the overlap $U_1 \cap U_2$.

So we are looking for a definition of a new mathematical object $\overline{M}_{g;n}$ such that

- it makes sense to speak of a morphism $S \to \overline{M}_{g;n}$ from a scheme S to $\overline{M}_{g;n}$, and such morphisms are in bijective correspondence to isomorphism classes of families of stable curves over S,
- given two morphism $f_1; f_2 : S \to \overline{M}_{g;n}$ corresponding to families $_1 : C_1 \to S$ and $_2 : C_2 \to S$ of curves, we have a notion of isomorphisms $f_1 \to f_2$ corresponding to the set of isomorphisms $C_1 \to C_2$ of families of curves over S.

Then in the example above, for the scheme $S = U_1 \cup U_2$, a morphism $S \to \overline{M}_{g;n}$ is determined by its restrictions $f_1 : U_1 \to \overline{M}_{g;n}$ and $f_2 : U_2 \to \overline{M}_{g;n}$ together with an isomorphism $f_1|_{U_1 \cap U_2} \to f_2|_{U_1 \cap U_2}$ (telling us how to glue the families on the overlap).

Looking at the requirements above, I claim that we have already seen mathematical objects with such "morphisms between morphisms", namely categories! Given categories $C_1; C_2$ we have a notion of a functor $f : C_1 \to C_2$, and given two functors $f_1; f_2 : C_1 \to C_2$ we have the notion of a natural transformation (or a natural equivalence) $f_1 \to f_2$.

So the idea is that a stack is a category (with some extra data, satisfying suitable properties). In our favorite example, the stack[31] $\overline{M}_{g;n}$ is the category whose objects are families of stable curves over a scheme:

$$\mathrm{Ob}(\overline{M}_{g;n}) : (\ : C \to S; p_1; \ldots; p_n : S \to C) \text{ family of stable genus } g \text{ curves} \tag{61}$$

The morphisms in the category are a bit peculiar (we'll see in a few lines why this makes sense): they are given by fibre diagrams (or pullbacks) of families of curves[32]

$$\mathrm{Mor}(\ ^0 : C^0 \to S^0, \ : C \to S) = \left\{ \begin{tikzcd} C^0 \arrow[r, "\hat{f}"] \arrow[d, "\ ^0"] & C \arrow[d] \\ S^0 \arrow[r, "f"] & S \end{tikzcd} : (\hat{f}, f) \text{ make } C^0 = S^0 \times_S C \text{ a pullback of } C = S \right\} \tag{62}$$

Note that there is a functor $F : \overline{M}_{g;n} \to \mathrm{Sch}_C$ to the category of schemes, sending $\ : C \to S$ to the scheme S and sending the morphism $(\hat{f}, f)$ in (62) to the morphism $f : S^0 \to S$. This functor has some very nice properties:

- the preimage of $S \in \mathrm{Sch}_C$ (i.e. the objects of $\overline{M}_{g;n}$ mapping to S) are precisely the families (61) of stable genus g curves over S,
- given $\ : C \to S$ and any morphism $f : S^0 \to S$ of schemes, there exists $\ ^0 : C^0 \to S^0$ and a morphism $(\hat{f}, f)$ mapping to the given morphism f under F,
- given the family $\ : C \to S$, the set of morphisms $(\hat{f}, f)$ from this family to itself which map to the morphism $f = \mathrm{id}_S : S \to S$ are precisely the automorphisms of the family $\ : C \to S$ of stable curves. This requires the small check that a map $\hat{f} : C \to C$ such that $(\hat{f}, \mathrm{id}_S)$ makes the diagram in (62) into a fibre diagram is an isomorphism. This is the reason why we chose the morphisms in $\overline{M}_{g;n}$ to be pullback diagrams.

We see that the fibre $F^{-1}(S)$ of F over S, defined as the category whose objects are the objects of $\overline{M}_{g;n}$ mapping to S and whose morphisms are the morphisms $(\hat{f}, f)$ sitting over the identity $f = \mathrm{id}_S$, is a groupoid, i.e. a category in which all morphisms are isomorphisms. Together with some more technical assumptions (see Fan[01, Section 3]) this makes $(\overline{M}_{g;n}; F : \overline{M}_{g;n} \to \mathrm{Sch}_C)$ into a category fibred in groupoids over Sch_C.

Note that given a scheme M over C, the category Sch_M of schemes over M is also a category fibred in groupoids: its map $\mathrm{Sch}_M \to \mathrm{Sch}_C$ sends $X \to M$ to X and a morphism $f : X^0 \to X$ of schemes over M to the morphism $f : X^0 \to X$ of schemes over C. Note that the fibre of Sch_M over a scheme $X \in \mathrm{Sch}_C$ is precisely the category of morphisms $X \to M$. This allows us to draw a diagram as follows:

$$\begin{tikzcd} & \text{categories fibred in groupoids} \arrow[d, "(M;F) \mapsto (S \mapsto F^{-1}(S) = \text{iso})"] \\ \mathrm{Sch}_C \arrow[ur, "M \mapsto \mathrm{Sch}_M"] \arrow[r, "M \mapsto h^M"'] & \text{moduli functors} \end{tikzcd} \tag{63}$$

[31] From now on, the symbol $\overline{M}_{g;n}$ stands for the category we describe here, no longer the moduli functor from before.

[32] Below we omit the sections $p_1; \ldots; p_n$ from the notation since it becomes too complicated otherwise, but they are always part of the data and need to satisfy similar compatibilities.

Here $F^{-1}(S)/\text{iso}$ is the set[33] of objects in the category $F^{-1}(S)$ up to (iso)morphisms of the category. With what we said before, this immediately makes obvious that the diagram (63) commutes. So we see that, extending the Yoneda embedding from Lemma 2.1, the category of schemes can be embedded in the category[34] of categories $\mathcal{M}$ bred in groupoids. A morphism from a scheme S to a category $(\mathcal{M}, F)$ bred in groupoids is then a functor f fitting in the diagram

$$\begin{tikzcd} \mathrm{Sch}_S \arrow[rr, "f"] \arrow[dr] & & \mathcal{M} \arrow[dl, "F"] \\ & \mathrm{Sch}_C & \end{tikzcd}$$

and you can check that a morphism $f : S \to \overline{\mathcal{M}}_{g,n}$ is exactly equivalent to specifying a family $\pi : C \to S$ of curves over S. So the fundamental idea of fine moduli spaces (that morphisms to them are equivalent to families for the moduli functor) is already baked into the category theory above. But now, we also have a notion of morphisms between morphisms: given $f, f^0 : \mathrm{Sch}_S \to \overline{\mathcal{M}}_{g,n}$ corresponding to families $\pi : C \to S$ and $\pi^0 : C^0 \to S$, we say that an isomorphism from f to f^0 is a natural equivalence of functors $f^0 \to f$ making the diagram above commute. This can be seen to be equivalent to giving an isomorphism $C \to C^0$ of families of stable curves over S.

To be useful, we need to make sure that we can "do algebraic geometry with the categories above". It turns out that looking at arbitrary categories $\mathcal{M}$ bred in groupoids is too general to do this. Therefore, we only look at particular types of such categories, called (algebraic) stacks. These are categories $\mathcal{M}$ bred in groupoids that satisfy some additional conditions (which allow us to do reasonable algebraic geometry with such categories). The category $\overline{\mathcal{M}}_{g,n}$ above turns out to be an algebraic stack, and instead of giving the definition in full generality, let me just describe them in the specific example of $\overline{\mathcal{M}}_{g,n}$.

$\overline{\mathcal{M}}_{g,n}$ is a stack: Being a stack means that $\overline{\mathcal{M}}_{g,n}$ has a "sheaf-like" property. Assume we are given a scheme S and an (etale) cover $(U_i \to S)_i$ together with families $(\pi_i : C_i \to U_i)_i$ of curves. Assume moreover that we have a family $(\varphi_{ij} : C_i|_{U_i \setminus U_j} \to C_j|_{U_i \setminus U_j})_{ij}$ of isomorphisms of the families of curves on the overlaps $U_i \setminus U_j = U_i \times_S U_j$, which are compatible on overlaps (i.e. they satisfy the cocycle condition $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on triple overlaps). Then these glue to a family $\pi : C \to S$ over S and the family is unique up to unique isomorphism[35], see Figure 26.

Given stacks X, Y, Z and morphisms $X \to Y$ and $Z \to Y$, one can define the fibre product $X \times_Y Z$. Then we say that a morphism $\varphi : X \to Y$ is representable if for every scheme U and morphism $U \to Y$, the fibre product $X \times_Y U$ in the diagram

$$\begin{tikzcd} X \times_Y U \arrow[r, "\varphi_U"] \arrow[d] & U \arrow[d] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd} \tag{64}$$

is isomorphic to a scheme $X \times_Y U = S$. Let P be a property of morphisms of schemes invariant under pullback/base change (e.g. being a smooth morphism). Then we say

[33] If you like general nonsense: one probably has to restrict to small categories $\mathcal{M}$ bred in groupoids, let's ignore this here.

[34] There is a natural notion of a functor between two categories $(\mathcal{M}_1, F_1), (\mathcal{M}_2, F_2)$ bred in groupoids: it is a functor $G : \mathcal{M}_1 \to \mathcal{M}_2$ between the underlying categories such that $F_2 \circ G = F_1$.

[35] In the general definition of a stack $\mathcal{M}$, the families of curves over the U_i are replaced by functors $\mathrm{Sch}_{U_i} \to \mathcal{M}$ and the isomorphisms φ_{ij} correspond to natural equivalences between the restrictions of these morphisms to $\mathrm{Sch}_{U_i \setminus U_j}$.

Figure 26: Gluing two families of curves on an open cover $S = U_1 \cup U_2$ along an isomorphism of the restrictions of the families on the overlap $U_1 \cap U_2$

that a representable morphism $\varphi : X \to Y$ of stacks has property P if for all $U \to Y$ the morphism φ'_U (of schemes) in (64) has property P.

If you know something about manifolds, you should see the morphism $U \to Y$ as a type of chart for Y by objects we understand well (we cover a stack using schemes, and a manifold using open subsets of $\mathbb{R}^n$). Then the above says that we can check properties of morphisms between stacks by checking them on all charts (and representable morphisms have the magic property that the preimage of a chart of Y is a chart of X).

$\overline{M}_{g;n}$ is an algebraic stack: Being an algebraic stack means that X has a particularly nice chart: there exists a scheme U and a representable, smooth and surjective morphism $U \to X$ (smooth and surjective are both properties invariant under pullback, so this statement makes sense). To check this for $\overline{M}_{g;n}$, note that a morphism $U \to \overline{M}_{g;n}$ is given by a family $\pi : C \to U$ of stable curves over the scheme U. The fact that this is surjective means that every stable curve appears as a fibre in the morphism, while smoothness is slightly harder to interpret[36]. Proving the existence of such a $U \to \overline{M}_{g;n}$ requires some work (see [DM69, Section 5]), but we have seen an example of this at the very start of the course: the family E_t of cubic curves parametrized by $t \in \mathbb{C}$ from (3) gives rise to a family of 1-pointed stable curves of genus 1

$$\begin{tikzcd} E \arrow[r, equal] \arrow[d, shift right] & \{([X:Y:Z];t) \in \mathbb{P}^2 \times \mathbb{A}^1 : Y^2Z + X(X-Z)(X-tZ) = 0\} \\ \mathbb{A}^1 \arrow[u, bend right, "p_1"'] & \end{tikzcd} \tag{65}$$

where the section p_1 is given by $p_1(t) = ([0:1:0];t)$. The corresponding map $\mathbb{A}^1 \to \overline{M}_{1;1}$

[36] If you know the definition of formal smoothness: the property of $U \to \overline{M}_{g;n}$ being smooth requires that for a ring R and an R-point $R \to U$ with fibre C_R under π and a deformation C_{R^0} of C_R over a square-zero extension $R \to R^0$ of R, we can find a morphism $R^0 \to U$ such that the composition $R^0 \to U \to \overline{M}_{g;n}$ is induced by the family C_{R^0} of curves.

is a representable, smooth, surjective morphism.

Given an algebraic stack X with a representable, smooth, surjective cover $U \to X$, we say that X is smooth if we can choose the scheme U to be smooth (over $\mathbb{C}$). More generally, given any property Q of schemes that can be checked on a smooth cover, we say that X has Q if and only if U has Q.

For $\overline{M}_{g;n}$ we can indeed choose U to be smooth, so $\overline{M}_{g;n}$ is a smooth stack. In fact, the cover $U \to \overline{M}_{g;n}$ can be chosen to be etale (not just smooth). An algebraic stack with this stronger property is called a Deligne-Mumford stack. Such stacks are "really close to being a scheme" and for the most part this will allow us to pretend that $\overline{M}_{g;n}$ behaves just like a scheme which is a fine moduli space for the functor of families of stable curves. One can show that for a Deligne-Mumford stack M and a point $x : \mathrm{Spec}(\mathbb{C}) \to M$, the stabilizer group Stab_x of x (defined as the set of isomorphisms $x \to x$ from the morphism to itself) is finite. Under mild conditions (M separated and in characteristic 0) it is conversely true that an algebraic stack M with finite stabilizer groups at geometric points is Deligne-Mumford. This exactly brings us back to the definition of $\overline{M}_{g;n}$, since we asked that the automorphism group $\mathrm{Aut}(C; p_1; \ldots; p_n)$ of a stable curve is finite. It was the insight of Deligne and Mumford that this condition exactly ensures that the resulting stack $\overline{M}_{g;n}$ has nice properties.

Finally, there exists a morphism $\overline{M}_{g;n} \to \overline{M}_{g;n}$ from the stack $\overline{M}_{g;n}$ to the coarse moduli space $\overline{M}_{g;n}$ we talked about earlier. This morphism is proper, induces a bijection on geometric points and has the property that any other morphism $\overline{M}_{g;n} \to M$ to a scheme M must factor through $\overline{M}_{g;n} \to \overline{M}_{g;n}$ (this is the augmented version of the notion of a coarse moduli space for a moduli functor).

Where to learn more about stacks

Here is a list of resources, ordered in increasing comprehensiveness, which you can use to learn more about stacks:

- the paper "Stacks for Everybody" [Fan01] by Barbara Fantechi (11 pages, a few hours to work through, highly recommended),
- the course on the topic given by Prof. Georg Oberdieck in the Winter semester 2020 (one semester, also highly recommended),
- the book "Algebraic Stacks" (in preparation, by Behrend, Conrad, Edidin, Fantechi, Fulton, Gottsche und Kresch), found on the website of an old course by Andrew Kresch (220 pages, a few months, a great resource for self-study),
- the Stacks project [Sta13] (about 7000 pages, several years of intense study, great to look up results and particular topics, highly non-recommended to read from beginning to end).

5.2 Upgrades of previous results

By using the language of stacks, many results about the moduli spaces of curves that we saw before have a better version (i.e. nicer properties) when talking about the moduli stacks. One caveat: of course, the brief and informal introduction to stacks given in Section 5.1 is not enough to give a precise meaning to all the properties listed below. I still hope you get an idea of their meaning, but you can take those as black boxes for now (we will make them more precise as we need them).

Theorem 5.1. Let $g; n \geq 0$ with $2g - 2 + n > 0$.

a) The categories bred in groupoids $M_{g;n}$ and $\overline{M}_{g;n}$ are (algebraic) Deligne-Mumford stacks.

b) They are irreducible, proper and smooth of dimension $3g\ \ 3 + n$ and there is a natural inclusion $M_{g;n}\ \ \overline{M}_{g;n}$ as a nonempty, open substack.

c) The boundary $@\overline{M}_{g;n} = \overline{M}_{g;n} \ n\ M_{g;n}$ is an e ective Cartier divisor and even a normal crossings divisor[37].

d) The forgetful morphism $\ : \overline{M}_{g;n+1}\ !\ \ \overline{M}_{g;n}$ makes $\overline{M}_{g;n+1}$ the universal curve over $\overline{M}_{g;n}$. In particular, this morphism is representable, proper, at and of relative dimension 1.

e) For a stable graph of genus g with n legs, the gluing morphism

$$: \overline{M}\ \ = \prod_{v2V(\ )} \overline{M}_{g(v);n(v)}\ !\ \ \overline{M}_{g;n}$$

is representable, nite and a local complete intersection[38]. It has generic degree $\#\mathrm{Aut}(\ )$ onto its image $\overline{M}\ \ .$

Note that e.g. for the forgetful morphism, we originally constructed it by rst giving a natural transformation of the corresponding moduli functors (e.g. how to take an $(n+1)$-pointed family of curves and construct an n-pointed family from this). You can check that the same construction de nes a functor $\overline{M}_{g;n+1}\ !\ \ \overline{M}_{g;n}$ between the corresponding categories. The same discussion applies to the gluing morphisms

References and further reading

The origin of the notion of a (Deligne-Mumford) stack is the original paper [DM69] by Deligne and Mumford and Section 4 of this paper gives an introduction to this notion.

[37] Essentially, this means that etale locally the boundary looks like a union of some coordinate hyperplanes in $\mathbb{C}^N$, but see [Sta13, Tag 0CBN] for a formal de nition.

[38] This means it can be factored into a regular embedding followed by a smooth morphism.

6 Intersection theory on the moduli of stable curves

As the genus g and the number n of markings increase, the spaces $\overline{M}_{g;n}$ quickly become extremely complicated geometric objects. One way to study such complicated spaces is by computing some topological invariants and in particular to study their singular cohomology groups. By this last part we mean that you consider the set $\overline{M}_{g;n}(C)$ of complex points of the moduli space, with its complex topology, and study the cohomology groups

$$H\ (\overline{M}_{g;n}) = H\ (\overline{M}_{g;n}(C); Q): \tag{66}$$

In Section 6.2 we will see that given a closed algebraic subset S of $\overline{M}_{g;n}$ of complex codimension c, we can associate to S a cohomology class $[S] \ 2\ H^{2c}(\overline{M}_{g;n})$. For a second algebraic set S^0 meeting S transversally[39], we then have that the class $[S \ \backslash\ S^0]$ associated to the intersection of $S; S^0$ is equal to the cup product $[S] \wedge\ [S^0]$ of their classes $[S]; [S^0]$. We will introduce these notions carefully in Section 6.2, but this is the origin of the word "intersection theory". We will also see a few places where it is more convenient to work with the smooth stacks $\overline{M}_{g;n}$ instead of the moduli spaces $\overline{M}_{g;n}$, but we'll stick with $\overline{M}_{g;n}$ for the most part and only use the stacks when we need them.

One class of examples of closed subsets S of $\overline{M}_{g;n}$ are the closures $S = \overline{M}\ \ $ of the strata of $\overline{M}_{g;n}$ associated to a given stable graph . So we begin by studying these sets and their intersections in more detail.

6.1 Intersections of strata

In Section 4.2 we saw that the moduli spaces $\overline{M}_{g;n}$ admit a strati cation by locally closed subsets

$$\overline{M} = \bigcup\ M$$

according to stable graphs . In Proposition 4.15 we showed that the closures $\overline{M}\ \ $ are parametrized by the gluing maps

$$: \overline{M}\ \ = \prod_{v2V()} \overline{M}_{g(v);n(v)} \ !\ \ \overline{M}_{g;n}:$$

In this section we want to answer two natural questions concerning the sets $\overline{M}\ \ $:

a) Given a stable graph , what are the stable graphs ^{0} of curves in the closure $\overline{M}\ \ $?

b) Given two stable graphs $_1;\ _2$, what is the intersection $\overline{M}\ ^1 \backslash\ \overline{M}\ ^2\ \ \ \overline{M}_{g;n}$?

As we saw above, question b) in particular will be related to the computation of the intersection product $[\overline{M}\ ^1] \wedge\ [\overline{M}\ ^2]$. A good reference for these questions is [GP03, Appendix A], where the answers were worked out in a formal way for the rst time.

For question a) we can use that $\overline{M}\ \ = \ \ (\overline{M}\ \ )$, so we need to understand how the stable graph of the curve

$$((C_v; p_{1;v}; : : : ; p_{n(v);v})_{v2V()}) \tag{67}$$

depends on and the stable graphs $_v$ of the curves $(C_v; p_{1;v}; : : : ; p_{n(v);v})$. In Figure 27 you see an example of this.

[39] If $S; S^0$ are smooth, this means that at any point $p\ 2\ S \backslash\ S^0$ we have $T_p\overline{M}_{g;n} = T_pS + T_pS^0$.

Figure 27: The stable graph ^{0} of a curve obtained by gluing curves with stable graphs $_{v_1}$; $_{v_2}$ via is obtained by inserting the graphs $_{v_1}$; $_{v_2}$ at the vertices of

Intuitively, it is quite clear what happens: you start with the dual graph and glue the graphs $_v$ into the vertices v of . Making this precise in the formal language of stable graphs is a bit of a headache, which is why I give it as an exercise[40]

Exercise 6.1. Let be a stable graph and for $v \in V(\ )$ let $_v$ be a stable graph of genus $g(v)$ with $n(v)$ legs together with an identification $h : L(\ _v) \to H(v)$ of the legs of $_v$ with the half-edges $H(v)$ of v in . Define the graph ^{0} obtained by gluing the $_v$ into the vertices of and show that it is a stable graph. Convince yourself that for curves $(C_v; p_{1;v}; \ldots; p_{n(v);v})_{v \in V(\ )}$ with stable graphs $_v$, the dual graph of the curve (67) is equal to ^{0}.

Instead of describing the gluing of the graphs $_v$ into explicitly, we define the notion of a morphism $^0 \to$ of stable graphs which makes precise the notion that ^{0} can be obtained from by gluing some graphs $_v$ at the vertices v of .

Definition 6.2. Let ; ^{0} be stable graphs of genus g with n legs. A morphism $' : {}^0 \to$ is defined by two maps[41]

$$'_V : V(\ ^0) \to V(\ ); \quad '_H : H(\ ) \to H(\ ^0); \tag{68}$$

satisfying the following conditions:

a) the map $'_H$ is injective,

[40] Bwahaha.

[41] Notice the direction of the maps: $'_V$ goes from vertices of ^{0} to vertices of while $'_H$ goes from half-edges of to half-edges of ^{0}!

b) $'_H$ sends edges of to edges of ^{0}

$$f\,h;h^0g\ 2\ E(\ )\ =\ )\ f\ \ '_H(h);'_H(h^0)g\ 2\ E(\ ^0):$$

Below we will denote by $'_E : E(\ )\ !\ \ E(\ ^0)$ the corresponding (injective) map of edges.

c) $'_H$ sends the legs of to the corresponding legs of ^{0}

$$`_0('_H(h)) = `\ (h) \text{ for } h\ 2\ L(\ )\ ;$$

d) the map $'_V$ is surjective and compatible with $'_H$

$$'_V(v_0('_H(h))) = v\ (h) \text{ for } h\ 2\ H(\ )\ ;$$

e) given $v_0\ 2\ V(\ )$, the preimage of v_0 under $'_V$ is a stable graph $^0_{v_0}$ of genus $g(v_0)$ with $n(v_0)$ legs. More precisely, the vertices $V_{v_0} = '^1_V(v_0)$ mapping to v_0 under $'_V$ together with the half-edges $H_{v_0} = v_0^1(V_{v_0})$ incident to these vertices and all edges $f\,h;h^0g\ 2\ E(\ ^0)\ n\ '_E(E(\ ))$ with $h;h^0\ 2\ H_{v_0}$ form a stable graph $_{v_0}$ and this graph has genus $g(v_0)$ and a number $n(v_0)$ of legs.

We illustrate a morphism $^0\,!$ of stable graphs in Figure 28.

Figure 28: The data of a morphism $^0\,!$ illustrated; this is the morphism coming from the gluing in Figure 27

Remark 6.3. a) The existence of a morphism $^0\,!$ is precisely equivalent to saying that ^{0} can be obtained from by gluing some stable graphs at the vertices of (and these are the stable graphs $^0_{v_0}$ appearing in part e) of De nition 6.2).

b) Given a morphism $' : {}^0 \to$, there exists a natural gluing morphism

$$' : \overline{M}_{{}^0} \to \overline{M} ;$$

where for each $v \in V( )$ the component of $'$ to the factor $\overline{M}_{g(v);n(v)}$ of $\overline{M}$ is given by the gluing map ${}^0_v$ (de ned on the factors of $\overline{M}_{{}^0}$ associated to vertices of ^{0} in $'_V{}^{1}(v)$). See Figure 30 for two examples of such gluing morphisms.

c) In the literature, a morphism ${}^0 \to$ is sometimes called a -structure on ^{0}. Other names you might nd are that ^{0} is a specialization of or that is a contraction of ^{0}.

d) You can check that there is a category whose objects are stable graphs of genus g with n legs and whose morphisms ${}^0 \to$ are as described in De nition 6.2. In particular there is a natural way to compose morphisms $'; '^0$ of stable graphs, by setting

$$(' \quad '^0)_V = '_V \quad '^0_V ; \ (' \quad '^0)_H = '^0_H \quad '_H :$$

You can also check that the notion of an isomorphism of stable graphs de ned in De nition 4.5 is equivalent to the notion of an isomorphisms of this category.[42] This makes precise the hint given in Exercise 4.18 *c).

Exercise 6.4. a) Given a stable graph ^{0} and a set $E_0 \quad E( )$ of edges of , show that there exists a stable graph and a morphism $' : {}^0 \to$ with $'_E(E( )) = E_0$. Show that for a second graph ^{e} and morphism $'e : {}^0 \to {}^e$ with $'e_E(E(^e)) = E_0$, there exists a unique isomorphism $\to {}^e$ tting into the diagram

$$\begin{tikzcd} & \\ {}^0 \arrow[ur, "'"] \arrow[dr, "'e"'] & \\ & {}^e \end{tikzcd}$$

In other words, the map ${}^0 \to$ is unique up to isomorphism. It is called the contraction of the edges in $E({}^0) \cap E_0$.

b) Show that every morphism ${}^0 \to$ can be factored as a composition

$${}^0 = \quad {}_0 \xrightarrow{'_1} \quad {}_1 \xrightarrow{'_2} \quad \xrightarrow{'_d} \quad {}_d =$$

of morphisms $'_1; : : : ; '_d$ such that each $'_i$ contracts a single edge of ${}_{i \ 1}$.

Proposition 6.5. Let be a stable graph in genus g with n legs. Then a curve $(C; p_1; : : : ; p_n) \in \overline{M}_{g;n}$ with stable graph ^{0} lies in the closed set $\overline{M}$ if and only if there exists a morphism ${}^0 \to$. In particular, we have

$$\overline{M} = \bigcup_{{}^0 \to} M_{{}^0} : \tag{69}$$

[42] Unfortunately, the map $'_H$ now goes in the opposite direction compared to the convention of De nition 4.5, sorry about that.

Proof. By Proposition 4.15, the set $\overline{M}$ is the image of the gluing morphism . By Exercise 6.1 the stable graphs ^{0} of curves in the image of are precisely those obtained by gluing stable graphs $_v$ into the vertices of and by Remark 6.3 a) this is equivalent to the existence of a morphism $^0 !$. □

Since the closures $\overline{M}$ are union of strata M^0, it is now easy to answer question b) above.

Corollary 6.6. Let $_1; _2$ be two stable graphs of genus g with n legs. Then we have

$$\overline{M}^1 \setminus \overline{M}^2 = \bigcup_{9\ ^0 ! \ ^0_1; \ ^0 ! \ _2} M^0; \tag{70}$$

where the union goes over stable graphs ^{0} admitting a morphism to $_1$ and $_2$.

Example 6.7. For $g = 1; n = 2$ we show in Figure 29 all isomorphism classes of stable graphs and which morphisms exist between them. Note the cases where there are two or four morphisms between these graphs (can you write them all down?). You can compare this to the picture of $\overline{M}_{1;2}$ from Figure 24 and check that these morphisms precisely tell you how the closures of the strata intersect. As an example of the statement of Corollary 6.6, we see that

$$\overline{M}^1 \setminus \overline{M}^2 = M^3:$$

Figure 29: The stable graphs $_0; : : : ; _4$ in genus $g = 1$ with $n = 2$ legs and morphisms between them. There are four morphisms $_4 ! \ _2$ uniquely determined by the image $_H(h)$ of one of the half-edges h of $_2$, which can map to each of the four half-edges of $_4$. Note that we did not draw automorphisms of the graphs.

We answered questions a) and b) above to our satisfaction, but in the following sections we will see that it's useful to answer a re ned version of question b).

b') Given two stable graphs $_1;\ _2$, what is the bre product

$$\begin{tikzcd} F_{_1;\ _2} \arrow[r] \arrow[d] & \overline{M}_{_2} \arrow[d, "_2"] \\ \overline{M}_{_1} \arrow[r, "_1"'] & \overline{M}_{g;n} \end{tikzcd}$$

Notice how I sneakily formulated question b') for the moduli stacks instead of the moduli spaces? There is a good reason for this, which we will see below. To answer the question, we will need to consider in more detail the stable graphs ^{0} and maps $^0 ! \quad _1;\ _2$ above.

De nition 6.8. Given stable graphs $_1;\ _2;\ $, a $(\ _1;\ _2)$-structure on is a tuple

$$(\ ;'_1;'_2) = (\ '_1 : \quad ! \quad _1;'_2 : \quad ! \quad _2) \tag{71}$$

of morphisms from to $_1$ and $_2$. The $(\ _1;\ _2)$-structure is called generic if

$$E(\ ) = \ '_{1;E}(E(\ _1))\ [\ '_{2;E}(E(\ _2)); \tag{72}$$

i.e. every edge $e \ 2\ E(\ )$ is the image of an edge in $_1$ or $_2$ under the morphisms $'_1;'_2$.

Given a second stable graph ^{0} with a $(\ _1;\ _2)$-structure $(\ ^0;\ ^0_1;\ ^0_2)$ we say that this structure is isomorphic to (71) if there exists an isomorphism $! \quad ^0$ tting into the diagram

$$\begin{tikzcd} & 1 & \\ & & 0 \arrow[ul, "'_1"] \arrow[dl, "'_2"'] \arrow[ll] \\ & 2 \arrow[ur, "^0_1"'] \arrow[ur, "^0_2"] & \end{tikzcd}$$

Denote by $G_{_1;\ _2}$ the set of generic $(_1;\ _2)$-structures $(\ ;'_1;'_2)$ up to isomorphism.

Example 6.9. In Figure 29, let $'_{3!\ 1} : \quad _3 ! \quad _1$ and $'_{3!\ 2} : \quad _3 ! \quad _1$ be morphisms as indicated (we have two choices for $'_{3!\ 2}$). Then

$$(\ _3;'_{3!\ 1};'_{3!\ 2})$$

is a generic $(_1;\ _2)$-structure on $_3$.

Now we can answer question b').

Theorem 6.10. Given stable graphs $_1;\ _2$ of genus g with n legs, the bre product

$$\begin{tikzcd} F_{_1;\ _2} \arrow[r] \arrow[d] & \overline{M}_{_2} \arrow[d, "_2"] \\ \overline{M}_{_1} \arrow[r, "_1"'] & \overline{M}_{g;n} \end{tikzcd} \tag{73}$$

of the gluing morphisms $_1;\ _2$ is given by

$$F_{_1;\ _2} = \coprod_{(\ ;'_1;'_2) 2 G_{_1;\ _2}} \overline{M} \ : \tag{74}$$

The restriction of the diagram (73) to the connected component $\overline{M}$ of $F_{1;\ 2}$ associated to $(\ ;'_1;'_2) \ 2\ G_{1;\ 2}$ is given by

$$\begin{tikzcd} \overline{M} \arrow[r, "'_2"] \arrow[d, "'_1"'] & \overline{M}_2 \arrow[d, "1"] \\ \overline{M}_1 \arrow[r, "1"'] & \overline{M}_{g;n} \end{tikzcd} \tag{75}$$

Proof. We are going to explain how the proof works on the level of $\mathbb{C}$-points. For the more general treatment of families of curves (which you need to de ne the isomorphism (74)), see e.g. [Sv18, Proposition 2.14].

What is a point in the bre product (73) above? It is the data of points

$$((C_v^0;(q_h^0)_{h2H(v)}))_{v2V(\ _1)}\ 2\ \overline{M}_1 \text{ and } ((C_v^{00};(q_h^{00})_{h2H(v)}))_{v2V(\ _2)}\ 2\ \overline{M}_2 \tag{76}$$

together with an isomorphism

$$_1((C_v^0;(q_h^0)_h)_v)! \qquad _2((C_v^{00};(q_h^{00})_h)_v) = (C;p_1;::::;p_n)\ 2\ \overline{M}_{g;n}: \tag{77}$$

Note that here we use that we take the bre diagram of stacks! If we had written everything with the coarse moduli spaces $\overline{M}_1;\overline{M}_2$ and $\overline{M}_{g;n}$, the data of a point in the bre product would be given by two points (76) such that there exists some isomorphism (77). For the stacky bre product, the isomorphism (77) is part of the data! For more on the slightly subtle de nition of stacky bre products, see [Fan01, Section 6.1].

We illustrate the data that we have so far on the left side of Figure 30 (we know so far the collections of curves on the bottom left and top right and how to identify their images under the gluing maps $_1$ and $_2$).

Figure 30: An illustration how starting from curves in $\overline{M}_1$ and $\overline{M}_2$ and an identi cation of their images under $_1;\ _2$ with $(C;p_1;::::;p_n)$, we construct a graph (on the right) and an element of $\overline{M}$ (top left)

Now observe that there is a particular subset $N_{\Gamma_1;\Gamma_2}$ of the nodes of the curve C which is the set of those nodes $n_{\{h;h'\}} \in C$ created either by identifying two markings $q^0_h; q^0_{h'}$ in the map Γ_1 (for $\{h;h'\} \in E(\Gamma_1)$) or from markings $q^{00}_h; q^{00}_{h'}$ under Γ_2 (for $\{h;h'\} \in E(\Gamma_2)$). These are the purple nodes in Figure 30 (which are the images of the red, green and blue nodes).

We claim that there is a unique stable graph Γ whose edges correspond to the nodes in $N_{\Gamma_1;\Gamma_2}$. It is the graph obtained from the dual graph of C by contracting all edges not corresponding to nodes of $N_{\Gamma_1;\Gamma_2}$ (according to Exercise 6.4). The vertices v of Γ correspond to the connected components C_v of the partial normalization of C at the nodes in $N_{\Gamma_1;\Gamma_2}$ (you see this partial normalization at the top left of Figure 30). The genus $g(v)$ is the arithmetic genus of the nodal curve C_v. The graph Γ has natural morphisms $\varphi_1 : \Gamma \to \Gamma_1$, $\varphi_2 : \Gamma \to \Gamma_2$. For instance, on the level of edges the morphism $\varphi_{1;E} : E(\Gamma_1) \to E(\Gamma)$ sends $\{h;h'\} \in E(\Gamma_1)$ to the edge of Γ corresponding to the node $n_{\{h;h'\}}$ as above. Clearly, the $(\Gamma_1;\Gamma_2)$-structure $(\Gamma;\varphi_1;\varphi_2)$ is generic: the edges of Γ were defined to correspond to nodes of the form $n_{\{h;h'\}}$, so each is either in the image of $\varphi_{1;E}$ or $\varphi_{2;E}$. Moreover, the $(\Gamma_1;\Gamma_2)$-structure $(\Gamma;\varphi_1;\varphi_2)$ we constructed is unique up to isomorphism.

Finally, the curves C_v (for $v \in V(\Gamma)$) together with all marked preimages q_h of nodes in $N_{\Gamma_1;\Gamma_2}$ and markings $p_1;\dots;p_n$ give an element

$$((C_v;(q_h)_{h\in H(\Gamma)}))_{v\in V(\Gamma)} \in \overline{M}_\Gamma : \tag{78}$$

To summarize, what we described above is how to start with the data (76, 77) of a point in $F_{\Gamma_1;\Gamma_2}$ and use it to construct $(\Gamma;\varphi_1;\varphi_2) \in G_{\Gamma_1;\Gamma_2}$ and the point (78) of $\overline{M}_\Gamma$. This recipe defines you a map from the left-hand side of (74) to the right. On the other hand, the maps $\varphi_1;\varphi_2$ together with the universal property of the fibre diagram $F_{\Gamma_1;\Gamma_2}$ define you a morphism $\overline{M}_\Gamma \to F_{\Gamma_1;\Gamma_2}$. In this way, you define a map from the right to the left side of (74). Using a finite amount of work (which you find in the proof of [Sv18, Proposition 2.14]) you can check that these maps are inverse to each other, finishing the proof. □

6.2 A crash course in intersection theory of complex algebraic varieties

In this section we give an overview of the intersection theory (formulated in the language of singular (co)homology) for algebraic varieties X over the complex numbers. For simplicity, we will formulate things in the setting where X is a smooth, proper variety.

In the end we want to apply this to $X = \overline{M}_{g;n}$, which is not a variety. Now it is possible to define singular cohomology groups for stacks (see these lecture notes by Behrend) and then one finds that e.g. for $X = \overline{M}_{g;n}$ the map $\overline{M}_{g;n} \to \overline{M}_{g;n}$ induces (via pushforward, see below) an isomorphism of cohomology groups $H^*(\overline{M}_{g;n}) \to H^*(\overline{M}_{g;n})$. So in the end it does not matter where you do your computations, but many of them are nicer on the smooth stack $\overline{M}_{g;n}$. However, instead of learning about cohomology groups of stacks properly, we will essentially pretend that they work just like those for schemes, with a few minor adaptions that we will point out. I realize this is not optimal, but it does work surprisingly well and in the references section I'll point out where you can learn how to fill in the missing pieces.

Singular homology and cohomology

Let X be a connected, smooth, proper variety over the complex numbers of complex dimension d. We define its singular homology[43] and singular cohomology groups as

$$H_\bullet(X) = H_\bullet(X(\mathbb{C}); \mathbb{Q}) \text{ and } H^\bullet(X) = H^\bullet(X(\mathbb{C}); \mathbb{Q});$$

where the set $X(\mathbb{C})$ of $\mathbb{C}$-points of X is equipped with the complex topology. The fact that X is smooth means that $X(\mathbb{C})$ actually has the structure of a smooth manifold, of dimension $\dim_{\mathbb{R}} X(\mathbb{C}) = 2d$. In particular, we have

$$H_k(X) = 0; \; H^k(X) = 0 \text{ for } k < 0 \text{ or } k > 2d:$$

The fact that X is defined over the complex numbers implies that $X(\mathbb{C})$ has a natural orientation.

Cap product and cup product

Given a homology class $\in H_k(X)$ and a cohomology class $\in H^\ell(X)$, we can form their cap product $_ \in H_{k-\ell}(X)$ and often say that acts on via this cap product. This gives rise to a perfect pairing

$$H_k(X) \quad H^k(X) \to H_0(X) = \mathbb{Q}; \qquad \mapsto _ ; \tag{79}$$

allowing us to identify $H^k(X) = H_k(X)^\vee$ in a natural way. Here the isomorphism $H_0(X) = \mathbb{Q}$ is given by the degree map

$$\deg : H_0(X) \to \mathbb{Q}; \quad \sum_{i=1}^N a_i [P_i] \mapsto \sum_{i=1}^N a_i; \tag{80}$$

where we use the connectedness of X to conclude that all points $P_i \in X(\mathbb{C})$ are homologous.

On the other hand, the group $H^\bullet(X)$ has a natural ring structure with multiplication given by the cup product $\wedge$. This product respects the cohomological grading, i.e. we have

$$H^k(X) \quad H^\ell(X) \to H^{k+\ell}(X); \qquad \mapsto \wedge :$$

The cap and cup-product satisfy the compatibility

$$_ \; (\wedge \quad) = (_ \quad) _ :$$

Remark 6.11. For Deligne-Mumford stacks such as $X = \overline{M}_{g;n}$, the degree map (80) needs to be slightly adapted. It turns out that for a point $P = (C; p_1; \ldots; p_n) \in \overline{M}_{g;n}$, the class $[P]$ of P should have degree

$$\deg([P]) = \frac{1}{\#\mathrm{Aut}(P)} = \frac{1}{\#\mathrm{Aut}(C; p_1; \ldots; p_n)};$$

and thus the degree map (80) becomes

$$\deg : H_0(X) \to \mathbb{Q}; \quad \sum_{i=1}^N a_i [P_i] \mapsto \sum_{i=1}^N \frac{a_i}{\#\mathrm{Aut}(P_i)}: \tag{82}$$

Example 6.12. For $n \quad 0$ consider the projective space $X = \mathbb{P}^n$. Its cohomology ring is isomorphic to

$$H^\bullet(\mathbb{P}^n) = \mathbb{Q}[H]/(H^{n+1}); \tag{83}$$

generated by $H \in H^2(\mathbb{P}^n)$. Thus $H^{2j}(\mathbb{P}^n) = \mathbb{Q} \quad H^j$ for $j = 0; \ldots; n$ and all other cohomology groups vanish. We'll see several interpretations for the generator H below.

[43] When talking about non-compact varieties, it is often more natural to consider the so-called Borel-Moore homology, see [Ful84, Chapter 19.1] for a discussion.

Poincaré duality

Our assumptions on X (connected, smooth, proper) imply that the cup-product of cycles of complementary dimension

$$H^k(X) \quad H^{2d\ k}(X) \ ! \quad H^{2d}(X) = Q; \qquad 7! \quad \wedge \tag{84}$$

is a perfect pairing. Here the isomorphism $H^{2d}(X) = Q$ is given by sending a $2d$-class $\ \ $ to

$$\int_X := \deg(\ \ ) := \deg([X] _ \ \ );$$

where $[X]$ is the fundamental class of X (see the next paragraph).

The pairing (84) allows us to identify $H^k(X) = H^{2d\ k}(X)^-$ and combining with the pairing (79), we have a natural isomorphism $H^k(X) = H_{2d\ k}(X)$. Tracing through the de nitions, it is easy to check that this isomorphism is given by

$$H^k(X) ! \quad H_{2d\ k}(X); \quad 7! \ [X] _ \ : \tag{85}$$

Example 6.13. In our example of $X = P^n$ the pairing (84) is given by

$$(Q \ \ H^j) \quad (Q \ \ H^{n\ j}) \ ! \quad Q \ \ H^n; (H^j) \quad (H^{n\ j}) \ 7! \quad H \ \ ^n$$

for $k = 2j \ \ 2 \ f \ 0; : : : ; 2ng$ even (and $H^k(P^n) = 0$ for k odd).

Fundamental classes of subvarieties

The fact that $X(C)$ is a connected, closed and oriented manifold allows us to de ne a fundamental class $[X] \ 2 \ H_{2d}(X)$, which is a generator of the one-dimensional vector space $H_{2d}(X)$.

This can be generalized to (not-necessarily smooth) subvarieties $Z \ \ X$. Any such Z admits a nite triangulation in which the singular locus is a subcomplex (see [EH16, Section C.2.1] and references there for details, see Figure 31 for a picture). This can be used to de ne a fundamental class $[Z] \ 2 \ H_{2e}(X)$, where $e = \dim_C Z$. Combining this with the Poincaré duality isomorphism (85) we obtain

$$[Z] \ 2 \ H^{2c}(X); \ \text{for } c = \operatorname{codim}_C Z = d \quad e: \tag{86}$$

Note that the fundamental class $[X] \ 2 \ H^0(X)$ is the neutral element for the cup product, i.e.

$$[X] \wedge \quad = \quad \text{for all} \quad 2 \ H \ \ (X):$$

If $Z \ \ X$ is not a subvariety, but a closed subscheme (i.e. possibly nonreduced), we can still de ne the cycle $[Z]$. For this, let $Z_1; : : : ; Z_r$ be the irreducible components of the reduced scheme Z^{red}, and for $i = 1; : : : ; r$ let

$$m_i = \text{length}_{O_{Z_i;Z}} O_{Z_i;Z}$$

be the multiplicity of Z at Z_i (see [Ful84, Appendix A.1] for a de nition, for $Z = V(f)$ a hypersurface the number m_i is just the order of vanishing of f at the generic point of Z_i). Then we de ne

$$[Z] = \sum_{i=1}^{r} m_i [Z_i] \ 2 \ H \ \ (X):$$

Example 6.14. The generator $H \ 2 \ H^2(P^n)$ from Example 6.12 is given by the fundamental class $H = [P^{n\ 1}]$ of any linear codimension 1 hyperplane $P^{n\ 1} \ \ P^n$. More generally, we have $H^j = [P^{n\ j}]$ for a linear codimension j subspace $P^{n\ j} \ \ P^n$. See Example 6.16 below for an argument why all linear subspaces $P^{n\ 1} \ \ P^n$ are homologous.

Figure 31: A triangulation of a complex variety X can be used to de ne a fundamental class as the sum over the singular simplices of the triangulation; the orientation of X tells you how the triangles need to be oriented, if X has singularities you can choose the triangulation in such a way, that the singular locus is a subcomplex

Proper pushforward and at pullback

Let $X; Y$ be connected, smooth, proper varieties of dimension d, and let $f : X \to Y$ be a morphism. Since the induced map $f : X(C) \to Y(C)$ is continuous, we have

a pushforward

$$f : H_k(X) \to H_k(Y) \tag{87}$$

of homology classes under f and

a pullback

$$f : H^{\ell}(Y) \to H^{\ell}(X) \tag{88}$$

of cohomology classes under f. Note that pullback is compatible with the cup product, i.e. $f(\ \wedge\ ) = f(\ ) \wedge f(\ )$.

Using the isomorphism of homology and cohomology groups from (85), we can also see the pushforward as a map

$$f : H^{\ell}(X) \to H^{\ell+2(e-d)}(Y): \tag{89}$$

Note that pushforward and pullback are *functorial*, i.e. for morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ we have

$$g(f\ ) = (g \circ f)\ , \text{ for } \ 2 H(X);$$
$$f(g\ ) = (g \circ f)\ , \text{ for } \ 2 H(Z):$$

The most important basic compatibility between those operations is the *projection formula*[44]. It says that for $\ 2 H(X)$ and $\ 2 H(Y)$ we have

$$f(f\ \wedge\ ) = \ \wedge f\ : \tag{90}$$

[44] Note that for arbitrary maps $f : X \to Y$ of topological spaces with $\ 2 H(X)$ and $\ 2 H(Y)$ it is true that $f(\ _ f\ ) = (f\ )_$ and the projection formula follows from this by inserting $= [X]_$.

For arbitrary topological spaces X, Y the story would end here. However, in our algebraic setting and assuming some additional properties of f, we can write down more explicitly how the maps f^*, f_* act on fundamental classes $[Z]$ of subvarieties of X, Y. A reference for the statements below is [Ful84, Chapter 1].

Assume[45] that $f : X \to Y$ is proper. In this case, given a subvariety $Z \subset X$ the image $Z' = f(Z) \subset Y$ is a subvariety of Y and we have

$$f_*[Z] = \begin{cases} \deg(Z/Z')\,[Z'] & \text{if } \dim Z = \dim Z', \\ 0 & \text{otherwise.} \end{cases} \tag{91}$$

Here $\deg(Z/Z')$ is the degree of Z over Z', which can be defined as the degree of the field extension $C(Z') \subset C(Z)$ induced by the restriction of f to Z. It is also the number of preimages in Z of a general point $z' \in Z'$.

On the other hand, for $f : X \to Y$ flat and $Z \subset Y$ a closed subvariety, let $f^{-1}(Z) = X \times_Y Z \to X$ be the closed subscheme[46] obtained by pullback via f. Then we have

$$f^*[Z] = [f^{-1}(Z)]; \tag{92}$$

where $[f^{-1}(Z)]$ is the fundamental class associated to the subscheme $f^{-1}(Z) \subset X$.

The operations of proper pushforward and flat pullback satisfy the following compatibility condition.

Proposition 6.15 (Proposition 1.7 in [Ful84]). Assume we have X, X', Y, Y' connected, smooth, proper and a fibre diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \tag{93}$$

with g flat and f proper. Then g' is flat and f' is proper and for all $\alpha \in H^*(X)$ we have

$$g^* f_*\alpha = (f')_*(g')^*\alpha \in H^*(Y').$$

Example 6.16. Given $F \in \mathbb{C}[X_0, \dots, X_n]_d$ a homogeneous degree d polynomial, let's show that the hypersurface $S = V(F) \subset \mathbb{P}^n$ cut out by F has class $[S] = dH$. For this, consider the universal hypersurface $\mathcal{H}$ over the space $\mathbb{P}^N = \mathbb{P}(H^0(\mathbb{P}^n; \mathcal{O}(d)))$:

$$\begin{tikzcd} \mathcal{H} = \{([F], p) \in \mathbb{P}^N \times \mathbb{P}^n : p \in V(F)\} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathbb{P}^n \\ \mathbb{P}^N & \end{tikzcd} \tag{94}$$

The variety $\mathcal{H}$ is smooth, projective and connected (since π_2 is a projective bundle) and π_1, π_2 are both flat and proper. Then we see that the composition $(\pi_2)_*(\pi_1)^*$ sends the class of a point $[F] \in H^{2N}(\mathbb{P}^N)$ to the fundamental class $[V(F)] \in H^2(\mathbb{P}^n)$ of its vanishing locus. Since $\mathbb{P}^N$ is connected, we have $[F] = [X_0^d] \in H^{2N}(\mathbb{P}^N)$ and thus

$$[V(F)] = (\pi_2)_*(\pi_1)^*[F] = (\pi_2)_*(\pi_1)^*[X_0^d] = [V(X_0^d)] = d\,[X_0] = d\,H \in H^2(\mathbb{P}^n).$$

[45] Given that we assume X, Y to be proper, this is actually automatic! I still write the condition since you can generalize this story to non-proper varieties, see [Ful84, Chapter 1.4].

[46] Note that indeed $f^{-1}(Z)$ can be non-reduced, e.g. for $f : \mathbb{P}^1 \to \mathbb{P}^1; [X_0 : X_1] \mapsto [X_0^2 : X_1^2]$ and $Z = V(X_0^2)$ we have $f^{-1}(Z) = V(X_0^2)$.

Chern classes of line bundles

Given a line bundle L on X, we can associate a cohomology class $c_1(L) \in H^2(X)$ called the *first Chern class* of L. While this makes sense for arbitrary complex line bundles on manifolds (see [BT82, Chapter IV]), in our situation the map is particularly easy.

$$\begin{array}{ccccc} \text{line bundles } L \text{ on } X & \longleftrightarrow & D = \sum_i a_i D_i \text{ divisors on } X & \longrightarrow & H^2(X) \\ \mathcal{O}_X(D) & \longleftarrow & D = \sum_i a_i D_i & \longrightarrow & \sum_i a_i [D_i] \end{array} \tag{95}$$

Easy exercise 6.17. Show that given line bundles L_1, L_2 we have

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2).$$

Show that the dual L^- of a line bundle L on X has first Chern class

$$c_1(L^-) = -c_1(L).$$

Higher Chern classes of vector bundles

The construction above can be generalized to vector bundles V of arbitrary ranks r, giving us Chern classes $c_k(V) \in H^{2k}(X)$ for $k = 1, \ldots, r$ (see [Ful84, Chapter 3]). We will only need the case $k = r$ applied to vector bundles V which are sums of line bundles $V = L_1 \oplus \cdots \oplus L_r$. In this case, the *top Chern class* is given by the cup product

$$c_{top}(V) = c_r(V) = c_1(L_1) \wedge \cdots \wedge c_1(L_r) \tag{96}$$

of the first Chern classes of the line bundles L_i.

Excess intersection formula

In the next section we will be interested in computing intersection products of cycles of the form $(\ )$ for a stable graph and $\in H(\overline{M}\ )$. The following (general) result will be essential for this. Recall that a *local complete intersection* (l.c.i.) is a morphism that can be factored into a regular embedding followed by a smooth morphism, see [Ful84, Appendix B.7.6]. If the domain of a morphism is smooth, this condition is automatic (in particular, the morphisms are local complete intersection morphisms (of stacks)).

Proposition 6.18 (Proposition 17.4.1 in [Ful84]). Assume we have X, X^0, Y, Y^0 connected[47], smooth, proper and a fibre diagram

$$\begin{array}{ccc} X^0 & \xrightarrow{g^0} & X \\ \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f} \\ Y^0 & \xrightarrow{g} & Y \end{array} \tag{97}$$

with g, g^0 l.c.i. morphisms of codimensions d, d^0 and f proper. Then for $\in H(X)$ we have

$$g^* f_* = (f^0)_* (c_{top}(E) \wedge (g^0)^*) \in H(Y^0), \tag{98}$$

where E is the rank $d - d^0$ bundle on X^0 given as the quotient

$$E = (f^0)^* N_{Y^0/Y} / N_{X^0/X} \tag{99}$$

of the pullback of the normal bundle $N_{Y^0/Y}$ of g by the normal bundle $N_{X^0/X}$ of g^0.

[47] It's easy to see how you can drop the assumption on X^0 being connected, and we will use the result in this form.

Concerning the de nition of the excess bundle E: we are slightly cheating here, since the de nition for general l.c.i. morphisms $g; g^0$ is slightly more involved (see [Ful84, Proposition 6.6]). However, iff; g are unrami ed the above de nition makes sense, where the (duals of the) normal bundles $N_{Y^0=Y}$ and $N_{X^0=X}$ can be de ned via

$$N_{\overline{Y}^0=Y} = g \quad {}^1_Y = {}^1_{Y^0} \text{ and } N_{\overline{X}^0=X} = (g^0) \quad {}^1_X = {}^1_{X^0}:$$

Example 6.19. Let $i : P^1 \ ! \ P^2$ be the inclusion of a line in P^2, then we know $i \ [P^1] = H$ and $H^2 = [pt]$ is the class of a point. Let's try to nd the most complicated way to prove this, by using Proposition 6.18. One sees that the bre diagram (97) becomes

$$\begin{tikzcd} P^1 \arrow[r, "id"] \arrow[d, "id"] & P^1 \arrow[d, "i"] \\ P^1 \arrow[r, "i"] & P^2 \end{tikzcd} \tag{100}$$

Then we obtain (using the projection formula (90) and the fact that the fundamental class $[P^1] \ 2 \ H^0(P^1)$ is the neutral element with respect to the cup product on P^1)

$$H^2 = (i \ [P^1]) \wedge \ (i \ [P^1]) = i \ (i \ i \ [P^1] \wedge \ [P^1]) = i \ (i \ i \ [P^1]) \tag{101}$$

By the excess intersection formula (98) we compute the term $i \ i \ [P^1]$ as

$$i \ i \ [P^1] = (id) \quad c_{top}(E) \wedge \ (id) \ [P^1] \ = c_{top}(E) \wedge \ [P^1] = c_{top}(E); \tag{102}$$

with $E = N_{P^1=P^2} = N_{P^1=P^1} = N_{P^1=P^2}$ the excess bundle. It remains to compute this normal bundle, associated to the inclusion $i : P^1 \ ! \ P^2$. We can make our lives easy and use the fact (see [Ful84, Example 2.5.5]) that for an e ective Cartier divisor $D \quad X$ we have $N_{D=X} = O_X(D)j_D$, and obtain

$$N_{P^1=P^2} = O_{P^2}([P^1])j_{P^1} = O_{P^2}(1)j_{P^1} = O_{P^1}(1) = O_{P^1}(pt): \tag{103}$$

If you instead like your life to be hard, you can also compute this normal bundle using the Euler sequence on P^2 (see Exercise 6.21 c)). In any case, we see

$$c_{top}(E) = c_1(O_{P^1}(pt)) = [pt] \ \ 2 \ H^2(P^1)$$

and pushing this forward via $i : P^1 \ ! \ P^2$ we indeed obtain $[pt] 2 \ H^4(P^2)$.

Exercise 6.20. Assume that $X; Y^0 \quad Y$ are smooth subvarieties of a connected, smooth, proper variety Y. Assume the intersection of $X; Y^0$ is transversal, i.e. for every $x^0 2 \ X \setminus Y$ we have

$$T_{x^0}Y = T_{x^0}X + T_{x^0}Y^0.$$

Show that the scheme-theoretic intersection $X \setminus Y^0 = X \quad {}_Y \ Y^0$ is reduced and of pure codimension $\operatorname{codim}_Y(X) + \operatorname{codim}_Y(Y^0)$. Conclude that

$$[X] \wedge \ [Y] = [X \setminus Y]:$$

In the example of $Y = P^2$ and $X; Y^0 \quad Y$ curves of degree $d; e$ meeting transversally, use this to show that $X; Y^0$ intersect in precisely $d \ e$ points (this is a variant of Bezout's theorem).

Exercise 6.21 (hard, but rewarding). The morphism

$$A^1 \to A^2; t \mapsto (t^2 - 1; t^3 - t)$$

extends to the normalization $f : P^1 \to E_0$ of the nodal cubic curve

$$E_0 = \{Y^2 Z = X^2(X + Z)\} \subset P^2;$$

see Figure 32.

a) Show that $f_*[P^1] = 3 \cdot H \in H^2(P^2)$ so that $(f_*[P^1])^2 = 9 \cdot H^2 = 9[pt] \in H^4(P^2)$.

b) Now let's show this the hard way. First, compute the fibre product $P^1 \times_{P^2} P^1$ of the map f with itself.

c) Use the Euler sequence on P^2 and the conormal exact sequence for the map f to show that the normal bundle $N_{P^1 \to P^2}$ of the map f has degree 7 on P^1.

d) Conclude that $(f_*[P^1])^2 = 9[pt] \in H^4(P^2)$.

Figure 32: The normalization of the nodal cubic curve E_0

6.3 The tautological ring of the moduli space of stable curves

Now we have all the ingredients we need to start our study of the cohomology groups of the moduli spaces of stable curves. Our first goal here is to write down some interesting cycle classes in $H^*(\overline{M}_{g;n})$. For this, we have a bunch of tools at our disposal.

As a modest start, we always have the fundamental class $[\overline{M}_{g;n}] \in H^0(\overline{M}_{g;n})$.

More generally, for every stable graph Γ we have the fundamental class

$$[\overline{M}_\Gamma] \in H^{2e}(\overline{M}_{g;n});$$

for $e = \#E(\Gamma)$, of the corresponding closed stratum in $\overline{M}_{g;n}$.

If we nd some natural line bundle[48] L on $\overline{M}_{g;n}$, we can take its Chern class

$$c_1(L) \ 2 \ H^2(\overline{M}_{g;n}):$$

Once we built a collection of classes $_i$ on various spaces $\overline{M}_{g_i;n_i}$, we can obtain even more classes by

- { taking cup products of existing classes,
- { taking pushforwards and pullbacks of (products of) classes, under the gluing morphisms and forgetful morphisms .

The next de nition makes precise what we mean by the system $RH \ (\overline{M}_{g;n}) \quad H \ (\overline{M}_{g;n})$ of classes that you can obtain using the ingredients above. Its history goes back to the original paper [Mum83] by Mumford, though the formulation presented below was rst given in [FP00] by Faber and Pandharipande. Due to the last point in the above list (we can combine classes $_i$ on di erent spaces $\overline{M}_{g_i;n_i}$ via the gluing and forgetful maps), it will be natural to de ne the sets $RH \ (\overline{M}_{g;n}) \quad H \ (\overline{M}_{g;n})$ simultaneously for all $g; n$.

De nition 6.22. The tautological rings $(RH \ (\overline{M}_{g;n}))_{g;n}$ are the smallest system of Q-subalgebras

$$RH \ (\overline{M}_{g;n}) \quad H \ (\overline{M}_{g;n})$$

containing the units $1 = [\overline{M}_{g;n}] \ 2 \ H^0(\overline{M}_{g;n})$ and which are closed under pushforward by all gluing morphisms

$$: \overline{M} \ = \ \prod_{v2V()} \overline{M}_{g(v);n(v)} \ ! \ \overline{M}_{g;n} \tag{104}$$

and all forgetful morphisms[49]

$$: \overline{M}_{g;n+1} \ ! \ \overline{M}_{g;n}: \tag{105}$$

The elements $\ 2 \ RH \ (\overline{M}_{g;n})$ are called *tautological classes*.

Remark 6.23. Let's look more closely at the various parts of the de nition and make a couple of comments. In particular, while it might look that the de nition omits some of the ingredients we mentioned above (e.g. the classes Δ []), we'll see that all of those are nonetheless contained in $RH \ (\overline{M}_{g;n})$.

a) The fancy word "Q-subalgebra" just means that $RH \ (\overline{M}_{g;n})$ is a Q-subvector space of $H \ (\overline{M}_{g;n})$ which is invariant under cup product. In other words, given tautological classes $; \quad$ and $\ 2 \ Q$, we have that

$$+ \ ; \qquad \text{and} \quad \wedge$$

are again tautological.

[48] A similar story works for natural vector bundles V and their Chern classes $c_{\cdot}(V) \ 2 \ H^{2\cdot}(\overline{M}_{g;n})$, though unfortunately we won't have time to see good examples of this.

[49] Note: instead of the classical morphism forgetting marking $n+1$, we must allow morphisms forgetting arbitrary markings i, i.e. de ned by $(C; p_1; \ldots; p_{n+1}) \ 7! \ (C; p_1; \ldots; p_{i \ 1}; p_{i+1}; \ldots; p_{n+1})$ when C is smooth.

b) Let's also expand what we mean by being "closed under pushforward by all gluing morphisms". For this let be a stable graph and assume we are given some cohomology classes $_v$ 2 H $(\overline{M}_{g(v);n(v)})$ for v 2 V( ). We can then form the class

$$= \prod_{v2V} (\ _v) \ _v; \text{ for the projections } \ _v : \overline{M} \ ! \ \overline{M}_{g(v);n(v)}: \tag{106}$$

In other words, we pull back the classes $_v$ from the factor $\overline{M}_{g(v);n(v)}$ to $\overline{M}$ and then take their cup product (this is sometimes called the *box product* of the classes $_v$). Then we require that if all $_v$ are tautological, the pushforward

$$(\ ) \ 2 \ H \ (\overline{M}_{g;n}) \tag{107}$$

is also tautological. Applying this to the fundamental classes $_v = [\overline{M}_{g(v);n(v)}]$, which are tautological by assumption, we have $= [\overline{M} \ ]$ is the fundamental class of $\overline{M}$. Combining the formula (91) for the proper pushforward of fundamental classes with Theorem 5.1 e), stating that has degree #Aut( ) onto its image $\overline{M}$, we see that

$$[\overline{M} \ ] = \frac{1}{\#\mathrm{Aut}(\ )} \ [\overline{M} \ ] \tag{108}$$

is a tautological class.

The above de nition of the tautological ring is short and elegant, but not very explicit (it does not give a convenient way to write down an arbitrary tautological class). Our main goal for the remainder of this section is to give an explicit, nite set of generators of RH $(\overline{M}_{g;n})$ as a Q-vector space. To state the corresponding result, we'll need two more ingredients : the so-called - and -classes. Their de nition uses in a crucial way the universal curve over $\overline{M}_{g;n}$.

$$\begin{array}{c} \overline{C}_{g;n} = \overline{M}_{g;n+1} \\ \downarrow \uparrow p_i \\ \overline{M}_{g;n} \end{array} \tag{109}$$

De nition 6.24. For i = 1; : : : ; n, the i-th cotangent line bundle L_i on $\overline{M}_{g;n}$ is de ned as

$$L_i = p_i \ ^1; \tag{110}$$

where ^{1} is the sheaf of relative di erentials for the morphism and $p_i : \overline{M}_{g;n} \ ! \ \overline{C}_{g;n}$ is the section of corresponding to the i-th marked point. We de ne the i-th -class

$$_i = c_1(L_i) \ 2 \ H^2(\overline{M}_{g;n}) \tag{111}$$

to be the rst Chern class of this line bundle.

To explain the name, consider Figure 33.

The preimage of $(C; p_1; : : : ; p_n)$ 2 $\overline{M}_{g;n}$ under is isomorphic to C, and for q 2 C a smooth[50] point of C, the sheaf ^{1} has bre

$$^1 j_q = T_q C:$$

[50] The assumption that q is a smooth point is important: the rank 1 sheaf ^{1} is not even locally free (i.e. a line bundle) at the nodes of C.

Figure 33: The fibre of the sheaf $\mathcal{L}^1$ at the image of the section p_i is equal to the cotangent space

Thus, pulling back $\mathcal{L}^1$ by setting $q = p_i$, we see that at a point $(C; p_1; \ldots; p_n) \in \overline{M}_{g;n}$ the line bundle L_i has fibre

$$L_i|_{(C;p_1;\ldots;p_n)} = T_{p_i}C$$

equal to the cotangent space of C at p_i. This is the kind of natural line bundle L on $\overline{M}_{g;n}$ we were talking about at the start of the section. And even though we did not mention the classes ψ_i in the definition of the tautological ring, it turns out that they are nonetheless contained in it.

Proposition 6.25. The ψ-classes ψ_i are tautological.

Proof. Remember from Proposition 4.25 that the morphisms p_i above are actually special cases of gluing morphisms (for the graph Γ_i having two vertices of genus 0, g connected by a single edge, with legs i, $n+1$ at the genus 0 vertex and all other legs at the genus g vertex). Thus we have

$$[\psi_i] = (p_i)_*[\overline{M}_{g;n}] \in RH^2(\overline{M}_{g;n+1}). \tag{112}$$

We claim that

$$\psi_i = \pi_*([\psi_i] \cap [\psi_i]), \tag{113}$$

which would finish the proof (the tautological ring is invariant under cup products and pushforwards by forgetful morphisms). To compute the cup product $[\psi_i] \cap [\psi_i]$ we use the excess intersection formula from Proposition 6.18. We start by computing the fibre product of p_i with itself. Since p_i is a section of the separated morphism, it is a closed embedding (see [Vak17, Exercise 10.1.M]). Thus the fibre product is just given by $\overline{M}_{g;n}$ itself.

$$\begin{tikzcd} \overline{M}_{g;n} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \overline{M}_{g;n} \arrow[d, "p_i"] \\ \overline{M}_{g;n} \arrow[r, "p_i"] & \overline{M}_{g;n+1} \end{tikzcd} \tag{114}$$

Alternatively, you can find the diagram (114) as a special case of Theorem 6.10 for $\psi_1 = \psi_2 = \psi_i$. In any case, since the normal bundle $N_{\overline{M}_{g;n};\overline{M}_{g;n}}$ of the identity is trivial,

the excess bundle E is given by the normal bundle $N_{\overline{M}_{g;n},\overline{M}_{g;n+1}}$ of the map p_i. But the dual of this bundle is precisely given by

$$N^{\vee}_{\overline{M}_{g;n},\overline{M}_{g;n+1}} = p_i^* \frac{1}{\overline{M}_{g;n+1}} = \frac{1}{\overline{M}_{g;n}} = p_i^* \quad {}^{1} = L_i : \tag{115}$$

Now we're in business and we can first apply the excess intersection formula to obtain

$$(p_i)_* [\ _i] = (p_i)_* (p_i)^* [\overline{M}_{g;n}] = c_1(N_{\overline{M}_{g;n},\overline{M}_{g;n+1}}) = c_1(L_i^{\vee}) = \quad c_1(L_i) = \quad {}_i ;$$

and then conclude as follows

$$\begin{aligned} ([\ _i] \wedge [\ _i]) &= \quad (p_i)_* [\overline{M}_{g;n}] \wedge [\ _i] = \quad (p_i)_* ((p_i)^* [\ _i]) \\ &= \quad (\quad p_i)_* \ _i = \quad (\mathrm{id}_{\overline{M}_{g;n}})_* \ _i = \quad {}_i : \end{aligned}$$

□

De nition 6.26. For $a \quad 0$ de ne the a-th $\ $-class $_a$ as the pushforward

$$_a = \quad ((\ _{n+1})^{a+1})_* \ 2 \ RH^{2a}(\overline{M}_{g;n}) : \tag{116}$$

Note that $(\ _{n+1})^{a+1}$ has cohomological degree $2(a+1)$ on $\overline{M}_{g;n+1}$ and since

$$\dim_{\mathbb{C}} \overline{M}_{g;n+1} \quad \dim_{\mathbb{C}} \overline{M}_{g;n} = 1 ;$$

we indeed have $_a$ in cohomological degree $2a$. Since $_{n+1}$ is tautological, it follows that $_a$ is tautological.

Now that we have $\ $- and $\ $-classes, we combine them with the gluing maps to obtain the generating set of $RH \ (\overline{M}_{g;n})$.

De nition 6.27. Let be a stable graph. A decoration on is a class $\ 2 \ H \ (\overline{M} \ )$ which is a product of $\ $- and $\ $-classes pulled back from the factors $\overline{M}_{g(v);n(v)}$ of $\overline{M} \ $. More formally, it is a class

$$= \prod_{v \in V(\ )} \ _v \ _v , \text{ for } \ _v = \ _{e_{v;1}} \ \ _{e_{v;\ell_v}} \ \ _{f_{v;1}} \ \ _{f_{v;n(v)}} \ 2 \ RH \ (\overline{M}_{g(v);n(v)}) : \tag{117}$$

Given and a decoration on , we de ne the decorated stratum class $[\ ;\ ]$ to be the pushforward

$$[\ ;\ ] = (\ \ ) \quad 2 \ RH \ (\overline{M}_{g;n}) : \tag{118}$$

Clearly, since all $_v$ are tautological, the de nition of the tautological ring implies that also the $[\ ;\ ]$ are tautological. Note that for dimension reasons we have $H^k(\overline{M}_{g(v);n(v)}) = 0$ for $k > 2(3g(v) \quad 3 + n(v))$. This implies that there are only nitely many nonzero classes $_v$ of the form above, since they vanish unless

$$\sum_j a_{v;j} \quad e_{v;j} + \sum_i f_{v;i} \quad 3g(v) \quad 3 + n(v) :$$

Note that since is of relative dimension equal to the number $e = \# \ E(\ )$ of edges of , we have $[\ ;\ ] \ 2 \ RH^{2(d+e)}(\overline{M}_{g;n})$ for $\ 2 \ H^{2d}(\overline{M} \ )$. We can represent a decorated stratum class by a picture of a stable graph decorated by powers of $\ $-classes at half-edges and a monomial in $\ $-classes at vertices, as illustrated in Figure 34.

Theorem 6.28. The decorated stratum classes $[\ ]$ form a nite generating set, as a $\mathbb{Q}$-vector space, of the tautological ring $RH \ (\overline{M}_{g;n})$.

Figure 34: A tautological class in $RH^{20}(\overline{M}_8)$

Proof. In Definition 6.22 we defined the tautological rings as the minimal system of subspaces of $H^{\bullet}(\overline{M}_{g;n})$ satisfying a bunch of properties. So, to conclude we need to show that the $\mathbb{Q}$-vector subspaces $S_{g;n} \subset H^{\bullet}(\overline{M}_{g;n})$ spanned by classes $[\Gamma, \alpha]$ have all these properties. Indeed, then they must contain the minimal system $RH^{\bullet}(\overline{M}_{g;n})$, but as we saw they are themselves contained in $RH^{\bullet}(\overline{M}_{g;n})$, proving equality.

Some parts are straightforward: the $S_{g;n}$ are closed under addition and scalar multiplication with elements of $\mathbb{Q}$, they contain the units $[\overline{M}_{g;n}]$ (taking Γ the trivial graph and $\alpha = 1$ the trivial product). Also, they are closed under pushforward by gluing maps ξ: given Γ and decorated classes $[\Gamma_w, \alpha_w] \in S_{g(w);n(w)}$ on the vertices $w \in V(\Gamma)$, we have

$$\xi_*\left(\prod_w [\Gamma_w, \alpha_w]\right) = [\Gamma', \alpha'], \tag{119}$$

where Γ' is the graph obtained from Γ by gluing in the Γ_w at vertices of Γ (see Exercise 6.1) and the decoration α' is obtained by distributing the decorations α_w to the vertices of Γ' (remember that the vertices of Γ' are the union of the vertices of all Γ_w). Thus we see that the class (119) is again a decorated stratum class.

The only parts of Definition 6.22 that require serious work are showing that the spaces $S_{g;n}$ are

a) closed under cup products,

b) closed under pullbacks by forgetful morphism $\pi : \overline{M}_{g;n+1} \to \overline{M}_{g;n}$.

We will prove part a) in Corollary 6.32 below, and you will show part b) in *Exercise 6.36. Thus, modulo these results, the proof is finished. □

For the proof of Corollary 6.32 we'll need a few more preparations.

Exercise 6.29. Prove that for Γ a stable graph, and $i = 1, \ldots, n$ we have

$$\psi_i = (\xi_v)^* \psi_h \in H^{\bullet}(\overline{M}_{\Gamma}), \tag{120}$$

where $h \in H(\Gamma)$ is the half-edge corresponding to the marking i, incident to vertex $v \in V(\Gamma)$. Likewise, for $a \geq 0$ show that

$$\kappa_a = \sum_{v \in V(\Gamma)} \xi_v^* \kappa_a \in H^{\bullet}(\overline{M}_{\Gamma}). \tag{121}$$

Hint: In particular for the statement about ψ-classes, you should have a look at Proposition 6.34 below.

For the proof of Proposition 6.31 below we will need one more fact, about the normal bundle for the gluing morphisms . It can be proved using deformation theory (we'll discuss this in the optional section about the proof of Theorem 5.1), but for now we'll have to take it as a black box.

Fact 6.30. Given a stable graph , the gluing morphism is unramified with normal bundle

$$N = \bigoplus_{\{h,h'\}\in E(\ )} L_{\bar{h}} \quad L_{\bar{h}'} \tag{122}$$

being the direct sum over the edges $\{h,h'\}$ of of the tensor products of the tangent bundles $L_{\bar{h}}, L_{\bar{h}'}$ associated to the half-edges h, h'.

You find the proof that the are unramified in [Knu83a, Corollary 3.9].

Proposition 6.31. Let $_1$ be a stable graph of genus g with n legs and let $[\ _2;\ ]$ be a decorated stratum class on $\overline{M}_{g,n}$. Then the pullback $_1[\ _2;\ ]$ is given by

$$_1[\ _2;\ ] = \sum_{(\ ;\ '_1;\ '_2)\in G_{1;\ 2}} (\ '_1)\quad '_2(\ )\quad _{ex}\ ; \tag{123}$$

where $_{ex}$ (depending on $(\ ;\ '_1;\ '_2)$) is the top Chern class of the excess bundle on $\overline{M}$ given by

$$_{ex} = \prod_{\{h,h'\}\in '_{1;E}(E(\ _1))\setminus '_{2;E}(E(\ _2))} (\quad h \quad h'): \tag{124}$$

In particular, the class $_1[\ _2;\ ]$ is contained in the tautological ring of $\overline{M}_{\ _1}$, i.e. a sum of terms

$$\prod_{v\in V(\ _1)} \quad _v \quad _v;\ \text{for}\ \ _v \in RH\ (\overline{M}_{g(v);n(v)}):$$

Proof. This result is an application of the excess intersection formula from Proposition 6.18. Indeed, the maps $_1;\ _2$ are proper and l.c.i. by Theorem 5.1. We computed their fibre product in Theorem 6.10 to be the disjoint union over spaces $\overline{M}$ for generic $(\ _1;\ _2)$-structures $(\ ;\ '_1;\ '_2)$. They fit in diagrams

$$\begin{tikzcd} \overline{M} \arrow[r, "'_2"] \arrow[d, "'_1"'] & \overline{M}_{\ _2} \arrow[d, "_2"] \\ \overline{M}_{\ _1} \arrow[r, "_1"'] & \overline{M}_{g;n} \end{tikzcd} \tag{125}$$

This explains the corresponding sum in the formula above. Applying Proposition 6.18, the only part of the result left to show is that $_{ex}$ is indeed the top Chern class of the excess bundle restricted to $\overline{M}$.

Using Fact 6.30, we have

$$N_{\ _1} = \bigoplus_{\{h,h'\}\in E(\ _1)} L_{\bar{h}} \quad L_{\bar{h}'}: \tag{126}$$

Using that $'_2$ is a product of gluing maps (see Remark 6.3 b)), for graphs whose edges correspond to edges in $E(\ )\ \cap\ '_{2;E}(E(\ _2))$, we obtain

$$N_{'_2} = \bigoplus_{\{h,h'\}\in E(\ )\ \cap\ '_{2;E}(E(\ _2))} L_{\bar{h}} \quad L_{\bar{h}'}: \tag{127}$$

Pulling back the normal bundle (126) under the map ι_1 and forming the quotient E of this pullback by (127), we obtain

$$E = \iota_1^* N_{\ } / N_{\iota_2} = \bigoplus_{\{h,h'\} \in E(\ ) \setminus E(\ )} L_{\bar{h}} \quad L_{\bar{h}'}. \tag{128}$$

The fact that $_{ex}$ is the top Chern class of this vector bundle follows from Easy exercise 6.17, using that $_h = c_1(L_h)$ and $_{h'} = c_1(L_{h'})$. Finally, it is also clear that the expression (123) is contained in the tautological ring of $\overline{M}$: the map ι_1 is a product of gluing maps, and by Exercise 6.29, the term $\iota_2^*(\ )$ $_{ex}$ is a combination of - and -classes on the factors of $\overline{M}$. □

Corollary 6.32. The product of two decorated stratum classes $[\ ;\ _1]$ and $[\ _2;\ _2]$ on $\overline{M}_{g;n}$ is tautological, and given by

$$[\ _1;\ _1]\ [\ _2;\ _2] = \sum_{(\ ;\ '_1;\ '_2) \in G_{\ _1;\ _2}} \ ;\ \iota_1^*(\ _1)\ \ \iota_2^*(\ _2)\ \ _{ex}\ ; \tag{129}$$

with $_{ex}$ as in (124).

Proof. By the projection formula we have

$$[\ _1;\ _1]\ [\ _2;\ _2] = (\ \ _1)\ \ _1\ [\ _2;\ _2] = (\ \ _1)\ ((\ \ _1)\ [\ _2;\ _2]\ _1):$$

Thus the result follows by taking the formula from Proposition 6.31, multiplying by $_1$, and pushing forward again. In the process we also use the projection formula for $_1$ together with the fact that $_1\ \ \iota_1^* = \ $. □

Exercise 6.33. Verify the computations in the tautological ring of $\overline{M}_{3;1}$ shown in Figure 35.

The formula from Corollary 6.32 has been implemented in the software package admcycles [DSv20]. You can check out some example computations here.

Proposition 6.34. Given a stable graph and $v \in V(\ )$ let $\overline{C}_v$ be the pullback

$$\begin{tikzcd} \overline{C}_v \arrow[r] \arrow[d] & \overline{C}_{g(v);n(v)} \arrow[d] \\ \overline{M}\ \ \ \

Figure 35: Products in the tautological ring of $\overline{M}_{3;1}$

where the square on the right is a bre diagram and the map is the map gluing the families $\overline{C}_v$ along sections corresponding to half-edges of . In particular, the map is surjective, proper, birational and an isomorphism over the locus in $\overline{C}$ where the map is smooth. In terms of fundamental classes, we have

$$\sum_v \overline{C}_v = \overline{C} : \tag{133}$$

Proof. Looking back at the proof (sketch) of Proposition 4.15 and using our new language of stacks, we can see that we de ned the morphism by constructing the family $\overline{C} \to \overline{M}$ of stable curves over $\overline{M}$. The fact that $\overline{M}_{g;n}$ is the moduli stack of stable curves then told us there is a unique map such that $\overline{C}$ is the pullback of the universal curve over $\overline{M}_{g;n}$. This explains the bre diagram on the right side of (132). But the construction of $\overline{C}$ precisely started with the disjoint union of curves $\overline{C}_v$ and glued them together under a map . This map is surjective and an isomorphism away from the locus of nodes in $\overline{C}$ (in particular, it is birational) and since all $\overline{C}_v$ are proper, the map is proper as well. For the equality (133) note that the $\overline{C}_v$ map birationally (in particular of generic degree 1) to the set of irreducible components of $\overline{C}$, so (133) follows from the de nition of proper pushforward[51]. □

See Figure 36 for an illustration of the diagram (132).

[51] Here we are lying a bit more than usual: the stack $\overline{C}$ is not smooth, so to make full sense of this statement one should use Chow groups (in the sense of [Ful84] or [Vis89]).

Figure 36: The commutative diagram (132) from Proposition 6.34 illustrated

Proposition 6.35. There exists a commutative diagram

$$\begin{tikzcd}
\overline{M}_{g;n+2} \arrow[dr, "G"] \arrow[drr, bend left, "{}_{n+2}"] \arrow[ddr, bend right, "{}_{n+1}"'] & & \\
& \overline{M}_{g;n+1} \;{}_{\overline{M}_{g;n}}\; \overline{M}_{g;n+1} \arrow[r] \arrow[d] & \overline{M}_{g;n+1} \arrow[d] \\
& \overline{M}_{g;n+1} \arrow[r] & \overline{M}_{g;n}
\end{tikzcd} \tag{134}$$

where ; $_{n+1}$ are the forgetful morphisms of marking $n+1$, $_{n+2}$ is the forgetful morphism of marking $n+2$ and the map G is proper and birational and in particular satis es

$$G\,[\overline{M}_{g;n+2}] = \left[\overline{M}_{g;n+1} \;{}_{\overline{M}_{g;n}}\; \overline{M}_{g;n+1}\right] : \tag{135}$$

Proof. The proof is very similar to the proof of Proposition 6.34, where now we use that the stable curve

$$\overline{M}_{g;n+1} \;{}_{\overline{M}_{g;n}}\; \overline{M}_{g;n+1} \;!\; \overline{M}_{g;n+1}$$

de ning the forgetful morphism $\;: \overline{M}_{g;n+1} \;!\; \overline{M}_{g;n}$ can be constructed by starting with the universal curve $\overline{M}_{g;n+2} \;!\; \overline{M}_{g;n+1}$ and contracting some components of its bres (which become unstable after forgetting the marking $n+1$) under the morphism G, as outlined in the proof of Proposition 4.25. □

*Exercise 6.36. Let $\;: \overline{M}_{g;n+1} \;!\; \overline{M}_{g;n}$ be the forgetful morphism of the marking $n+1$.

a) Show that

$$_i = \;_i\;[\;_i] \text{ and } \;_a = \;_a\;{}^a_{n+1}; \tag{136}$$

where $_i = (p_i)\;[\overline{M}_{g;n}]$ is again the section of associated to the i-th marked point. (Hint: You can show these formulas using Proposition 6.35 together with the formula (113) for the -class.)

b) Show that the pushforward

$$\begin{matrix} e_1 & & e & & f_1 & & f_n \\ a_1 & & a & & 1 & & n \end{matrix} \tag{137}$$

of a monomial in - and -classes is again a polynomial in- and -classes on $\overline{M}_{g;n}$. (Hint: Use the projection formula together with part a)).

c) Show that the pullback [;] of a decorated stratum class on $\overline{M}_{g;n}$ is a combination of decorated strata classes on $\overline{M}_{g;n+1}$.

d) Show that the pushforward [;] of a decorated stratum class on $\overline{M}_{g;n+1}$ is a combination of decorated strata classes on $\overline{M}_{g;n}$.

This exercise not only nishes the proof of Theorem 6.28, but together with Proposition 6.31 also shows that the tautological rings are invariant under pullback (not just pushforward) by gluing and forgetful morphisms.

To summarize the content of this section: we de ned the tautological ring $RH(\overline{M}_{g;n})$, a subring of the cohomology $H(\overline{M}_{g;n})$ of $\overline{M}_{g;n}$. It is genererated by decorated strata classes [;] and we obtained explicit formulas for

the product $[_1; _1] \wedge [_2; _2]$ of two decorated strata classes and

the pushforward and pullback of decorated strata classes under gluing morphisms and forgetful morphisms .

So inside the a priori mysterious cohomology ring $H(\overline{M}_{g;n})$ we have found a subring in which we can do computations, described purely in terms of combinatorics of stable graphs.

6.4 *A panorama of results about the tautological rings

The de nition of the tautological ring from the last section opens up a whole box of interesting questions: is every cohomology class on $\overline{M}_{g;n}$ tautological? Can we give the relations between the generators [ ] of $RH(\overline{M}_{g;n})$? While we won't have time to explore these in detail, I wanted to nish by giving you a bit of an overview of the eld.

The tautological ring in low genus

For $g = 0$ it is known that every cohomology class on $\overline{M}_{0;n}$ is tautological. In fact, it is shown in [Kee92] that the vector space $H(\overline{M}_{0;n})$ is generated by undecorated strata classes [; 1] and that all relations between these classes can be obtained from the WDVV-relation in $H^2(\overline{M}_{0;4})$, illustrated in Figure 37, using forgetful and gluing morphisms.

For $g = 1$ it was shown in [Pet14] that every cohomology class of even degree on $\overline{M}_{1;n}$ (i.e. those in $H^{2d}(\overline{M}_{1;n})$ for some d) is tautological, but it was known before that there are nonzero odd cohomology groups, the rst one being $H^{11}(\overline{M}_{1;11}) = Q$ (see [GP03, Section 4.1]). Since all tautological classes have even degree, this shows that there can be non-tautological cohomology classes on $\overline{M}_{g;n}$.

Finally, for $g = 2$ the paper [GP03] gives an (explicit) closed algebraic subset $\overline{B} \subset \overline{M}_{2;20}$ of complex codimension 11, such that its fundamental class $[\overline{B}] \in H^{22}(\overline{M}_{2;20})$ is not tautological. On the other hand, [Pet16] shows that this is the rst possible example in genus 2: all even cohomology classes on $\overline{M}_{2;n}$ are tautological for $n \quad 19$.

Figure 37: The WDVV relation on $\overline{M}_{0;4}$; can you see how to prove it?

Remark 6.37. In fact, we know (almost) everything to understand their proof that $[\overline{B}]$ is not tautological: for the boundary divisor gluing map

$$: \overline{M}_{1;11} \quad \overline{M}_{1;11} \to \quad \overline{M}_{2;20}$$

it is shown in [GP03] that the preimage of $\overline{B}$ under is precisely the diagonal $\overline{M}_{1;11} \quad \overline{M}_{1;11}$. Since this has the correct complex codimension 11, it follows from Proposition 6.18 that $([\overline{B}])$ is a nonzero multiple of $[\ ]$. But it is a general result from cohomology that the class $[\ ]$ of the diagonal has a formula

$$[\ ] = \sum_{e_i} e_i \quad e^i \ 2 \ H \ (\overline{M}_{1;11} \quad \overline{M}_{1;11});$$

where $(e_i)_i$ and $(e^i)^i$ are Poincare dual bases of $H \ (\overline{M}_{1;11})$. Since $H^{11}(\overline{M}_{1;11}) \ 6= 0$ by the remark above, we have a nonzero term from $H^{11}(\overline{M}_{1;11}) \quad H^{11}(\overline{M}_{1;11})$ appearing above. This shows that $[\overline{B}]$ is not tautological: otherwise, by Proposition 6.31 the pullback $[\overline{B}]$, a multiple of $[\ ]$, would have to lie in the tautological ring of $\overline{M}_{1;11} \quad \overline{M}_{1;11}$. But this only has terms coming from $RH^{2d_1}(\overline{M}_{1;11}) \quad RH^{2d_2}(\overline{M}_{1;11})$, so that no term from $H^{11}(\overline{M}_{1;11}) \quad H^{11}(\overline{M}_{1;11})$ could appear.

See [FP13] for more results about tautological and non-tautological cohomology classes on $\overline{M}_{g;n}$.

Tautological relations

From Theorem 6.28 we know that the decorated strata classes $[\ ]$ generate the tautological ring $RH \ (\overline{M}_{g;n})$ as a $\mathbb{Q}$-vector space. However, in general they will not form a basis, but there will be linear relations between them (as we saw in the example of the WDVV-relation in Figure 37). To verify if two tautological classes are equal, we need to understand these.

A rst set of relations (for the restrictions of the generators $[\ ;\ ]$ to the moduli space $M_g \quad \overline{M}_g$ of smooth curves, for $= 0$) was conjectured by Faber and Zagier and proved by Pandharipande and Pixton (see [FP13]). Later Pixton proposed a generalization of these relations to all $\overline{M}_{g;n}$ ([Pix12]). These have by now been veri ed to hold rst in cohomology ([PPZ15]) and later in the Chow rings[52] ([Jan17]). It is conjectured that the system of relations proposed by Pixton is complete, e.g. that it contains all tautological relations. However, as of now this conjecture is still quite open!

[52] If you know about Chow rings: analogous to De nition 6.22, the tautological rings $R \ (\overline{M}_{g;n})$ in Chow can be de ned as the smallest system of subrings of the Chow rings of $\overline{M}_{g;n}$ closed under pushforwards by gluing and forgetful maps. All the proofs we presented can still be carried out in this setting, in particular $R \ (\overline{M}_{g;n})$ is generated by decorated stratum classes $[\ ;\ ]$. The cycle class map $R \ (\overline{M}_{g;n}) \to RH \ (\overline{M}_{g;n})$ is surjective (essentially by de nition) and we do not know a single example where it is not injective.

The intersection pairing and integrals of -classes

While we have seen in Corollary 6.32 how to express the intersection products $[\ ;\ ][\ _1;\ _1]$ $[\ _2;\ _2]$ of tautological classes in terms of decorated strata classes $[\ ]$, we haven't yet seen how to compute the intersection pairing

$$RH^{2d}(\overline{M}_{g;n}) \quad RH^{2(3g\ \ 3+n\ \ d)}(\overline{M}_{g;n}) \ !\ \ Q;\ [\ _1;\ _1]\ \ [\ _2;\ _2] \ 7!\ \int_{\overline{M}_{g;n}} [\ _1;\ _1]\ \ [\ _2;\ _2]; \tag{138}$$

since this requires a formula for the degree

$$\int_{\overline{M}_{g;n}} [\ ;\ ] \ 2\ Q$$

of a decorated stratum class $[;\ ]$. Since $[\ ;\ ]$ is a pushforward of a product of monomials $_v$ in - and -classes on the factors $\overline{M}_{g(v);n(v)}$ of $\overline{M}$, it follows that

$$\int_{\overline{M}_{g;n}} [\ ;\ ] = \prod_{v2V(\ )} \int_{\overline{M}_{g(v);n(v)}} {}_v:$$

Thus we are reduced to computing integrals of the form

$$\int_{\overline{M}_{g;n}} {}^{e_1}_{1} \quad {}^{e_{\cdot}}_{\ \ `} \quad {}^{f_1}_{1} \quad {}^{f_n}_{n}: \tag{139}$$

But -classes are essentially forgetful pushforwards of powers of -classes, and using Exercise 6.36 b) you can verify that it is possible to write (139) as a linear combination of integrals

$$h\ {}^{f_1}\ \ {}^{f_N}\ i := \int_{\overline{M}_{g;N}} {}^{f_1}_{1} \quad {}^{f_N}_{N} \tag{140}$$

involving only -classes, for some N n. In [Kon92], Kontsevich proved an earlier conjecture of Witten about the intersection numbers (140), essentially giving a way to determine all of them recursively. See [Koc01, Section 3.3] for a nice introduction.

Tautological formulas for interesting cycle classes on $\overline{M}_{g;n}$

One of the nice things about the tautological ring is that many cohomology classes on $\overline{M}_{g;n}$ arising from some geometric construction happen to be tautological. In this case, we can find a formula for them, i.e. express them as a linear combination of classes $[\ ]$. Having such a formula makes it easier to perform computations with them (e.g. compute intersection numbers, compare them with other classes, etc.).

A first example are the so-called -classes. They are obtained from the Hodge-bundle E, a vector bundle of rank g on $\overline{M}_{g;n}$ whose fibres are given by

$$E|_{(C;p_1;\dots;p_n)} = H^0(C;\ {}^1_C); \tag{141}$$

for C smooth[53]. We define the -classes as the Chern classes

$$_i = c_i(E) \ 2\ H^{2i}(\overline{M}_{g;n}); \ \text{for } i = 1;\dots;g:$$

In [Mum83], Mumford uses the Grothendieck Riemann-Roch formula (a vast generalization of the classical Riemann-Roch formula we applied to line bundles on curves) to calculate

[53] For a nodal curve C, the cotangent sheaf ${}^1_C$ in (141) must be replaced by the canonical line bundle $!_C$.

an explicit formula for the $_i$, showing that they are indeed contained in the tautological ring.

A second method for obtaining interesting cohomology classes is to nd some algebraic subset of $\overline{M}_{g;n}$ and to take its fundamental class. A particularly nice example are the hyperelliptic loci $\overline{Hyp}_g \quad \overline{M}_g$. They are de ned as the (Zariski) closure of the locus

$$Hyp_g = f C : C \text{ is hyperelliptic} g \quad M_g \tag{142}$$

of smooth hyperelliptic curves[54]. It turns out that Hyp_g is an irreducible, closed algebraic subset of M_g of codimension $g \quad 2$, so that we obtain

$$[\overline{Hyp}_g] \; 2 \; H^{2(g \quad 2)}(\overline{M}_g): \tag{143}$$

For $g = 2$ we nd that every smooth genus 2 curve C is hyperelliptic, so that $[\overline{Hyp}_2] = [\overline{M}_2]$. For $g = 3$, the cycle $[\overline{Hyp}_3]$ was rst computed by Eisenbud and Harris in [EH87]. Their method was very simple: it was known that $H \; (\overline{M}_3) = RH \; (\overline{M}_3)$ and so by Poincare duality, the intersection pairing

$$RH^2(\overline{M}_3) \quad RH^{10}(\overline{M}_3) \; ! \quad Q$$

is perfect. Hence the class $[\overline{Hyp}_3] \; 2 \; RH^2(\overline{M}_{g;n})$ is uniquely determined by its intersection with a generating set of $RH^{10}(\overline{M}_3)$. Using di erent techniques, Faber and Pandharipande compute the class $[\overline{Hyp}_4]$ in [FP05] and in fact show that $[\overline{Hyp}_g]$ is tautological for all g! Finally, explicit formulas for $[\overline{Hyp}_5]$ and $[\overline{Hyp}_6]$ were found in the paper [Sv18] by myself and van Zelm, using essentially the same technique as Eisenbud and Harris, but now aided by our computer program [DSv20].

References and further reading (and rewatching)

A great reference for morphisms of stable graphs, bre products of gluing maps and the formula for the intersection product of tautological classes is [GP03, Appendix A]. See also Section 2 of my own paper [Sv18] for a slightly more detailed discussion of the same material.

Good references for intersection theory are the classical book [Ful84] by Fulton and a more modern treatment [EH16] by Eisenbud and Harris. Reading them, you will notice that what we do in Section 6.2 is actually just a slightly disguised version of algebraic intersection theory: the theory of the Chow groups $A \; (X)$ of an algebraic variety X. It can be formulated purely in terms of algebraic geometric notions (no need for a complex topology). Roughly, for X smooth, the group $A \; (X)$ is generated by classes $[S]$ of subvarieties $S \quad X$ and you divide by an equivalence relation called rational equivalence. These groups have an intersection product (de ned algebraically) and in the setting of Section 6.2 (X connected, smooth and proper and de ned over C) admit a ring homomorphism

$$cl : A \; (X) \; ! \quad H^2 \; (X)$$

sending the class $[S] \; 2 \; A \; (X)$ to the fundamental class $[S] \; 2 \; H^2 \; (X)$ we discussed above. Many of the operations we introduced (proper pushforward, at pullback, etc) can be de ned already on the level of Chow groups $A \; (X)$.

The theory of Chow groups has also been generalized to algebraic stacks (see [Vis89] for the treatment of Deligne-Mumford stacks and [Kre99] for more general algebraic stacks). This is the appropriate theory to use when studying intersection theory on $\overline{M}_{g;n}$.

[54] Recall that C is hyperelliptic if it admits a map $C \; ! \quad P^1$ which is generically of degree 2.

Starting July 6th 2020 there will be an online reading group "Intersection theory on stacks" organized by Reinier Kramer from the MPI Bonn. If you are interested, you can write an email to Reinier.

For a much more comprehensive overview about the tautological rings, see the survey paper [Pan18] by Pandharipande. Now that you are more familiar with the moduli spaces of curves, you can also consider rewatching his ICM lecture, which I recommended in Section 1.

7 *Ideas of proof for the main theorem

Until now, we have treated the two main theorems about the properties of the moduli spaces $\overline{M}_{g;n}$ and the moduli stacks $\overline{\mathcal{M}}_{g;n}$ (Theorems 3.19 and 5.1) essentially as black boxes, without discussing how to prove them. In this optional section, we'll try to see the main ingredients of the proofs. These require advanced techniques from di erent areas of modern algebraic geometry, each of which merits their own lecture course. So we'll focus on the global picture and give references which contain the details.

7.1 Construction of the coarse moduli space of curves

Let's start by explaining how to construct the moduli space $\overline{M}_g$ of curves of genus $g \quad 2$ using Geometric Invariant Theory (GIT). A reference for this section is [HM98, Section 4.C].

The basic idea is that we rst construct an auxilliary moduli space $\overline{K}_g$, parametrizing a stable curve C together with some additional data, then form a quotient of $\overline{K}_g$ by the action of an algebraic group, dividing out all possible choices of this extra data. We'll explain most of the story starting with the case of smooth curves C, and in the end comment what needs to be done for stable curves. Fixing an integer $k \quad 5$, the auxilliary space for the moduli M_g of smooth curves is

$$K_g = \quad (C; ' \ : C \ ! \ \ P^r) : \ , \ \begin{array}{c} C \text{ smooth, irreducible curve of genus } g \\ \text{non-degenerate embedding; } O_{P^r}(1) = \ ! \ _C{}^k \end{array} \quad ;$$

where $! \ _C = \ ^1_C$ is the canonical line bundle of C and

$$r = (2k \quad 1)(g \quad 1) \quad 1:$$

Also recall that $C \quad P^r$ is non-degenerate if C is not contained in any hyperplane of P^r.

Now why should we like K_g better than M_g? The reason is that K_g parametrizes subschemes $C \quad P^r$ of the xed variety P^r and Grothendieck ([Gro61]) constructed[55] a ne moduli space $\mathrm{Hilb}(P^r)$, the Hilbert scheme of P^r, parametrizing all subschemes of P^r. Then K_g is simply a locally closed subscheme[56] of $\mathrm{Hilb}(P^r)$ and forms a ne moduli space parametrizing tuples $(C; ' \ : C \ ! \ \ P^r)$ as above.

Now let's see how to recover M_g from K_g. First we have

$$\deg(! \ _C{}^k) = 2k(g \quad 1):$$

Since $k \quad 5$, the line bundle $! \ _C{}^k$ has degree greater than $2g + 1$ and thus it is very ample ([Vak17, Section 19.2.11]), its rst cohomology vanishes ([Vak17, Section 19.2.5]) and we have

$$h^0(C; ! \ _C{}^k) = 2k(g \quad 1) + 1 \quad g = r + 1:$$

Thus the morphism $' \ : C \ ! \ \ P^r$ is a non-degenerate embedding with $O_{P^r}(1) = \ ! \ _C{}^k$ if and only if $'$ is given by a complete linear system of $! \ _C{}^k$. In other words (see also Example 2.4), the data of $'$ is equivalent to the choice of a basis $s_0; : : : ; s_r \ 2 \ H^0(C; ! \ _C{}^k)$

[55] If you are unhappy that we construct the space M_g by using the existence of yet another moduli space $\mathrm{Hilb}(P^r)$: after a long and technical proof, the existence of the Hilbert scheme follows from the existence of the Grassmannian variety $Gr(m; n)$, and this can really be constructed by hand, gluing a ne spaces along explicit open subschemes.

[56] Roughly, being smooth, irreducible and non-degenerate is an open condition and satisfying $O_{P^r}(1)|_C = \ ! \ _C{}^k$ is a closed condition.

up to scaling. The fact that ω_C^k is very ample automatically implies that any such ι is a closed embedding. Thus we see

$$K_g = \left\{ (C; [s_0 : \cdots : s_r]) : \begin{array}{c} [s_0 : \cdots : s_r] \in \mathbb{P}(H^0(C; \omega_C^k)^{r+1}) \\ s_0, \ldots, s_r \text{ form a basis of} \\ H^0(C; \omega_C^k) \end{array} \right\}.$$

The group PGL_{r+1} acts on K_g by leaving the curve C invariant and acting in a natural way on the tuples $[s_0 : \cdots : s_r]$. This action comes from a bigger action $PGL_{r+1} \curvearrowright \mathrm{Hilb}(\mathbb{P}^r)$ associated to the natural action $PGL_{r+1} \curvearrowright \mathbb{P}^r$. Inside K_g, for a fixed C this action is simply transitive on the set of all possible choices $[s_0 : \cdots : s_r]$ (any two choices of basis are related by a unique base change in PGL_{r+1}). Thus, if we manage to define a reasonable notion of "quotient" in algebraic geometry we should have

$$K_g / PGL_{r+1} = M_g.$$

The fact that K_g indeed admits such a quotient which is again a scheme can be proved using Geometric Invariant Theory. This is a theory developed by Mumford (see [MFK94]) to prove the existence of such quotients under suitable conditions. Checking these conditions for K_g is the hard technical core of the proof, which we will not discuss further.

To go from the case of smooth curves to the case of stable curves, we should take ω_C above to be the canonical sheaf of C (see [Vak17, Chapter 30]). The canonical sheaf is a coherent sheaf associated to any projective variety. For a nodal, connected projective curve C it turns out to be a line bundle ω_C and it has the amazing property that ω_C is ample if and only if C is stable. Moreover, in this case ω_C^k is very ample for $n \geq 3$. Then we can define $\overline{K}_g \subset \mathrm{Hilb}(\mathbb{P}^r)$ similar as above and the argument still works.

Several properties of the coarse moduli space $\overline{M}_g$ now follow from this construction. For instance, the fact that each component of the Hilbert scheme is projective together with the machinery of Geometric Invariant Theory implies that the quotient $\overline{M}_g = \overline{K}_g / PGL_{r+1}$ is in fact also a projective variety.

The above construction is also interesting from the perspective of algebraic stacks: it turns out that $\overline{K}_g$ is smooth (this can be shown using methods from Section 7.2 below, see [HM98, Claim (4.39)]) and the morphism $\overline{K}_g \to \overline{M}_g$ is representable, smooth and surjective (see [DM69, Section 5]). So the construction of $\overline{K}_g$ is part of the proof that $\overline{M}_g$ is an algebraic stack. Moreover, this atlas $\overline{K}_g \to \overline{M}_g$ shows that the stack $\overline{M}_g$ is smooth since $\overline{K}_g$ is smooth.

7.2 Dimension and smoothness

To understand why the stack $M_{g;n}$ has dimension $3g - 3 + n$, a powerful tool is given by deformation theory. This is a theory developed for understanding the local structure (e.g. dimension and smoothness) of moduli spaces or stacks M. Its core idea is to consider the functor of points of M on schemes of the form $\mathrm{Spec}(A)$ for A a local Artinian[57] $\mathbb{C}$-algebra with residue field $\mathbb{C}$. The simplest such algebra (apart from $\mathbb{C}$ itself) is given by the dual numbers

$$A = \mathbb{C}[\varepsilon]/(\varepsilon^2).$$

The following exercise shows that for the dual numbers A, understanding the A-points of M (for M a scheme) allows us to understand the Zariski tangent spaces at points of M.

[57] Recall that a finitely generated $\mathbb{C}$-algebra is Artinian if it is of finite dimension as a $\mathbb{C}$-vector space.

Exercise 7.1 (see Exercise 12.1.I in [Vak17]). Let M be a complex scheme and let $p : \mathrm{Spec}(\mathbb{C}) \to M$ be a $\mathbb{C}$-point of M. Show that the elements of the Zariski tangent space

$$T_p M = (m_{M,p}/m_{M,p}^2)^{-}, \text{ with } m_{M,p} \subset \mathcal{O}_{M,p} \text{ the maximal ideal}$$

are in bijection with the space

$$\mathrm{Mor}_{\mathbb{C}}(\mathrm{Spec}(\mathbb{C}[\ ]/(\ ^2)), M)_p$$

of morphisms $\mathrm{Spec}(\mathbb{C}[\ ]/(\ ^2)) \to M$ such that the composition with

$$\mathrm{Spec}(\mathbb{C}[\ ]/(\ ^2)) \to \mathrm{Spec}(\mathbb{C}), \quad \mapsto 0$$

equals the inclusion $p : \mathrm{Spec}(\mathbb{C}) \to M$.

So let's try to see why for a smooth curve $C \in M_g$ (for $g \geq 2$) the tangent space $T_C M_g$ has dimension $3g - 3$. By the exercise, elements of this tangent space correspond to maps $\mathrm{Spec}(A) \to M_g$ with image $\{C\}$, for $A = \mathbb{C}[\ ]/(\ ^2)$. Since M_g is a moduli stack of smooth genus g curves, these are equivalent to families $\mathcal{C} \to \mathrm{Spec}(A)$ of smooth genus g curves such that the pullback $C_0 \to \mathrm{Spec}(\mathbb{C})$ by the closed embedding $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(A)$ is isomorphic to C.

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, hook] & \mathrm{Spec}(A) \end{tikzcd} \tag{144}$$

Actually, this is a special case of a so-called *first order deformation*, and it makes sense to treat it in some more generality. For this, let X be a complex variety, then a first order deformation of X is a flat morphism $\mathcal{X} \to \mathrm{Spec}(A)$ together with an identification $\mathcal{X}_0 = X$ of X with the fibre $\mathcal{X}_0$ of $\mathcal{X}$ over $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(A)$. We illustrate the situation in Figure 38.

To understand the set of first-order deformations of a smooth variety X, we'll make a bunch of observations.

a) Since $(\ ) \subset A$ is the unique prime ideal, the underlying topological space of $\mathrm{Spec}(A)$ is a single point, and $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(A)$ is an isomorphism on the level of topological spaces. Similarly, since $\mathcal{X}_0 \subset \mathcal{X}$ is cut out by the square-zero ideal $(\ )$, it is not difficult to see that the morphism $\mathcal{X}_0 \to \mathcal{X}$ is also an isomorphism of topological spaces. So the Zariski-open subsets $\mathcal{U}$ of $\mathcal{X}_0$ are in bijection with the open subsets U of X. Moreover, it is true that U is affine if and only if $\mathcal{U}$ is affine (see [Ser06, Lemma 1.2.3]).

b) It turns out that first-order deformations $\mathcal{U} \to \mathrm{Spec}(A)$ of smooth, affine schemes U are very simple: each such deformation is isomorphic to the trivial deformation $\mathcal{U} = U \times \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ (see [Ser06, Theorem 1.2.4]). Thus if we choose an affine open cover U_i of X, each element of the corresponding affine cover $\mathcal{U}_i$ of $\mathcal{X}$ will be the trivial deformation $\mathcal{U}_i = U_i \times \mathrm{Spec}(A)$ of U_i. Note that this is independent of the particular first-order deformation $\mathcal{X}$: we can first choose the cover $\mathcal{U}_i$ of $\mathcal{X}$ and know that every deformation $\mathcal{X}$ will be trivial on each of the patches $\mathcal{U}_i$ of $\mathcal{X}$.

   Thus all the information of the deformation $\mathcal{X} \to \mathrm{Spec}(A)$ is contained in the data of the isomorphisms

$$\varphi_{ij} : \mathcal{U}_{ij} \times \mathrm{Spec}(A) \to \mathcal{U}_{ij} \times \mathrm{Spec}(A) \tag{145}$$