

MONOIDAL AND ENRICHED DERIVATORS

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ABSTRACT. In this paper, we develop the theory of monoidal derivators and the related notions of derivators being tensored, cotensored, or enriched over a monoidal derivator. The passage from model categories to derivators respects these notions and hence gives rise to natural examples. We also introduce the notion of the center of additive derivators which allows for a convenient formalization of linear structures on additive derivators and graded variants thereof in the stable situation. As an illustration, we discuss some derivators related to chain complexes and symmetric spectra.

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0. INTRODUCTION

In this paper, we develop the monoidal (cf. also to [Cis08]) and enriched aspects of the theory of derivators. As we saw in the companion paper [Gro10a], two important classes of derivators are given by the derivators associated to combinatorial model categories and the derivators represented by bicomplete categories. Both classes of examples can be refined to give corresponding statements about situations where the ‘input is suitably monoidal, tensored, cotensored, or enriched’. We formalize the notions of monoidal, tensored, cotensored, and enriched (pre)derivators and make these statements precise. The author is not aware of a place in the literature where linear structures on a derivator are considered. It is for this purpose that we introduce the notion of an additive derivator and of the center of a derivator. This latter notion gives a compact definition of a derivator which is linear over some ring (and a graded variant thereof in the stable situation).

It is well-known that the homotopy categories of (combinatorial) monoidal model categories (in the sense of Hovey [Hov99], as opposed to the slightly different notion of [SS00]) can be canonically endowed with monoidal structures and similarly for suitably monoidal Quillen adjunctions. These statements are truncations of more structured results as we will see below. Once we define a monoidal derivator as a monoidal object in the Cartesian monoidal 2-category Der of derivators we will show that the derivator associated to a combinatorial monoidal model category can be canonically endowed with a monoidal structure. These results generalize to model categories which are suitably (co)tensored over a monoidal model category. As a consequence of the general theory, we show that the 2-categories of prederivators and derivators are Cartesian closed monoidal. As expected, we see that the internal hom $\text{HOM}(\mathbb{D}, \mathbb{D})$ coming from this Cartesian structure gives the universal example of a derivator acting on \mathbb{D} .

Since the passage from combinatorial model categories to derivators respects monoidal structures we obtain a very conceptual explanation for the existence of linear structures on certain naturally occurring derivators. In fact, the linear structures are obtained by specializing to a small part of the structure available on a derivator \mathbb{D} which is left tensored over a monoidal derivator \mathbb{E} : under suitable additivity assumptions the left action restricts to an algebra structure on the center $Z(\mathbb{D})$. We also have a corresponding result in the stable situation where one has to add some exactness assumptions and where the outcome is a graded-linear structure on the stable derivator. As special cases we obtain, e.g., that the derivator of spectra is linear over the stable homotopy groups of spheres and that the derivator of chain complexes is linear over the ground ring. It is easy to extend this in both cases to modules over commutative monoids. We also have such a result for modules over non-commutative monoids. In these cases the derivators are linear over the homotopy groups of the topological Hochschild cohomology of the ring spectrum resp. over the Hochschild cohomology of the differential-graded algebra. These examples were our original motivation for studying these questions.

Since we are restricting attention to combinatorial model categories, hence presentable categories, the special adjoint functor theorem (SAFT) of Freyd can in certain situations be applied to deduce that there is canonically more structure available. For example, let us assume that we have a combinatorial model category \mathcal{M} which is left tensored over a combinatorial monoidal model category \mathcal{N} such that the action preserves colimits separately in each variable. Then it is a consequence of SAFT that the category \mathcal{M} is also cotensored and enriched over \mathcal{N} . To capture these additional structures at the level of derivators, we also introduce the notion of cotensored and enriched derivators and establish the relevant examples.

Along the development of the theory, we will see that the notions introduced below extend the corresponding ones from classical category theory. The 2-functor which sends a bicomplete

category to the represented derivator is a faithful 2-functor and one should guarantee that the notions introduced here are compatible with the ones from classical category theory. For example, it is straightforward to see that a bicomplete category is additive if and only if the represented derivator is additive. There are similar observations for the other notions we discuss in this paper.

We now turn to a description of the content by sections. In Section 1, we begin by considering the Cartesian monoidal 2-categories of derivators and prederivators. We introduce the notion of bimorphisms between derivators and remark that the Cartesian product corepresents this bimorphism functor. We then consider the basic notions of monoidal (pre)derivators, monoidal morphisms, and monoidal transformations between such which are organized in the 2-categories MonPDer and MonDer . We define a monoidal prederivator by making explicit the notion of a monoidal object in the Cartesian 2-category of prederivators. One then remarks that monoidal prederivators can be identified with 2-functors $\text{Cat}^{\text{op}} \rightarrow \text{CAT}$ which factor over the 2-category MonCAT of monoidal categories. Since the 2-functor which sends a category to the associated represented prederivator preserves 2-products, we obtain an induced 2-functor $\text{MonCAT} \rightarrow \text{MonPDer}$. We then show that derivators associated to combinatorial monoidal model categories can be canonically endowed with monoidal structures. This is done by showing, more generally, that a Brown functor between model categories (cf. Definition 1.15) induces a morphism of associated derivators. Some relevant examples related to simplicial sets, chain complexes, and symmetric spectra are given, before we turn, in the last subsection, to the center $Z(\mathbb{D})$ of a derivator. This notion allows for a convenient formalization of linear structures on a derivator. We establish the result that suitably additive monoidal derivators are linear over the ring of self-maps of the monoidal unit of the underlying monoidal category.

In Section 2, we turn to prederivators tensored or cotensored over monoidal prederivators. We begin with a short technical subsection in which we construct the 2-Grothendieck fibration of tensored categories. In the next subsection, we introduce the notions of tensored and cotensored derivators as certain module objects. We show that the Cartesian monoidal 2-categories PDer resp. Der of prederivators resp. derivators are closed and that the internal hom $\text{HOM}(\mathbb{D}, \mathbb{D})$ together with the canonical action on \mathbb{D} provides the universal example of a module structure on \mathbb{D} . The latter part is, in fact, a special case of a general 2-categorical statement which we prove as Theorem B.11. In the last subsection we give some interesting examples. We show that if a combinatorial model category \mathcal{M} is tensored over a combinatorial monoidal model category \mathcal{N} , then the derivator $\mathbb{D}_{\mathcal{M}}$ associated to \mathcal{M} is canonically tensored over $\mathbb{D}_{\mathcal{N}}$. The result of Section 1 on the linear structures on suitably additive monoidal derivators can be generalized to the situation of a suitably tensored additive derivator.

In Section 3, we introduce the notion of derivators enriched over a monoidal derivator. In order to have a compact definition of such a gadget we start by considering the 2-Grothendieck opfibration of enriched categories. Elaborating a bit on the fact that enriched category theory admits base change along monoidal functors we obtain the 2-category of enriched categories. In the next subsection, we use this to give a compact definition of an enriched derivator. Our main source of examples of enriched derivators is the following result (Theorem 3.10): an action of a monoidal derivator \mathbb{E} on a derivator \mathbb{D} which is part of an adjunction of two variables exhibits the derivator \mathbb{D} as being canonically enriched over \mathbb{E} . This is, in particular, the case for closed monoidal derivators. In the last subsection, we show that if a combinatorial model category \mathcal{M} is suitably tensored over a combinatorial monoidal model category \mathcal{N} then the associated derivator $\mathbb{D}_{\mathcal{M}}$ is canonically enriched over $\mathbb{D}_{\mathcal{N}}$. We close by mentioning some derivators related to chain complexes and symmetric spectra as more specific examples of enriched derivators.

Finally, in the appendices we recall and establish some 2-categorical notions and results. In Appendix A, we quickly recall the classical Grothendieck construction associated to a category-valued functor. We then give a variant thereof in the 2-categorical setting. These constructions are used in Section 2 and Section 3. Appendix B has two subsections. In the first one, we recall the notions of monoidal objects and modules in monoidal 2-categories and construct the 2-category of all modules using the 2-categorical Grothendieck construction. In the second subsection, we show that in a closed monoidal 2-category the canonical actions of internal endomorphism objects give us the terminal module structures (in a bicategorical sense, cf. Theorem B.11). This allows us to put the results on the linear structures of Section 1 and Section 2 into perspective.

Before we begin with the proper content of this paper let us make two more comments. The first comment concerns set-theoretical issues. In what follows we will frequently consider the ‘category of categories’ and similar gadgets. Strictly speaking these are not honest categories in the sense that they would be locally small, i.e., have hom-sets as opposed to more general hom-classes. These problems could be circumvented by a use of Grothendieck’s language of universes. Since we do not wish to add an additional technical layer to the exposition by keeping track of the different universes we decided to ignore these issues.

The second remark concerns the different kinds of ‘hom-objects’ which will show up frequently. Let \mathcal{C} be a category and let $X, Y \in \mathcal{C}$ be two objects. The set of categorical morphisms from X to Y will be denoted by $\text{hom}_{\mathcal{C}}(X, Y)$. If the category \mathcal{C} is enriched over a monoidal category \mathcal{D} , we will usually write $\text{Hom}_{\mathcal{C}}(X, Y) \in \mathcal{D}$ for the enriched hom-objects. Finally, in the case of a closed monoidal category \mathcal{D} and two objects $X, Y \in \mathcal{D}$, the internal hom will be denoted by $\text{HOM}_{\mathcal{D}}(X, Y) \in \mathcal{D}$. The author is aware of the fact that these three situations are of course not disjoint but we will apply these conventions as a rule of thumb.

1. MONOIDAL DERIVATORS

1.1. **The Cartesian monoidal 2-categories Der and PDer.** For the basic notions of the theory of 2-categories we refer to [Bor94a, ML98, Kel05b] which will be used more systematically here than in the companion paper [Gro10a]. In order to establish some notation we begin by quickly recalling the definitions of a prederivator and morphisms of prederivators. By contrast, the notion of a derivator will not be recalled and we refer instead to [Gro10a]. Original references for derivators are [Gro, Hel88]. Other references for the theory of derivators and stable derivators include [Fra96, Kel91, Mal07a, Mal01, CN08].

We recall that a *prederivator* is a 2-functor $\mathbb{D}: \mathbf{Cat}^{\text{op}} \rightarrow \mathbf{CAT}$ where \mathbf{Cat} denotes the 2-category of small categories and \mathbf{CAT} denotes the 2-category of (not necessarily small) categories. Spelling out this definition, we thus have for every small category J an associated category $\mathbb{D}(J)$, for a functor $u: J \rightarrow K$ an induced functor $\mathbb{D}(u) = u^*: \mathbb{D}(K) \rightarrow \mathbb{D}(J)$ and for a natural transformation $\alpha: u \rightarrow v$ of two such functors a natural transformation $\mathbb{D}(\alpha) = \alpha^*: u^* \rightarrow v^*$ as indicated in the following diagram:

$$J \begin{array}{c} \xrightarrow{u} \\ \Downarrow \alpha \\ \xrightarrow{v} \end{array} K, \quad \mathbb{D}(K) \begin{array}{c} \xrightarrow{u^*} \\ \Downarrow \alpha^* \\ \xrightarrow{v^*} \end{array} \mathbb{D}(J).$$

These associations are compatible with compositions and units in a strict sense, i.e., we have equalities of the respective expressions. One can of course also consider 2-functors which are only defined on certain 2-subcategories $\mathbf{Dia} \subseteq \mathbf{Cat}$ (for example finite categories, finite and finite-dimensional categories or posets) subject to certain closure properties (cf. Section 4 of [Gro10a]). This would then lead to the notion of a (pre)derivator *of type Dia*. For simplicity, we will stick to the case of all small categories but everything that we do in this paper can also be done for prederivators of type \mathbf{Dia} .

A *morphism* $F: \mathbb{D} \rightarrow \mathbb{D}'$ of prederivators is a pseudo-natural transformation of 2-functors. Thus, such a morphism consists of a family of functors $F_J: \mathbb{D}(J) \rightarrow \mathbb{D}'(J)$ together with specified isomorphisms $\gamma_u^F: u^* \circ F_K \rightarrow F_J \circ u^*$ for each functor $u: J \rightarrow K$. These isomorphisms have to be suitably compatible with compositions and identities. More precisely, given a pair of composable functors $J \xrightarrow{u} K \xrightarrow{v} L$ and a natural transformation $\alpha: u_1 \rightarrow u_2: J \rightarrow K$, we then have the following relation resp. commutative diagrams:

$$\gamma_{\text{id}_J} = \text{id}_{F_J} \quad \begin{array}{ccc} u^*v^*F & \xrightarrow{\gamma_v} & u^*Fv^* \\ & \searrow \gamma_{vu} & \downarrow \gamma_u \\ & & Fu^*v^* \end{array} \quad \begin{array}{ccc} u_1^*F & \xrightarrow{\alpha^*} & u_2^*F \\ \gamma_{u_1} \downarrow & & \downarrow \gamma_{u_2} \\ Fu_1^* & \xrightarrow{\alpha^*} & Fu_2^* \end{array}$$

Here, we suppressed the indices of F and the upper indices of the natural transformation γ (as we will frequently do in the sequel) to avoid awkward notation. Moreover, we will not distinguish notationally between the natural transformations γ and their inverses. If all the components γ_u^F are identities then F will be called a *strict morphism*.

We will later introduce the notion of an adjunction of two variables between prederivators and in that context it will be important that we also have a lax version of morphisms. So, let us call a lax natural transformation $F: \mathbb{D} \rightarrow \mathbb{D}'$ a *lax morphism of prederivators*. Thus, such a lax morphism consists of a similar datum as a morphism satisfying the same coherence conditions with the difference that the natural transformations $\gamma_u^F: u^* \circ F \rightarrow F \circ u^*$ are not necessarily invertible.

For simplicity we will also apply the same terminology for ‘extranatural’ variants thereof as in the context of adjunctions of two variables (cf. Lemma 1.11).

Finally, let $F, G: \mathbb{D} \rightarrow \mathbb{D}'$ be two morphisms of prederivators. A *natural transformation* $\tau: F \rightarrow G$ is a family of natural transformations $\tau_J: F_J \rightarrow G_J$ which are compatible with the coherence isomorphisms belonging to the functors F and G . Thus, for every functor $u: J \rightarrow K$ the following diagram commutes

$$\begin{array}{ccc} u^*F & \xrightarrow{\tau} & u^*G \\ \gamma \downarrow & & \downarrow \gamma \\ Fu^* & \xrightarrow{\tau} & Gu^*. \end{array}$$

One checks that a natural transformation is precisely the same as a *modification* of pseudo-natural transformations (see [Bor94a, Definition 7.5.3]). Given two parallel morphisms F and G of prederivators let us denote by $\text{nat}(F, G)$ the natural transformations from F to G . Thus, with prederivators as objects, morphisms as 1-cells and natural transformations as 2-cells we obtain the 2-category \mathbf{PDer} of prederivators. In fact, this is just a special case of the 2-category of 2-functors, pseudo-natural transformations and modifications. The full sub-2-category spanned by the derivators is denoted by \mathbf{Der} . Given two (pre)derivators \mathbb{D} and \mathbb{D}' let us denote the category of morphisms by $\mathbf{Hom}(\mathbb{D}, \mathbb{D}')$ while we will write $\mathbf{Hom}^{\text{strict}}(\mathbb{D}, \mathbb{D}')$ for the full subcategory spanned by the strict morphisms.

Example 1.1. The Yoneda embedding $y: \mathbf{CAT} \rightarrow \mathbf{PDer}$ sends a category \mathcal{C} to the represented prederivator $y(\mathcal{C}): J \mapsto \mathbf{Fun}(J, \mathcal{C})$. Here, $\mathbf{Fun}(J, \mathcal{C})$ denotes the category of functors from J to \mathcal{C} . The 2-categorical Yoneda lemma implies that for an arbitrary prederivator \mathbb{D} we have a natural isomorphism of categories

$$Y: \mathbf{Hom}_{\mathbf{PDer}}^{\text{strict}}(y(J), \mathbb{D}) \xrightarrow{\cong} \mathbb{D}(J).$$

For simplicity, we will sometimes drop the embedding y from notation and again just write \mathcal{C} for the prederivator represented by a category \mathcal{C} .

In every 2-category we have the notion of adjoint 1-morphisms, equivalences, and Kan extensions (see Sections 1 and 2 of [Str72]). Let us consider the first two notions in the 2-categories \mathbf{PDer} and \mathbf{Der} . So, let $L: \mathbb{D} \rightarrow \mathbb{D}'$ and $R: \mathbb{D}' \rightarrow \mathbb{D}$ be two morphisms of (pre)derivators and let $\eta: \text{id}_{\mathbb{D}} \rightarrow R \circ L$ and $\epsilon: L \circ R \rightarrow \text{id}_{\mathbb{D}'}$ be two natural transformations. Then one can check that the 4-tuple (L, R, η, ϵ) defines an adjunction $\mathbb{D} \dashv \mathbb{D}'$ resp. an equivalence $\mathbb{D} \xrightarrow{\sim} \mathbb{D}'$ if and only if for each category J we obtain an adjunction $\mathbb{D}(J) \dashv \mathbb{D}'(J)$ resp. an equivalence $\mathbb{D}(J) \xrightarrow{\sim} \mathbb{D}'(J)$ by evaluation. Given such an adjunction $(L, R): \mathbb{D} \dashv \mathbb{D}'$ the adjunctions at the different levels are compatible in the sense that for a functor $u: J \rightarrow K$ we obtain the following commutative diagram:

$$\begin{array}{ccc} \text{hom}_{\mathbb{D}'(K)}(L_K X, Y) & \xrightarrow{\cong} & \text{hom}_{\mathbb{D}(K)}(X, R_K Y) \\ u^* \downarrow & & \downarrow u^* \\ \text{hom}_{\mathbb{D}'(J)}(u^* L_K X, u^* Y) & & \text{hom}_{\mathbb{D}(J)}(u^* X, u^* R_K Y) \\ \gamma^L \downarrow & & \downarrow \gamma^R \\ \text{hom}_{\mathbb{D}'(J)}(L_J u^* X, u^* Y) & \xrightarrow{\cong} & \text{hom}_{\mathbb{D}(J)}(u^* X, R_J u^* Y) \end{array}$$

Here, the morphisms γ^L resp. γ^R are the natural transformations which belong to the morphisms L resp. R .

The fact that adjoint morphisms of derivators behave in the expected way with respect to homotopy Kan extensions is the content of the following lemma. Recall that given a functor $u: J \rightarrow K$ and a derivator \mathbb{D} , one of the axioms of a derivator guarantees that the *restriction functor* $u^*: \mathbb{D}(K) \rightarrow \mathbb{D}(J)$ has an adjoint on either side. We denote any left resp. right adjoint functor of u^* by $u_!: \mathbb{D}(J) \rightarrow \mathbb{D}(K)$ resp. $u_*: \mathbb{D}(J) \rightarrow \mathbb{D}(K)$ and call such a functor a *homotopy left resp. homotopy right Kan extension functor*.

Lemma 1.2. *Let $(L, R): \mathbb{D} \rightarrow \mathbb{D}'$ be an adjunction of derivators. Then L preserves homotopy left Kan extensions and R preserves homotopy right Kan extensions.*

Proof. By duality it suffices to give the proof for homotopy left Kan extensions. Let $u: J \rightarrow K$ be a functor between small categories and let u^* denote the induced functors in \mathbb{D} and \mathbb{D}' . Similarly, let us denote both respective homotopy left Kan extension functors by $u_!$. For objects $X \in \mathbb{D}(J)$ and $Y \in \mathbb{D}'(K)$ we have the following chain of natural isomorphisms:

$$\begin{aligned} \mathrm{hom}_{\mathbb{D}'(K)}(u_!L_J(X), Y) &\cong \mathrm{hom}_{\mathbb{D}'(J)}(L_J(X), u^*(Y)) \\ &\cong \mathrm{hom}_{\mathbb{D}(J)}(X, R_J u^*(Y)) \\ &\cong \mathrm{hom}_{\mathbb{D}(J)}(X, u^* R_K(Y)) \\ &\cong \mathrm{hom}_{\mathbb{D}(K)}(u_!(X), R_K(Y)) \\ &\cong \mathrm{hom}_{\mathbb{D}'(K)}(L_K u_!(X), Y) \end{aligned}$$

By the Yoneda lemma, this natural isomorphism is induced by an isomorphism between the corepresenting objects. Taking $Y = L_K u_!(X)$ and tracing the map $\mathrm{id}: L_K u_!(X) \rightarrow L_K u_!(X)$ through this chain of isomorphisms we obtain a natural isomorphism $\beta: u_!L_J \rightarrow L_K u_!$. But this natural isomorphism is easily identified with a base change morphism occurring in the definition of a homotopy left Kan extension preserving morphism of derivators (cf. Section 3 of [Gro10a]). This concludes the proof. \square

Let us define the ('internal') product of two prederivators. Thus, let \mathbb{D} and \mathbb{D}' be prederivators, then their product $\mathbb{D} \times \mathbb{D}'$ is defined to be the composition of the 2-functors

$$\mathrm{Cat}^{\mathrm{op}} \xrightarrow{\Delta} \mathrm{Cat}^{\mathrm{op}} \times \mathrm{Cat}^{\mathrm{op}} \xrightarrow{\mathbb{D} \times \mathbb{D}'} \mathrm{CAT} \times \mathrm{CAT} \xrightarrow{\times} \mathrm{CAT}$$

where Δ denotes the diagonal. The product of morphisms of prederivators and natural transformations is defined similarly and this gives us the 2-product in the 2-category PDer of prederivators. Recall from [Gro10a] that we also have the notions of a pointed resp. stable derivator.

Lemma 1.3. *Let \mathbb{D} and \mathbb{D}' be derivators. Then $\mathbb{D} \times \mathbb{D}'$ is again a derivator. Moreover, if \mathbb{D} and \mathbb{D}' are in addition pointed, resp. stable then the product $\mathbb{D} \times \mathbb{D}'$ is also pointed, resp. stable.*

Proof. Since isomorphisms in product categories are detected pointwise and since a product of two functors is an adjoint functor resp. an equivalence if and only if this is the case for the two factors the axioms (Der1)-(Der3) are immediate. Also the base change axiom holds since the base change morphism in $\mathbb{D} \times \mathbb{D}'$ can be taken to be the product of the base change morphisms in \mathbb{D} and \mathbb{D}' which are isomorphisms by assumption. Thus, with \mathbb{D} and \mathbb{D}' also the product $\mathbb{D} \times \mathbb{D}'$ is a derivator. Similarly, since the product of pointed categories is again pointed we obtain the result for pointed derivators. For stable derivators, note that $\mathbb{D} \times \mathbb{D}'$ is strong since the product of two full resp. essentially surjective functors is again a full resp. essentially surjective functor. Finally, an

object $X = (Y, Y') \in \mathbb{D}(\square) \times \mathbb{D}'(\square)$ is (co)Cartesian if and only if the components $Y \in \mathbb{D}(\square)$ and $Y' \in \mathbb{D}'(\square)$ are (co)Cartesian. Hence, if \mathbb{D} and \mathbb{D}' are stable, the product $\mathbb{D} \times \mathbb{D}'$ is also stable. \square

The product endows the 2-categories \mathbf{PDer} and \mathbf{Der} with the structure of a symmetric monoidal 2-category, called the Cartesian monoidal structure. The unit e of the monoidal structure is the prederivator with constant value the terminal category e (consisting of one object and its identity morphism only) and the symmetry constraint is given by the twist morphism $T: \mathbb{D} \times \mathbb{D}' \longrightarrow \mathbb{D}' \times \mathbb{D}$. To simplify notation we will suppress the canonical associativity isomorphisms and hence also brackets from notation. In the next subsection, we will introduce monoidal (pre)derivators as monoidal objects in the respective 2-categories.

Before we turn to monoidal derivators let us introduce *bimorphisms* between (pre)derivators. Since the product of two derivators is the 2-categorical product we understand morphisms into them. But also maps out of a product of two derivators are easy to describe: up to an equivalence of categories these are just the bimorphisms as we will define them now.

Definition 1.4. Let \mathbb{D} , \mathbb{E} , and \mathbb{F} be prederivators. A *bimorphism* B from (\mathbb{D}, \mathbb{E}) to \mathbb{F} , denoted $B: (\mathbb{D}, \mathbb{E}) \longrightarrow \mathbb{F}$, consists of a family of functors

$$B_{J_1, J_2}: \mathbb{D}(J_1) \times \mathbb{E}(J_2) \longrightarrow \mathbb{F}(J_1 \times J_2), \quad J_1, J_2 \in \mathbf{Cat},$$

and for each pair of functors $(u_1, u_2): (J_1, J_2) \longrightarrow (K_1, K_2)$ a natural isomorphism γ_{u_1, u_2}^B as indicated in:

$$\begin{array}{ccc} \mathbb{D}(K_1) \times \mathbb{E}(K_2) & \xrightarrow{B_{K_1, K_2}} & \mathbb{F}(K_1 \times K_2) \\ u_1^* \times u_2^* \downarrow & \Leftarrow & \downarrow (u_1 \times u_2)^* \\ \mathbb{D}(J_1) \times \mathbb{E}(J_2) & \xrightarrow{B_{J_1, J_2}} & \mathbb{F}(J_1 \times J_2) \end{array}$$

These data have to satisfy the following coherence conditions. Given a pair of composable pairs $(u_1, u_2): (J_1, J_2) \longrightarrow (K_1, K_2)$ and $(v_1, v_2): (K_1, K_2) \longrightarrow (L_1, L_2)$ and a pair of natural transformations $(\alpha_1, \alpha_2): (u_1, u_2) \longrightarrow (u'_1, u'_2)$ we have $\gamma_{\text{id}_{J_1}, \text{id}_{J_2}} = \text{id}_{B_{J_1, J_2}}$ and the commutativity of the following two diagrams:

$$\begin{array}{ccc} (u_1 \times u_2)^*(v_1 \times v_2)^* B & \xrightarrow{\gamma} & (u_1 \times u_2)^* B(v_1^* \times v_2^*) & & (u_1 \times u_2)^* B & \longrightarrow & (u'_1 \times u'_2)^* B \\ & \searrow \gamma & \downarrow \gamma & & \downarrow \gamma & & \downarrow \gamma \\ & & B(u_1^* \times u_2^*)(v_1^* \times v_2^*) & & B(u_1^* \times u_2^*) & \longrightarrow & B(u_1'^* \times u_2'^*) \end{array}$$

Now, given two parallel bimorphism $B, B': (\mathbb{D}, \mathbb{E}) \longrightarrow \mathbb{F}$, a natural transformation $\tau: B \longrightarrow B'$ of bimorphisms consists of a family of natural transformations $\tau_{J_1, J_2}: B_{J_1, J_2} \longrightarrow B'_{J_1, J_2}$. These have to be compatible in the sense that given a pair of functors $(u_1, u_2): (J_1, J_2) \longrightarrow (K_1, K_2)$ the following diagram commutes:

$$\begin{array}{ccc} (u_1 \times u_2)^* B & \xrightarrow{\tau} & (u_1 \times u_2)^* B' \\ \gamma \downarrow & & \downarrow \gamma \\ B(u_1^* \times u_2^*) & \xrightarrow{\tau} & B'(u_1^* \times u_2^*) \end{array}$$

Given three prederivators \mathbb{D} , \mathbb{E} , and \mathbb{F} we obtain a category of bimorphism from (\mathbb{D}, \mathbb{E}) to \mathbb{F} which we denote by $\mathbf{BiHom}((\mathbb{D}, \mathbb{E}), \mathbb{F})$. In fact, given three such prederivators we can consider the *exterior*

product $\mathbb{D} \boxtimes \mathbb{E}$ of \mathbb{D} and \mathbb{E} and the 2-functor $\mathbb{F} \circ (- \times -)$ which are defined by

$$(\mathbb{D} \boxtimes \mathbb{E})(J_1, J_2) = \mathbb{D}(J_1) \times \mathbb{E}(J_2) \quad \text{and} \quad (\mathbb{F} \circ (- \times -))(J_1, J_2) = \mathbb{F}(J_1 \times J_2).$$

Then, we have an equality of categories

$$\mathbf{BiHom}((\mathbb{D}, \mathbb{E}), \mathbb{F}) = \mathbf{PsNat}(\mathbb{D} \boxtimes \mathbb{E}, \mathbb{F} \circ (- \times -))$$

where $\mathbf{PsNat}(-, -)$ denotes the category of pseudo-natural transformations and modifications. This observation shows that $\mathbf{BiHom}((-, -), -)$ is functorial in all three arguments. Let us now show that $\mathbf{BiHom}((-, -), -)$ is corepresentable. For two prederivators \mathbb{D} and \mathbb{E} , the universal bimorphism $(\mathbb{D}, \mathbb{E}) \longrightarrow \mathbb{D} \times \mathbb{E}$ has components induced by the projections:

$$\mathbb{D}(J_1) \times \mathbb{E}(J_2) \xrightarrow{\text{pr}_1^* \times \text{pr}_2^*} \mathbb{D}(J_1 \times J_2) \times \mathbb{E}(J_1 \times J_2)$$

This bimorphism induces the right adjoint in the following proposition.

Proposition 1.5. *For prederivators \mathbb{D} , \mathbb{E} , and \mathbb{F} we have a natural equivalence of categories*

$$\mathbf{BiHom}((\mathbb{D}, \mathbb{E}), \mathbb{F}) \xrightarrow{\simeq} \mathbf{Hom}(\mathbb{D} \times \mathbb{E}, \mathbb{F}).$$

Proof. Let us begin by defining a natural functor $l: \mathbf{BiHom}((\mathbb{D}, \mathbb{E}), \mathbb{F}) \longrightarrow \mathbf{Hom}(\mathbb{D} \times \mathbb{E}, \mathbb{F})$ so let us consider a bimorphism $B: (\mathbb{D}, \mathbb{E}) \longrightarrow \mathbb{F}$. The component $l(B)_J$ of $l(B): \mathbb{D} \times \mathbb{E} \longrightarrow \mathbb{F}$ is defined by:

$$l(B)_J: \mathbb{D}(J) \times \mathbb{E}(J) \xrightarrow{B_{J,J}} \mathbb{F}(J \times J) \xrightarrow{\Delta_J^*} \mathbb{F}(J)$$

The structure morphism belonging to $u: J \longrightarrow K$ is defined by $\gamma_u^{l(B)} = \Delta_J^* \gamma_{u,u}^B$:

$$\begin{array}{ccccc} \mathbb{D}(K) \times \mathbb{E}(K) & \xrightarrow{B_{K,K}} & \mathbb{F}(K \times K) & \xrightarrow{\Delta_K^*} & \mathbb{F}(K) \\ u^* \times u^* \downarrow & \not\Downarrow & \downarrow (u \times u)^* & & \downarrow u^* \\ \mathbb{D}(J) \times \mathbb{E}(J) & \xrightarrow{B_{J,J}} & \mathbb{F}(J \times J) & \xrightarrow{\Delta_J^*} & \mathbb{F}(J) \end{array}$$

It is immediate that $l(B)$ is in fact a morphism of prederivators and one checks that this assignment can be completed to the definition of a functor l .

We now construct a functor $r: \mathbf{Hom}(\mathbb{D} \times \mathbb{E}, \mathbb{F}) \longrightarrow \mathbf{BiHom}((\mathbb{D}, \mathbb{E}), \mathbb{F})$ so let us consider a morphism $G: \mathbb{D} \times \mathbb{E} \longrightarrow \mathbb{F}$. The component $r(G)_{J_1, J_2}$ is defined to be the following composition:

$$r(G)_{J_1, J_2}: \mathbb{D}(J_1) \times \mathbb{E}(J_2) \xrightarrow{\text{pr}_1^* \times \text{pr}_2^*} \mathbb{D}(J_1 \times J_2) \times \mathbb{E}(J_1 \times J_2) \xrightarrow{G_{J_1 \times J_2}} \mathbb{F}(J_1 \times J_2)$$

For a pair of functors $(u_1, u_2): (J_1, J_2) \longrightarrow (K_1, K_2)$ we set $\gamma_{(u_1, u_2)}^{r(G)} = \gamma_{u_1 \times u_2}^G (\text{pr}_1^* \times \text{pr}_2^*)$:

$$\begin{array}{ccccc} \mathbb{D}(K_1) \times \mathbb{E}(K_2) & \xrightarrow{\text{pr}_1^* \times \text{pr}_2^*} & \mathbb{D}(K_1 \times K_2) \times \mathbb{E}(K_1 \times K_2) & \xrightarrow{G_{K_1 \times K_2}} & \mathbb{F}(K_1 \times K_2) \\ u_1^* \times u_2^* \downarrow & & (u_1 \times u_2)^* \times (u_1 \times u_2)^* \downarrow & \longleftarrow & \downarrow (u_1 \times u_2)^* \\ \mathbb{D}(J_1) \times \mathbb{E}(J_2) & \xrightarrow{\text{pr}_1^* \times \text{pr}_2^*} & \mathbb{D}(J_1 \times J_2) \times \mathbb{E}(J_1 \times J_2) & \xrightarrow{G_{J_1 \times J_2}} & \mathbb{F}(J_1 \times J_2) \end{array}$$

This completes the definition of a bimorphism $r(G): (\mathbb{D}, \mathbb{E}) \longrightarrow \mathbb{F}$. One checks again that this assignment can be extended to a functor r as intended.

Let us next show that the composition $r \circ l$ is naturally isomorphic to the identity. For a bimorphism $B: (\mathbb{D}, \mathbb{E}) \rightarrow \mathbb{F}$ the component of $(r \circ l)(B)$ at (J_1, J_2) is given by $\Delta^* \circ B \circ (\text{pr}_1^* \times \text{pr}_2^*)$ as depicted in:

$$\begin{array}{ccc} \mathbb{D}(J_1) \times \mathbb{E}(J_2) & \xrightarrow{B} & \mathbb{F}(J_1 \times J_2) \\ \text{pr}_1^* \times \text{pr}_2^* \downarrow & \Downarrow & \downarrow (\text{pr}_1 \times \text{pr}_2)^* \\ \mathbb{D}(J_1 \times J_2) \times \mathbb{E}(J_1 \times J_2) & \xrightarrow{B} & \mathbb{F}(J_1 \times J_2 \times J_1 \times J_2) \xrightarrow{\Delta_{J_1 \times J_2}^*} \mathbb{F}(J_1 \times J_2) \end{array}$$

Since $(\text{pr}_1 \times \text{pr}_2) \circ \Delta_{J_1 \times J_2}^* = \text{id}$ this diagram shows that we have an isomorphism from B_{J_1, J_2} to $(r \circ l)(B)_{J_1, J_2}$ given by $\tau_{J_1, J_2} = \Delta_{J_1 \times J_2}^* \gamma_{\text{pr}_1, \text{pr}_2}^B$. Let us check that these assemble into an isomorphism of bimorphisms $\tau: B \rightarrow rl(B)$. Thus, let us consider a pair of functors $(u_1, u_2): (J_1, J_2) \rightarrow (K_1, K_2)$ and let us check that the following diagram commutes:

$$\begin{array}{ccc} (u_1 \times u_2)^* B_{K_1, K_2} & \xrightarrow{\tau_{K_1, K_2}} & (u_1 \times u_2)^* rl(B)_{K_1, K_2} \\ \gamma_{u_1, u_2}^B \downarrow & & \downarrow \gamma_{u_1, u_2}^{rl(B)} \\ B_{J_1, J_2}(u_1^* \times u_2^*) & \xrightarrow{\tau_{J_1, J_2}} & rl(B)_{J_1, J_2}(u_1^* \times u_2^*) \end{array}$$

Unraveling definitions we see that this diagram can be rewritten in the following form where we omit the indices of B for simplicity:

$$\begin{array}{ccc} (u_1 \times u_2)^* \Delta_{K_1 \times K_2}^* (\text{pr}_1 \times \text{pr}_2)^* B & \xrightarrow{\gamma^B} & (u_1 \times u_2)^* \Delta_{K_1 \times K_2}^* B(\text{pr}_1^* \times \text{pr}_2^*) \\ \downarrow = & & \downarrow = \\ \Delta_{J_1 \times J_2}^* (u_1 \times u_2 \times u_1 \times u_2)^* (\text{pr}_1 \times \text{pr}_2)^* B & \xrightarrow{\gamma^B} & \Delta_{J_1 \times J_2}^* (u_1 \times u_2 \times u_1 \times u_2)^* B(\text{pr}_1^* \times \text{pr}_2^*) \\ \downarrow = & & \downarrow \gamma^B \\ \Delta_{J_1 \times J_2}^* (\text{pr}_1 \times \text{pr}_2)^* (u_1 \times u_2)^* B & & \Delta_{J_1 \times J_2}^* B((u_1 \times u_2)^* \times (u_1 \times u_2)^*) (\text{pr}_1^* \times \text{pr}_2^*) \\ \downarrow \gamma^B & & \downarrow = \\ \Delta_{J_1 \times J_2}^* (\text{pr}_1 \times \text{pr}_2)^* B(u_1^* \times u_2^*) & \xrightarrow{\gamma^B} & \Delta_{J_1 \times J_2}^* B(\text{pr}_1^* \times \text{pr}_2^*)(u_1^* \times u_2^*) \end{array}$$

But by the coherence property of the bimorphism B we deduce that this diagram commutes and thus that we have constructed an isomorphism of bimorphisms $\tau: B \rightarrow rl(B)$.

Finally, let us construct a natural isomorphism $l \circ r \rightarrow \text{id}$. So, let G be a morphism $\mathbb{D} \times \mathbb{E} \rightarrow \mathbb{F}$. The component $(l \circ r)(G)_J$ is given by $\Delta^* \circ G \circ (\text{pr}_1^* \times \text{pr}_2^*)$ as in:

$$\begin{array}{ccc} \mathbb{D}(J) \times \mathbb{E}(J) & \xrightarrow{\text{pr}_1^* \times \text{pr}_2^*} & \mathbb{D}(J \times J) \times \mathbb{E}(J \times J) \xrightarrow{G} \mathbb{F}(J \times J) \\ & & \Delta_J^* \times \Delta_J^* \downarrow \quad \Leftarrow \quad \downarrow \Delta_J^* \\ & & \mathbb{D}(J) \times \mathbb{E}(J) \xrightarrow{G} \mathbb{F}(J) \end{array}$$

By the equality $(\Delta_J^* \times \Delta_J^*) \circ (\text{pr}_1^* \times \text{pr}_2^*) = \text{id}$ it follows that this diagram gives us an isomorphism $\sigma_J = \gamma_{\Delta_J^*}^G(\text{pr}_1^* \times \text{pr}_2^*): lr(G)_J \rightarrow G_J$. Let us check that these isomorphisms assemble into a natural

isomorphism $lr(G) \rightarrow G$. Thus, let us consider a functor $u: J \rightarrow K$. Unraveling definitions we have to show this time that the following square commutes:

$$\begin{array}{ccc}
 u^* \Delta_K^* G_{K \times K}(\text{pr}_1^* \times \text{pr}_2^*) & \xrightarrow{\gamma^G} & u^* G_K(\Delta_K^* \times \Delta_K^*)(\text{pr}_1^* \times \text{pr}_2^*) \\
 \downarrow = & & \downarrow \gamma^G \\
 \Delta_J^*(u \times u)^* G_{K \times K}(\text{pr}_1^* \times \text{pr}_2^*) & & G_J(u^* \times u^*)(\Delta_K^* \times \Delta_K^*)(\text{pr}_1^* \times \text{pr}_2^*) \\
 \downarrow \gamma^G & & \downarrow = \\
 \Delta_J^* G_{J \times J}((u \times u)^* \times (u \times u)^*)(\text{pr}_1^* \times \text{pr}_2^*) & \xrightarrow{\gamma^G} & G_J(\Delta_J^* \times \Delta_J^*)((u \times u)^* \times (u \times u)^*)(\text{pr}_1^* \times \text{pr}_2^*)
 \end{array}$$

But this diagram is commutative by the coherence conditions of the 2-cells belonging to a morphism of prederivators. Thus σ is an isomorphism and we can now conclude that the functor $\text{BiHom}(\mathbb{D}, \mathbb{E}, -)$ is corepresentable by $\mathbb{D} \times \mathbb{E}$. \square

The proof shows that these natural equivalences give us natural isomorphisms of categories if we restrict to strict bimorphisms (in the sense that all 2-cells are identities) on the left-hand-side and to strict morphisms on the right-hand-side.

1.2. Monoidal prederivators, monoidal morphisms, and monoidal transformations. Emphasizing similarity to the fact that a monoidal category ([EK66] or [ML98]) is just a monoidal object (called a pseudo-monoid in [DS97]) in the Cartesian 2-category CAT , we could just say that a monoidal prederivator is a monoidal object in the Cartesian 2-category PDer (cf. Appendix B). We prefer to make this more explicit:

Definition 1.6. Let \mathbb{D} be a prederivator. A *monoidal structure* on \mathbb{D} is a 5-tuple $(\otimes, \mathbb{S}, a, l, r)$ consisting of two morphisms of prederivators

$$\otimes: \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D} \quad \text{and} \quad \mathbb{S}: e \rightarrow \mathbb{D}$$

and natural isomorphisms $l, a,$ and r as indicated in the diagrams:

$$\begin{array}{ccc}
 \begin{array}{ccc} e \times \mathbb{D} & \xrightarrow{\mathbb{S} \times \text{id}} & \mathbb{D} \times \mathbb{D} \\ & \nearrow \cong & \downarrow \otimes \\ & & \mathbb{D} \end{array} & \begin{array}{ccc} \mathbb{D} \times \mathbb{D} \times \mathbb{D} & \xrightarrow{\text{id} \times \otimes} & \mathbb{D} \times \mathbb{D} \\ \otimes \times \text{id} \downarrow & \leftarrow & \downarrow \otimes \\ \mathbb{D} \times \mathbb{D} & \xrightarrow{\otimes} & \mathbb{D} \end{array} & \begin{array}{ccc} \mathbb{D} \times \mathbb{D} & \xleftarrow{\text{id} \times \mathbb{S}} & \mathbb{D} \times e \\ \otimes \downarrow & \nearrow \cong & \downarrow \\ \mathbb{D} & \xleftarrow{\cong} & \mathbb{D} \end{array}
 \end{array}$$

This structure has to satisfy the usual coherence conditions as in Appendix B. A *symmetric monoidal structure* on \mathbb{D} is a 6-tuple $(\otimes, \mathbb{S}, a, l, r, t)$ where $(\otimes, \mathbb{S}, a, l, r)$ is a monoidal structure and t is a natural isomorphism as in

$$\begin{array}{ccc}
 \mathbb{D} \times \mathbb{D} & \xrightarrow{T} & \mathbb{D} \times \mathbb{D} \\
 & \searrow \cong & \downarrow \otimes \\
 & & \mathbb{D} \\
 & \nearrow \otimes & \\
 \mathbb{D} & & \mathbb{D}
 \end{array}$$

which satisfies additional coherence conditions as specified in [ML98, Bor94b]. A *monoidal* resp. *symmetric monoidal prederivator* is a prederivator endowed with a monoidal resp. symmetric monoidal structure.

We will often denote a monoidal prederivator simply by $(\mathbb{D}, \otimes, \mathbb{S})$ or even by \mathbb{D} . Moreover, we apply the same terminology for derivators, i.e., a derivator is monoidal if and only if the underlying prederivator is monoidal. The prederivator e can also be considered as the prederivator represented by the terminal category e . So, the 2-categorical Yoneda lemma provides a natural isomorphism of categories

$$Y: \mathbf{Hom}_{\mathbf{PDer}}^{\text{strict}}(e, \mathbb{D}) \xrightarrow{\cong} \mathbb{D}(e).$$

The left-hand-side denotes the full subcategory of $\mathbf{Hom}_{\mathbf{PDer}}(e, \mathbb{D})$ spanned by the strict morphisms of derivators, i.e., those morphisms for which the coherence isomorphisms γ are identities. Thus, in particular, a strict morphism $e \rightarrow \mathbb{D}$ amounts to the choice of an object in $\mathbb{D}(e)$. A not necessarily strict morphism $e \rightarrow \mathbb{D}$ contains more information but see Lemma 1.46 (this reflects the fact that we should work with the *bicategorical* Yoneda lemma as opposed to the *2-categorical* one since we are working with pseudo-natural transformations instead of the more restrictive 2-natural transformations).

Let \mathbb{D} be a (symmetric) monoidal prederivator and let J be a category. Then, by definition, we have a functor $\otimes: \mathbb{D}(J) \times \mathbb{D}(J) \rightarrow \mathbb{D}(J)$, an object $\mathbb{S}(J) \in \mathbb{D}(J)$, and also natural transformations which endow $\mathbb{D}(J)$ with the structure of a (symmetric) monoidal category. Moreover, for a functor $u: J \rightarrow K$ we have an induced natural isomorphism γ^{\otimes} as indicated in:

$$\begin{array}{ccc} \mathbb{D}(K) \times \mathbb{D}(K) & \xrightarrow{\otimes} & \mathbb{D}(K) \\ u^* \times u^* \downarrow & \nearrow & \downarrow u^* \\ \mathbb{D}(J) \times \mathbb{D}(J) & \xrightarrow[\otimes]{} & \mathbb{D}(J) \end{array}$$

Similarly, since $\mathbb{S}: e \rightarrow \mathbb{D}$ is a morphism of derivators we have a canonical natural isomorphism $\gamma^{\mathbb{S}}$ as in the following diagram:

$$\begin{array}{ccc} e & \xrightarrow{\mathbb{S}(K)} & \mathbb{D}(K) \\ & \searrow \nearrow & \downarrow u^* \\ \mathbb{S}(J) & \xrightarrow{\mathbb{S}(J)} & \mathbb{D}(J) \end{array}$$

It is easy to check that these two natural isomorphisms endow $u^*: \mathbb{D}(K) \rightarrow \mathbb{D}(J)$ with the structure of a strong (symmetric) monoidal functor. For example, the definition of a natural transformation between morphisms of prederivators implies that the following diagram commutes:

$$\begin{array}{ccc} (\otimes \circ (\otimes \times \text{id})) \circ u^* & \xrightarrow{a} & (\otimes \circ (\text{id} \times \otimes)) \circ u^* \\ \downarrow \gamma & & \downarrow \gamma \\ u^* \circ (\otimes \circ (\otimes \times \text{id})) & \xrightarrow{a} & u^* \circ (\otimes \circ (\text{id} \times \otimes)) \end{array}$$

Evaluating this at three objects $X, Y,$ and $Z \in \mathbb{D}(K)$ gives us precisely the first coherence condition as imposed on a strong (symmetric) monoidal structure on a functor:

$$\begin{array}{ccc}
 (u^*X \otimes u^*Y) \otimes u^*Z & \xrightarrow{a} & u^*X \otimes (u^*Y \otimes u^*Z) \\
 \downarrow \gamma & & \downarrow \gamma \\
 u^*(X \otimes Y) \otimes u^*Z & & u^*X \otimes u^*(Y \otimes Z) \\
 \downarrow \gamma & & \downarrow \gamma \\
 u^*((X \otimes Y) \otimes Z) & \xrightarrow{a} & u^*(X \otimes (Y \otimes Z))
 \end{array}$$

The other coherence axioms are checked similarly. Moreover, there is a corresponding result for natural transformations. Let $\alpha: u \rightarrow v$ be a natural transformation of functors $J \rightarrow K$. Then it follows immediately that $\alpha^*: u^* \rightarrow v^*$ is a monoidal transformation with respect to the canonical monoidal structures. For example the fact that $\mathbb{S}: e \rightarrow \mathbb{D}$ is a morphism of prederivators encodes that α^* is compatible with the unitality constraints of u^* and v^* . In fact, the commutative square on the left reduces to the triangle on the right:

$$\begin{array}{ccc}
 \mathbb{S}_J u^* & \xrightarrow{\gamma} & u^* \mathbb{S}_K \\
 \alpha^* \downarrow & & \downarrow \alpha^* \\
 \mathbb{S}_J v^* & \xrightarrow{\gamma} & v^* \mathbb{S}_K
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{S}(J) & \longrightarrow & u^*(\mathbb{S}(K)) \\
 \searrow & & \downarrow \alpha^* \\
 & & v^*(\mathbb{S}(K))
 \end{array}$$

Thus, a monoidal prederivator resp. a symmetric monoidal prederivator \mathbb{D} factors canonically as

$$\mathbb{D}: \text{Cat}^{\text{op}} \longrightarrow \text{MonCAT} \longrightarrow \text{CAT} \qquad \text{resp.} \qquad \mathbb{D}: \text{Cat}^{\text{op}} \longrightarrow \text{sMonCAT} \longrightarrow \text{CAT}.$$

Here, MonCAT resp. sMonCAT denotes the 2-category of monoidal resp. symmetric monoidal categories with strong (symmetric) monoidal functors and monoidal transformations. Note that the dual \mathbb{D}^{op} of a monoidal prederivator \mathbb{D} is also canonically endowed with a monoidal structure. Before we turn to some interesting examples, let us quickly give the adapted classes of morphisms and natural transformations. Again, the same terminology will also apply for derivators.

Definition 1.7. Let \mathbb{D} and \mathbb{D}' be monoidal prederivators. A *monoidal structure on a morphism* $F: \mathbb{D} \rightarrow \mathbb{D}'$ of prederivators is a pair of natural transformations

$$\begin{array}{ccc}
 \mathbb{D} \times \mathbb{D} & \xrightarrow{\otimes} & \mathbb{D} \\
 F \times F \downarrow & \nearrow & \downarrow F \\
 \mathbb{D}' \times \mathbb{D}' & \xrightarrow{\otimes} & \mathbb{D}'
 \end{array}
 \qquad
 \begin{array}{ccc}
 e & \xrightarrow{\mathbb{S}} & \mathbb{D} \\
 \searrow & \nearrow & \downarrow F \\
 & & \mathbb{D}'
 \end{array}$$

such that the coherence diagrams of Appendix B are satisfied. A monoidal structure is called *strong* if these natural transformations are isomorphisms. A *(strong) monoidal morphism* $F: \mathbb{D} \rightarrow \mathbb{D}'$ between monoidal prederivators is a morphism endowed with a (strong) monoidal structure.

There is an obvious variant for the case of symmetric monoidal prederivators [Bor94b] which demands for an additional coherence property but which again will not be made precise. For completeness we include the definition of a monoidal natural transformation.

Definition 1.8. Let \mathbb{D} and \mathbb{D}' be monoidal prederivators and let $F, G: \mathbb{D} \rightarrow \mathbb{D}'$ be monoidal morphisms. A natural transformation $\phi: F \rightarrow G$ is called *monoidal* if the following two diagrams commute:

$$\begin{array}{ccc} \otimes \circ (F \times F) & \longrightarrow & F \circ \otimes \\ \phi \times \phi \downarrow & & \downarrow \phi \\ \otimes \circ (G \times G) & \longrightarrow & G \circ \otimes \end{array} \qquad \begin{array}{ccc} \mathbb{S} & \longrightarrow & F \circ \mathbb{S} \\ & \searrow & \downarrow \phi \\ & & G \circ \mathbb{S} \end{array}$$

As in classical category theory, there is no additional assumption on a monoidal transformation of symmetric monoidal functors. Thus, with these notions we have the 2-categories of (symmetric) monoidal (pre)derivators together with the strong monoidal morphisms and monoidal transformations, which are denoted by:

$$\text{MonPDer}, \quad \text{sMonPDer}, \quad \text{MonDer} \quad \text{resp.} \quad \text{sMonDer}$$

For a summary, let us use the following notation of Appendix B: Given a monoidal 2-category \mathcal{C} , let us denote by $\text{Mon}(\mathcal{C})$ the 2-category of monoidal objects in \mathcal{C} . For the case of the Cartesian monoidal 2-category CAT we have $\text{Mon}(\text{CAT}) = \text{MonCAT}$, the 2-category of monoidal categories. Thus, we may summarize our discussion as the following isomorphism of 2-categories:

$$\text{MonPDer} = \text{Mon}(\text{CAT}^{\text{Cat}^{\text{op}}}, \times, e) \cong \text{Mon}(\text{CAT}, \times, e)^{\text{Cat}^{\text{op}}} = (\text{MonCAT})^{\text{Cat}^{\text{op}}}$$

There is an analogous variant for symmetric monoidal prederivators.

Using the equivalence of categories $\text{BiHom}((\mathbb{D}, \mathbb{D}), \mathbb{D}) \simeq \text{Hom}(\mathbb{D} \times \mathbb{D}, \mathbb{D})$ of Proposition 1.5 a monoidal structure on a prederivator \mathbb{D} induces a bimorphism $\otimes: (\mathbb{D}, \mathbb{D}) \rightarrow \mathbb{D}$. This bimorphism is then also coherently associative and unital, which gives us the associated exterior version of the monoidal structure.

Similarly to the theory of ordinary derivators, also in the monoidal context there are the two important classes of examples coming from categories and model categories. Let us recall that given a small category J and a category \mathcal{C} we denote the associated functor category by $\text{Fun}(J, \mathcal{C})$.

Example 1.9. The 2-functor $y: \text{CAT} \rightarrow \text{PDer}$ sending a category \mathcal{C} to the represented prederivator \mathcal{C} defined by $\mathcal{C}: J \mapsto \text{Fun}(J, \mathcal{C})$ preserves 2-products and hence monoidal objects. Thus, we obtain induced 2-functors

$$y: \text{MonCAT} \rightarrow \text{MonPDer} \quad \text{and} \quad y: \text{sMonCAT} \rightarrow \text{sMonPDer}.$$

The monoidal structure on the prederivator represented by a monoidal category \mathcal{C} sends two objects $X, Y \in \text{Fun}(J, \mathcal{C})$ to the composition

$$J \xrightarrow{\Delta} J \times J \xrightarrow{X \times Y} \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C},$$

where Δ is the diagonal functor. The monoidal unit is given by

$$J \longrightarrow e \xrightarrow{\mathbb{S}} \mathcal{M}.$$

If the monoidal category \mathcal{C} is bicomplete, then we obtain a corresponding monoidal derivator. There is a similar result for symmetric monoidal categories.

The second class of examples of monoidal derivators comes from combinatorial monoidal model categories and will be treated in Subsection 1.4. But let us first extend the last example to include results about biclosed monoidal categories. This will be done in the next subsection.

1.3. Adjunctions of two variables and closed monoidal derivators. Our next aim is to introduce the notion of an *adjunction of two variables between prederivators* (which will, in particular, allow us to talk about *closed monoidal derivators*). Similarly to the theory of monoidal structures this can be given in two equivalent ways: there is an exterior version using bimorphisms and an interior version using morphisms of two variables. Let us give the details for the exterior version.

We begin by recalling the following. Let \mathcal{D} , \mathcal{E} , and \mathcal{F} be categories and let us agree that we call a bifunctor $\otimes: \mathcal{D} \times \mathcal{E} \rightarrow \mathcal{F}$ a left adjoint of two variables if there are functors Hom_l and Hom_r and natural isomorphisms as in:

$$\text{hom}_{\mathcal{F}}(X \otimes Y, Z) \cong \text{hom}_{\mathcal{E}}(Y, \text{Hom}_l(X, Z)) \cong \text{hom}_{\mathcal{D}}(X, \text{Hom}_r(Y, Z))$$

Let now \mathbb{D} , \mathbb{E} , and \mathbb{F} be prederivators and let $\otimes: (\mathbb{D}, \mathbb{E}) \rightarrow \mathbb{F}$ be a bimorphism of prederivators. The minimum we expect from the notion of an adjunction of two variables is the following. For two categories J_1 and J_2 we would like to obtain an adjunction of two variables by evaluation. Thus, for $X \in \mathbb{D}(J_1)$, $Y \in \mathbb{E}(J_2)$, and $Z \in \mathbb{F}(J_1 \times J_2)$ we would expect to have natural isomorphisms

$$\text{hom}_{\mathbb{F}(J_1 \times J_2)}(X \otimes Y, Z) \cong \text{hom}_{\mathbb{E}(J_2)}(Y, \text{Hom}_l(X, Z)) \cong \text{hom}_{\mathbb{D}(J_1)}(X, \text{Hom}_r(Y, Z)).$$

Here, $\text{Hom}_l(-, -)$ and $\text{Hom}_r(-, -)$ are certain functors

$$\text{Hom}_l(-, -): \mathbb{D}(J_1)^{\text{op}} \times \mathbb{F}(J_1 \times J_2) \rightarrow \mathbb{E}(J_2) \quad \text{and} \quad \text{Hom}_r(-, -): \mathbb{E}(J_2)^{\text{op}} \times \mathbb{F}(J_1 \times J_2) \rightarrow \mathbb{D}(J_1).$$

Moreover, these natural isomorphisms should be compatible with restriction of diagrams in the following sense. Let us focus on $\text{Hom}_l(-, -)$ but similar reasonings apply to $\text{Hom}_r(-, -)$. Thus, for a pair of functors $(u_1, u_2): (J_1, J_2) \rightarrow (K_1, K_2)$ we would like to have a commutative diagram of the following form:

$$\begin{array}{ccc} \text{hom}_{\mathbb{F}(K_1 \times K_2)}(X \otimes Y, Z) & \xrightarrow{\cong} & \text{hom}_{\mathbb{E}(K_2)}(Y, \text{Hom}_l(X, Z)) \\ \downarrow & & \downarrow \\ \text{hom}_{\mathbb{F}(J_1 \times J_2)}(u_1^* X \otimes u_2^* Y, (u_1 \times u_2)^* Z) & \xrightarrow{\cong} & \text{hom}_{\mathbb{E}(J_2)}(u_2^* Y, \text{Hom}_l(u_1^* X, (u_1 \times u_2)^* Z)) \end{array}$$

Here, the left vertical morphism is obtained by an application of the restriction of diagram functor $(u_1 \times u_2)^*: \mathbb{F}(K_1 \times K_2) \rightarrow \mathbb{F}(J_1 \times J_2)$ followed by a map which is induced by the structure *isomorphism* of the bimorphism $\otimes: (\mathbb{D}, \mathbb{E}) \rightarrow \mathbb{F}$:

$$\gamma_{u_1, u_2}^{\otimes}: u_1^* X \otimes u_2^* Y \rightarrow (u_1 \times u_2)^*(X \otimes Y)$$

Now, if we want to construct the vertical map on the right-hand-side we would certainly start by applying $u_2^*: \mathbb{E}(K_2) \rightarrow \mathbb{E}(J_2)$. But then we are in the situation that we need a map

$$\text{hom}_{\mathbb{E}(J_2)}(u_2^* Y, u_2^* \text{Hom}_l(X, Z)) \rightarrow \text{hom}_{\mathbb{E}(J_2)}(u_2^* Y, \text{Hom}_l(u_1^* X, (u_1 \times u_2)^* Z))$$

which would most naturally be induced by a morphism $u_2^* \text{Hom}_l(X, Z) \rightarrow \text{Hom}_l(u_1^* X, (u_1 \times u_2)^* Z)$. Let us check that such a map can be canonically constructed from the structure morphisms belonging to the bimorphism \otimes . By adjointness, a map $\gamma_{u_1, u_2}^{\text{Hom}_l}: u_2^* \text{Hom}_l(X, Z) \rightarrow \text{Hom}_l(u_1^* X, (u_1 \times u_2)^* Z)$ is equivalently given by a map $u_1^* X \otimes u_2^* \text{Hom}_l(X, Z) \rightarrow (u_1 \times u_2)^* Z$. Using the adjunction counit $\epsilon^K: X \otimes \text{Hom}_l(X, Z) \rightarrow Z$ we can consider the map:

$$u_1^* X \otimes u_2^* \text{Hom}_l(X, Z) \xrightarrow{\gamma^{\otimes}} (u_1 \times u_2)^*(X \otimes \text{Hom}_l(X, Z)) \xrightarrow{\epsilon^K} (u_1 \times u_2)^* Z$$

The adjoint of this map is taken as the definition of $\gamma_{u_1, u_2}^{\text{Hom}_l}$, i.e., we set:

$$\begin{array}{ccc} u_2^* \text{Hom}_l(X, Z) & \xrightarrow{\eta^J} & \text{Hom}_l(u_1^* X, u_1^* X \otimes u_2^* \text{Hom}_l(X, Z)) \\ \downarrow \gamma_{u_1, u_2}^{\text{Hom}_l} & & \downarrow \gamma_{u_1, u_2}^{\otimes} \\ \text{Hom}_l(u_1^* X, (u_1 \times u_2)^* Z) & \xleftarrow{\epsilon^K} & \text{Hom}_l(u_1^* X, (u_1 \times u_2)^*(X \otimes \text{Hom}_l(X, Z))) \end{array}$$

We now claim that this map can be used to show that the adjunctions at the different levels are compatible. Thus, we have to show that the following diagram is commutative:

$$\begin{array}{ccc} \text{hom}_{\mathbb{F}(K_1 \times K_2)}(X \otimes Y, Z) & \xrightarrow{\cong} & \text{hom}_{\mathbb{E}(K_2)}(Y, \text{Hom}_l(X, Z)) \\ \downarrow (u_1 \times u_2)^* & & \downarrow u_2^* \\ \text{hom}_{\mathbb{F}(J_1 \times J_2)}((u_1 \times u_2)^*(X \otimes Y), (u_1 \times u_2)^* Z) & & \text{hom}_{\mathbb{E}(J_2)}(u_2^* Y, u_2^* \text{Hom}_l(X, Z)) \\ \downarrow \gamma^{\otimes} & & \downarrow \gamma^{\text{Hom}_l} \\ \text{hom}_{\mathbb{F}(J_1 \times J_2)}(u_1^* X \otimes u_2^* Y, (u_1 \times u_2)^* Z) & \xrightarrow{\cong} & \text{hom}_{\mathbb{E}(J_2)}(u_2^* Y, \text{Hom}_l(u_1^* X, (u_1 \times u_2)^* Z)) \end{array}$$

For a map $f: X \otimes Y \rightarrow Z$ let us denote by $\phi_1(f)$ resp. $\phi_2(f)$ the image of f under the path passing through the upper right resp. lower left corner. By definition $\phi_1(f)$ is the map $\epsilon \gamma^{\otimes} \eta f$ as depicted in the next diagram precomposed by $\eta^K: u_2^* Y \rightarrow u_2^* \text{Hom}_l(X, X \otimes Y)$:

$$\begin{array}{ccc} u_2^* \text{Hom}_l(X, X \otimes Y) & \xrightarrow{f} & u_2^* \text{Hom}_l(X, Z) \\ \downarrow \eta^J & & \downarrow \eta^J \\ \text{Hom}_l(u_1^* X, u_1^* X \otimes u_2^* \text{Hom}_l(X, X \otimes Y)) & \xrightarrow{f} & \text{Hom}_l(u_1^* X, u_1^* X \otimes u_2^* \text{Hom}_l(X, Z)) \\ \downarrow \gamma^{\otimes} & & \downarrow \gamma^{\otimes} \\ \text{Hom}_l(u_1^* X, (u_1 \times u_2)^*(X \otimes \text{Hom}_l(X, X \otimes Y))) & \xrightarrow{f} & \text{Hom}_l(u_1^* X, (u_1 \times u_2)^*(X \otimes \text{Hom}_l(X, Z))) \\ \downarrow \epsilon^K & & \downarrow \epsilon^K \\ \text{Hom}_l(u_1^* X, (u_1 \times u_2)^*(X \otimes Y)) & \xrightarrow{f} & \text{Hom}_l(u_1^* X, (u_1 \times u_2)^* Z) \end{array}$$

Since the above diagram is commutative we can calculate

$$\phi_1(f) = f \epsilon^K \gamma^{\otimes} \eta^J \eta^K = f \epsilon^K \eta^K \gamma^{\otimes} \eta^J = f \gamma^{\otimes} \eta^J = \phi_2(f).$$

Here we used once more the triangular identity and the naturality of the transformations as expressed by the commutativity of the following diagram

$$\begin{array}{ccc}
 u_2^* Y & \xrightarrow{\eta^K} & u_2^* \mathbf{Hom}_l(X, X \otimes Y) \\
 \eta^J \downarrow & & \downarrow \eta^J \\
 \mathbf{Hom}_l(u_1^* X, u_1^* X \otimes u_2^* Y) & \xrightarrow{\eta^K} & \mathbf{Hom}_l(u_1^* X, u_1^* X \otimes u_2^* \mathbf{Hom}_l(X, X \otimes Y)) \\
 \gamma^\otimes \downarrow & & \downarrow \gamma^\otimes \\
 \mathbf{Hom}_l(u_1^* X, (u_1 \times u_2)^*(X \otimes Y)) & \xrightarrow{\eta^K} & \mathbf{Hom}_l(u_1^* X, (u_1 \times u_2)^*(X \otimes \mathbf{Hom}_l(X, X \otimes Y))) \\
 & \searrow \text{id} & \downarrow \epsilon^K \\
 & & \mathbf{Hom}_l(u_1^* X, (u_1 \times u_2)^*(X \otimes Y))
 \end{array}$$

together with the fact that ϕ_2 sends f in a short-hand-notation to $f\gamma^\otimes\eta^J$.

Thus, in order to express the compatibility of the adjunction isomorphisms with restriction of diagrams we have constructed a natural transformation $\gamma_{u_1, u_2}^{\mathbf{Hom}_l}$ as indicated in:

$$\begin{array}{ccc}
 \mathbb{D}(K_1)^{\text{op}} \times \mathbb{F}(K_1 \times K_2) & \xrightarrow{\mathbf{Hom}_l(-, -)_{K_1, K_2}} & \mathbb{E}(K_2) \\
 u_1^* \times (u_1 \times u_2)^* \downarrow & \Leftarrow & \downarrow u_2^* \\
 \mathbb{D}(J_1)^{\text{op}} \times \mathbb{F}(J_1 \times J_2) & \xrightarrow{\mathbf{Hom}_l(-, -)_{J_1, J_2}} & \mathbb{E}(J_2)
 \end{array}$$

But this time –as the examples will show– it is important to note that this natural transformation is not necessarily invertible! Moreover, these natural transformations satisfy certain coherence conditions which are very similar to the ones in the case of a bimorphism (we will show this below in the case of an internal adjunction of two variables (see Lemma 1.11) since this will be used in Section 3). Said differently the functors $\mathbf{Hom}_l(-, -)_{K_1, K_2}$ together with the natural transformations $\gamma_{u_1, u_2}^{\mathbf{Hom}_l}$ assemble into a lax dinatural transformation $\mathbf{Hom}_l(-, -)$.

In the case of $\mathbf{Hom}_r(-, -)$ a similar reasoning leads to the conclusion that we can construct natural transformations $\gamma_{u_1, u_2}^{\mathbf{Hom}_r}: u_1^* \circ \mathbf{Hom}_r(-, -)_{K_1, K_2} \rightarrow \mathbf{Hom}_r(-, -)_{J_1, J_2} \circ (u_2^* \times (u_1 \times u_2)^*)$ which satisfy suitable coherence conditions. Again it is important to note that these natural transformations are not necessarily invertible. Thus, also the $\mathbf{Hom}_r(-, -)_{K_1, K_2}$ together with the natural transformations $\gamma_{u_1, u_2}^{\mathbf{Hom}_r}$ assemble into a lax dinatural transformation $\mathbf{Hom}_r(-, -)$.

Definition 1.10. Let \mathbb{D} , \mathbb{E} , and \mathbb{F} be prederivators. A *left adjoint of two variables* from (\mathbb{D}, \mathbb{E}) to \mathbb{F} is a bimorphism $\otimes: (\mathbb{D}, \mathbb{E}) \rightarrow \mathbb{F}$ such that for all pairs of categories (K_1, K_2) the functor $\otimes_{K_1, K_2}: \mathbb{D}(K_1) \times \mathbb{E}(K_2) \rightarrow \mathbb{F}(K_1 \times K_2)$ is a left adjoint of two variables.

The discussion preceding the definition thus guarantees the following. Given a left adjoint of two variables $\otimes: (\mathbb{D}, \mathbb{E}) \rightarrow \mathbb{F}$, we can find functors $\mathbf{Hom}_l(-, -)$ and $\mathbf{Hom}_r(-, -)$ and natural isomorphisms:

$$\text{hom}_{\mathbb{F}(K_1 \times K_2)}(X \otimes Y, Z) \cong \text{hom}_{\mathbb{E}(K_2)}(Y, \mathbf{Hom}_l(X, Z)) \cong \text{hom}_{\mathbb{D}(K_1)}(X, \mathbf{Hom}_r(Y, Z))$$

Moreover, the functors $\mathbf{Hom}_l(-, -)$ and $\mathbf{Hom}_r(-, -)$ can both be extended to lax dinatural transformations which in turn can be used to show that the adjunctions of the different levels are compatible. Let us denote an adjunction of two variables by $\otimes: (\mathbb{D}, \mathbb{E}) \rightarrow \mathbb{F}$ or by $(\otimes, \mathbf{Hom}_l, \mathbf{Hom}_r): (\mathbb{D}, \mathbb{E}) \rightarrow \mathbb{F}$.

The internal version of a left adjoint of two variables is completely parallel. A morphism $\otimes: \mathbb{D} \times \mathbb{E} \rightarrow \mathbb{F}$ is called a *left adjoint of two variables* if for all categories J the induced functor $\otimes: \mathbb{D}(J) \times \mathbb{E}(J) \rightarrow \mathbb{F}(J)$ is a left adjoint of two variables. By similar arguments as in the exterior case this implies that we have compatible adjunctions of two variables at the different levels. For example let $\text{Hom}_r(-, -): \mathbb{E}(J)^{\text{op}} \times \mathbb{F}(J) \rightarrow \mathbb{D}(J)$ be chosen levelwise right adjoints to $-\otimes-$ then we can define natural transformations $\gamma_u^{\text{Hom}_r}$ by the following diagram:

$$\begin{array}{ccc} u^* \text{Hom}_r(Y, Z) & \xrightarrow{\eta} & \text{Hom}_r(u^*Y, u^* \text{Hom}_r(Y, Z) \otimes u^*Y) \\ \gamma_u^{\text{Hom}_r} \downarrow & & \downarrow \gamma_u^{\otimes} \\ \text{Hom}_r(u^*Y, u^*Z) & \xleftarrow{\epsilon} & \text{Hom}_r(u^*Y, u^*(\text{Hom}_r(Y, Z) \otimes Y)) \end{array}$$

Next, we want to show that these data assemble into a lax morphism. Let us allow ourselves to commit a slight abuse of notation and write Hom_r as a lax morphism $\mathbb{E}^{\text{op}} \times \mathbb{F} \rightarrow \mathbb{D}$ although, strictly speaking, this is not correct since $\mathbb{E}^{\text{op}}(K) = \mathbb{E}(K^{\text{op}})^{\text{op}} \neq \mathbb{E}(K)^{\text{op}}$.

Lemma 1.11. *Let \mathbb{D} , \mathbb{E} , and \mathbb{F} be prederivators and let $\otimes: \mathbb{D} \times \mathbb{E} \rightarrow \mathbb{F}$ be a left adjoint of two variables. The functors Hom_l resp. Hom_r together with the natural transformations $\gamma_u^{\text{Hom}_l}$ resp. $\gamma_u^{\text{Hom}_r}$ define a lax morphism of prederivators:*

$$\text{Hom}_l(-, -): \mathbb{D}^{\text{op}} \times \mathbb{F} \rightarrow \mathbb{E} \quad \text{resp.} \quad \text{Hom}_r(-, -): \mathbb{E}^{\text{op}} \times \mathbb{F} \rightarrow \mathbb{D}$$

Proof. Let us give the proof in the case of Hom_r . It is easy to see that $\gamma_{\text{id}}^{\text{Hom}_r} = \text{id}$ since this reduces to a triangular identity of adjunctions. So, let us consider two composable functors $J \xrightarrow{v} K \xrightarrow{u} L$. Using the fact that $\otimes: \mathbb{D} \times \mathbb{E} \rightarrow \mathbb{F}$ is a morphism of derivators it is easy to verify that the following diagram commutes for arbitrary objects $X \in \mathbb{D}(L)$, $Y \in \mathbb{E}(L)$, and $Z \in \mathbb{F}(L)$:

$$\begin{array}{ccc} \text{hom}_{\mathbb{F}(L)}(X \otimes Y, Z) & \xrightarrow{u^*} & \text{hom}_{\mathbb{F}(K)}(u^*(X \otimes Y), u^*Z) \xrightarrow{\gamma^{\otimes}} \text{hom}_{\mathbb{F}(K)}(u^*X \otimes u^*Y, u^*Z) \\ (uv)^* \downarrow & & \downarrow v^* \\ \text{hom}_{\mathbb{F}(J)}((uv)^*(X \otimes Y), (uv)^*Z) & & \text{hom}_{\mathbb{F}(J)}(v^*(u^*X \otimes u^*Y), v^*u^*Z) \\ \gamma^{\otimes} \downarrow & & \downarrow \gamma^{\otimes} \\ \text{hom}_{\mathbb{F}(J)}((uv)^*X \otimes (uv)^*Y, (uv)^*Z) & \xlongequal{\quad} & \text{hom}_{\mathbb{F}(J)}(v^*u^*X \otimes v^*u^*Y, v^*u^*Z) \end{array}$$

Using the fact that we have levelwise adjunctions and that these adjunctions are compatible we obtain the corresponding result for the ‘right-hand-side of the adjunction’. By this we mean that also the following diagram commutes in which we use H resp. γ^{H} as abbreviations for Hom_r resp. γ^{Hom_r} :

$$\begin{array}{ccc} \text{hom}_{\mathbb{D}(L)}(X, \text{H}(Y, Z)) & \xrightarrow{u^*} & \text{hom}_{\mathbb{D}(K)}(u^*X, u^* \text{H}(Y, Z)) \xrightarrow{\gamma^{\text{H}}} \text{hom}_{\mathbb{D}(K)}(u^*X, \text{H}(u^*Y, u^*Z)) \\ (uv)^* \downarrow & & \downarrow v^* \\ \text{hom}_{\mathbb{D}(J)}((uv)^*X, (uv)^* \text{H}(Y, Z)) & & \text{hom}_{\mathbb{D}(J)}(v^*u^*X, v^* \text{H}(u^*Y, u^*Z)) \\ \gamma^{\text{H}} \downarrow & & \downarrow \gamma^{\text{H}} \\ \text{hom}_{\mathbb{D}(J)}((uv)^*X, \text{H}((uv)^*Y, (uv)^*Z)) & \xlongequal{\quad} & \text{hom}_{\mathbb{D}(J)}(v^*u^*X, \text{H}(v^*u^*Y, v^*u^*Z)) \end{array}$$

Choosing $X = \text{Hom}_r(Y, Z)$ and tracing the identity $\text{id}: \text{Hom}_r(Y, Z) \rightarrow \text{Hom}_r(Y, Z)$ through the two possible ways to the lower right corner we obtain the second coherence condition of a lax morphism.

Finally, we also have to show a certain compatibility with 2-cells. So, let us consider a natural transformation $\alpha: u_1 \rightarrow u_2$ between parallel functors $J \rightarrow K$. We have to show that the following diagram commutes:

$$\begin{array}{ccc} u_1^* \circ \mathbf{H} & \xrightarrow{\alpha^*} & u_2^* \circ \mathbf{H} \\ \gamma_{u_1}^{\mathbf{H}} \downarrow & & \downarrow \gamma_{u_2}^{\mathbf{H}} \\ \mathbf{H} \circ (u_1^{*\text{op}} \times u_1^*) & \xrightarrow{\alpha^*} \mathbf{H} \circ (u_1^{*\text{op}} \times u_2^*) \xleftarrow{\alpha^{*\text{op}}} & \mathbf{H} \circ (u_2^{*\text{op}} \times u_2^*) \end{array}$$

But unraveling definitions we can see that the above diagram can be extended to the following one. For simplicity of notation we drop the ‘op’ in $u_1^{*\text{op}}$ and $u_2^{*\text{op}}$:

$$\begin{array}{ccccc} u_1^* \mathbf{H}(Y, Z) & \xrightarrow{\alpha^*} & & & u_2^* \mathbf{H}(Y, Z) \\ \eta_1 \downarrow & & & & \downarrow \eta_2 \\ \mathbf{H}(u_1^* Y, u_1^* \mathbf{H}(Y, Z) \otimes u_1^* Y) & \xrightarrow{\alpha^*} \mathbf{H}(u_1^* Y, u_2^* \mathbf{H}(Y, Z) \otimes u_1^* Y) & \xleftarrow{\eta_1} & & \mathbf{H}(u_2^* Y, u_2^* \mathbf{H}(Y, Z) \otimes u_2^* Y) \\ \gamma_{u_1}^{\otimes} \downarrow & & \alpha^* \downarrow & & \downarrow \gamma_{u_2}^{\otimes} \\ & & \mathbf{H}(u_1^* Y, u_2^* \mathbf{H}(Y, Z) \otimes u_2^* Y) & \xleftarrow{\alpha^{*\text{op}}} & \\ & & \downarrow \gamma_{u_2}^{\otimes} & & \\ \mathbf{H}(u_1^* Y, u_1^* (\mathbf{H}(Y, Z) \otimes Y)) & \xrightarrow{\alpha^*} \mathbf{H}(u_1^* Y, u_2^* (\mathbf{H}(Y, Z) \otimes Y)) & \xleftarrow{\alpha^{*\text{op}}} & & \mathbf{H}(u_2^* Y, u_2^* (\mathbf{H}(Y, Z) \otimes Y)) \\ \epsilon \downarrow & & \downarrow \epsilon & & \downarrow \epsilon \\ \mathbf{H}(u_1^* Y, u_1^* Z) & \xrightarrow{\alpha^*} \mathbf{H} \circ (u_1^{*\text{op}} \times u_2^*) \xleftarrow{\alpha^{*\text{op}}} & & & \mathbf{H} \circ (u_2^{*\text{op}} \times u_2^*) \end{array}$$

In this diagram, the upper right quadrilateral commutes by the extranaturality of the adjunction unit in the context of an adjunction with parameters (cf. [ML98, Section IX.4]) while the center left square commutes since \otimes is a morphism of prederivators. The remaining part of the diagram commutes by naturality which concludes the proof. \square

The interior and the exterior version of adjunctions of two variables are compatible. If we have a bimorphism $\otimes: (\mathbb{D}, \mathbb{E}) \rightarrow \mathbb{F}$ and a morphism of two variables $\otimes: \mathbb{D} \times \mathbb{E} \rightarrow \mathbb{F}$ which correspond to each other under the equivalence of Proposition 1.5 then the bimorphism \otimes is a left adjoint of two variables if and only if this is the case for the morphism \otimes .

We now turn to examples in the context of represented (pre)derivators. Let $\otimes: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ be a functor of two variables. We can extend \otimes to a (strict) bimorphism $\otimes: (\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{E}$ of the associated represented prederivators. In fact, for a pair of categories (J_1, J_2) let us define $\otimes_{J_1, J_2}: \mathcal{C}^{J_1} \times \mathcal{D}^{J_2} \rightarrow \mathcal{E}^{J_1 \times J_2}$ by sending a pair (X, Y) to:

$$X \otimes Y: J_1 \times J_2 \xrightarrow{X \times Y} \mathcal{C} \times \mathcal{D} \xrightarrow{\otimes} \mathcal{E}$$

Let us call this bimorphism \otimes the *bimorphism represented by* \otimes .

Proposition 1.12. *Let \mathcal{C} , \mathcal{D} be complete categories, \mathcal{E} a category and $\otimes: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ a left adjoint of two variables. The represented bimorphism $\otimes: (\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{E}$ is then also a left adjoint of two variables. In particular, adjunctions of two variables between bicomplete categories induce adjunctions of two variables between represented derivators.*

Proof. Let us content ourselves by giving the construction of $\text{Hom}_l(-, -)$ and the natural isomorphism expressing one half of the fact that we have an adjunction of two variables. So, let us consider a pair of categories (J_1, J_2) and let us construct a right adjoint

$$\text{Hom}_l(-, -): (\mathcal{C}^{J_1})^{\text{op}} \times \mathcal{E}^{J_1 \times J_2} \rightarrow \mathcal{D}^{J_2}.$$

Using $(\mathcal{C}^{J_1})^{\text{op}} = (\mathcal{C}^{\text{op}})^{J_1^{\text{op}}}$, as an intermediate step we can associate a pair (X, Z) to the functor

$$\text{Hom}_l(-, -) \circ (X \times Z): J_1^{\text{op}} \times J_1 \times J_2 \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{E} \rightarrow \mathcal{D}.$$

Here, Hom_l is a functor expressing the fact that \otimes is an adjunction of two variables. Forming the end over the category J_1 we can define $\text{Hom}_l(X, Z): J_2 \rightarrow \mathcal{D}$ by:

$$\text{Hom}_l(X, Z)(-) = \int_{j_1} \text{Hom}_l(X(j_1), Z(j_1 \times -))$$

Let us check that this gives us the desired adjunction. For this purpose let us consider a functor $Y \in \mathcal{D}^{J_2}$. Using the fact that natural transformations give an example of a further end construction we can make the following calculation:

$$\begin{aligned} \text{hom}_{\mathcal{E}^{J_1 \times J_2}}(X \otimes Y, Z) &\cong \int_{(j_1, j_2)} \text{hom}_{\mathcal{E}}(X(j_1) \otimes Y(j_2), Z(j_1, j_2)) \\ &\cong \int_{(j_1, j_2)} \text{hom}_{\mathcal{D}}(Y(j_2), \text{Hom}_l(X(j_1), Z(j_1, j_2))) \\ &\cong \int_{j_2} \text{hom}_{\mathcal{D}}(Y(j_2), \int_{j_1} \text{Hom}_l(X(j_1), Z(j_1, j_2))) \\ &= \int_{j_2} \text{hom}_{\mathcal{D}}(Y(j_2), \text{Hom}_l(X, Z)(j_2)) \\ &\cong \text{hom}_{\mathcal{D}^{J_2}}(Y, \text{Hom}_l(X, Z)) \end{aligned}$$

The third isomorphism follows from Fubini's theorem for ends and the fact that corepresented functors are end preserving, the second one is the adjunction isomorphism at the level of categories, while the first and the last one are given by the fact that natural transformations can be expressed as ends. This concludes the construction of an adjunction of two variables. \square

We use this example to illustrate that the structure maps belonging to the right adjoints are not necessarily isomorphisms, i.e., that we only obtain *lax* dinatural transformations as opposed to *pseudo* dinatural transformations. So, let us consider a pair of functors $(u_1, u_2): (J_1, J_2) \rightarrow (K_1, K_2)$, two diagrams $X: K_1 \rightarrow \mathcal{C}$ and $Z: K_1 \times K_2 \rightarrow \mathcal{E}$ and let us have a look at the diagram:

$$\begin{array}{ccc} \int_{k_1} \text{Hom}_l(X(k_1), Z(k_1, u_2(-))) & \xrightarrow{\text{pr}_{u_1(j'_1), u_1(j''_1)}} & \\ \downarrow & \searrow & \\ \int_{j_1} \text{Hom}_l(X(u_1(j_1)), Z(u_1(j_1), u_2(-))) & \xrightarrow{\text{pr}_{j'_1, j''_1}} & \text{Hom}_l(X(u_1(j'_1)), Z(u_1(j''_1), u_2(-))) \end{array}$$

The upper left object is $u_2^* \text{Hom}_l(X, Z)$ and the lower left one is $\text{Hom}_l(u_1^* X, (u_1 \times u_2)^* Z)$. The horizontal morphism belongs to the universal wedge of the lower end construction while the diagonal morphism is part of the universal wedge belonging to the upper end construction. By the universal property of the lower wedge there is a unique dashed arrow as indicated which is compatible with all projection morphisms. If we take these dashed arrows as a definition of $\gamma_{u_1, u_2}^{\text{Hom}_l(-, -)}: u_2^* \text{Hom}_l(-, -) \rightarrow \text{Hom}_l(u_1^*(-), (u_1 \times u_2)^*(-))$ one can check that Hom_l becomes a lax dinatural transformation. The fact that the adjunctions at the different levels are compatible with the restriction functors is expressed by the commutativity of the following diagram. In this diagram, we drop the arguments to simplify notation:

$$\begin{array}{ccc}
\int_{K_1 \times K_2} \text{hom}_\mathcal{E}(X \otimes Y, Z) & \xrightarrow{\quad\quad\quad} & \int_{K_2} \text{hom}_\mathcal{D}(Y, \int_{K_1} \text{Hom}_l(X, Z)) \\
\downarrow & & \downarrow \\
\int_{J_1 \times J_2} \text{hom}_\mathcal{E}((u_1 \times u_2)^*(X \otimes Y), (u_1 \times u_2)^* Z) & & \int_{J_2} \text{hom}_\mathcal{D}(u_2^* Y, \int_{K_1} \text{Hom}_l(X, (\text{id} \times u_2)^* Z)) \\
\downarrow & & \downarrow \\
\int_{J_1 \times J_2} \text{hom}_\mathcal{E}(u_1^* X \otimes u_2^* Y, (u_1 \times u_2)^* Z) & \xrightarrow{\quad\quad\quad} & \int_{J_2} \text{hom}_\mathcal{D}(u_2^* Y, \int_{J_1} \text{Hom}_l(u_1^* X, (u_1 \times u_2)^* Z))
\end{array}$$

This diagram commutes by the universal property of end constructions.

Let us recall from Example 1.9 that prederivators represented by monoidal categories can be canonically endowed with a monoidal structure. We can now obtain a similar result for biclosed monoidal categories under an additional completeness assumption. Of course there is a similar result for closed monoidal categories, i.e., symmetric biclosed monoidal categories.

Definition 1.13. Let $(\mathbb{D}, \otimes, \mathbb{S})$ be a monoidal prederivator. The monoidal prederivator or the monoidal structure is called *biclosed* if the morphism $\otimes: \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D}$ is a left adjoint of two variables. A symmetric monoidal prederivator having this additional property is called a *closed monoidal prederivator*.

Corollary 1.14. *Let \mathcal{C} be a (bi)closed monoidal, complete category. The represented monoidal structure on the represented prederivator \mathcal{C} is then also (bi)closed. In particular, derivators represented by (bi)closed monoidal, bicomplete categories are canonically (bi)closed monoidal.*

1.4. Monoidal model categories induce monoidal derivators. Before we turn to monoidal model categories let us make some more comments on the derivator $\mathbb{D}_\mathcal{M}$ associated to a combinatorial model category \mathcal{M} (cf. [Gro10a]). Recall that combinatorial model categories as introduced by Smith are cofibrantly generated model categories which have an underlying presentable category (for the theory of presentable categories cf. the original source [GU71] but also [AR94, MP89]). In the construction of the derivator $\mathbb{D}_\mathcal{M}$ we use the fact that the diagram categories \mathcal{M}^J can be endowed both with the injective and the projective model structure. The existence of the projective model structure follows from a general lifting result of cofibrantly generated model structures along a left adjoint functor ([Hir03]) while the existence of the injective model structure is, for example, shown in [Lur09]. Since both model structures have the same class of weak equivalences, it is not important which one we use in the definition of the value $\mathbb{D}_\mathcal{M}(J)$ as they have canonically isomorphic homotopy categories:

$$\text{Ho}(\mathcal{M}_{\text{proj}}^J) \cong \text{Ho}(\mathcal{M}_{\text{inj}}^J)$$

Now, for a functor $u: J \rightarrow K$, the induced precomposition functor $u^*: \mathcal{M}^K \rightarrow \mathcal{M}^J$ preserves weak equivalences with respect to both structures. Hence, by the universal properties of the localization functors $\gamma: \mathcal{M}^J \rightarrow \mathbf{Ho}(\mathcal{M}^J)$ and $\gamma: \mathcal{M}^K \rightarrow \mathbf{Ho}(\mathcal{M}^K)$ we obtain a unique induced functor u^* at the level of the homotopy categories such that the following diagram commutes on the nose:

$$\begin{array}{ccc} \mathcal{M}^K & \xrightarrow{u^*} & \mathcal{M}^J \\ \gamma \downarrow & & \downarrow \gamma \\ \mathbf{Ho}(\mathcal{M}^K) & \xrightarrow{u^*} & \mathbf{Ho}(\mathcal{M}^J) \end{array}$$

By definition, this induced functor is taken as the value $\mathbb{D}_{\mathcal{M}}(u)$.

Alternatively, one could also form the left derived functor $\mathbb{L}u^*$ with respect to the injective model structures or the right derived functor $\mathbb{R}u^*$ with respect to the projective model structures. Recall that these are functors endowed with natural transformations which turn $\mathbb{L}u^*$ into a right Kan extension of $\gamma \circ u^*$ along γ while $\mathbb{R}u^*$ becomes a left Kan extension of $\gamma \circ u^*$ along γ :

$$\begin{array}{ccc} \mathcal{M}^K & \xrightarrow{u^*} & \mathcal{M}^J \\ \gamma \downarrow & \nearrow & \downarrow \gamma \\ \mathbf{Ho}(\mathcal{M}^K) & \xrightarrow{\mathbb{L}u^*} & \mathbf{Ho}(\mathcal{M}^J) \end{array} \qquad \begin{array}{ccc} \mathcal{M}^K & \xrightarrow{u^*} & \mathcal{M}^J \\ \gamma \downarrow & \searrow & \downarrow \gamma \\ \mathbf{Ho}(\mathcal{M}^K) & \xrightarrow{\mathbb{R}u^*} & \mathbf{Ho}(\mathcal{M}^J) \end{array}$$

Since the localization functor $\gamma: \mathcal{M}^K \rightarrow \mathbf{Ho}(\mathcal{M}^K)$ is a 2-localization, we obtain, in particular, an isomorphism of categories:

$$\gamma^*: \mathbf{Ho}(\mathcal{M}^J)^{\mathbf{Ho}(\mathcal{M}^K)} \rightarrow \mathbf{Ho}(\mathcal{M}^J)^{(\mathcal{M}^K, W)}$$

Here, the right-hand-side is the full subcategory of $\mathbf{Ho}(\mathcal{M}^J)^{\mathcal{M}^K}$ spanned by the functors which invert the weak equivalences. For an arbitrary functor $F: \mathbf{Ho}(\mathcal{M}^K) \rightarrow \mathbf{Ho}(\mathcal{M}^J)$ this gives us the following two bijections

$$\mathbf{nat}(F, u^*) \xrightarrow{\gamma^*} \mathbf{nat}(F \circ \gamma, u^* \circ \gamma), \qquad \mathbf{nat}(u^*, F) \xrightarrow{\gamma^*} \mathbf{nat}(u^* \circ \gamma, F \circ \gamma).$$

But these bijections express that the induced functor $u^*: \mathbf{Ho}(\mathcal{M}^K) \rightarrow \mathbf{Ho}(\mathcal{M}^J)$ is simultaneously also a right Kan extension and a left Kan extension of $\gamma \circ u^*$ along γ . We thus obtain natural isomorphisms

$$\mathbb{L}u^* \cong u^* \cong \mathbb{R}u^*.$$

This observation will be useful in the construction of the monoidal derivator underlying a combinatorial monoidal model category. More generally, it allows for the construction of morphisms of derivators induced by Brown functors and hence, in particular, by Quillen functors or Quillen bifunctors. One motivation for the notion of Brown functors is the following. In order to form the derived functor of a –say– left Quillen functor not all of the defining properties of a left Quillen functor are needed as already emphasized in [Hov99, Hir03, Mal07b]. Thus, sometimes the following definition is useful (cf. also to [DHKS04] and [Shu11] where these are called deformable functors and derivable functors, respectively).

Definition 1.15. Let \mathcal{M} and \mathcal{N} be model categories and let $F: \mathcal{M} \rightarrow \mathcal{N}$ be a functor. F is a *left Brown functor* if F preserves weak equivalences between cofibrant objects. Dually, F is a *right Brown functor* if F preserves weak equivalences between fibrant objects.

As one sees from the constructions in [Hov99, Hir03], this suffices to obtain the respective derived functors which again will have the universal property of the respective Kan extensions. In what follows, we will only state and prove the results for left Brown functors (and left Quillen (bi)functors), but also the dual statements hold true.

Proposition 1.16. *Let \mathcal{M} and \mathcal{N} be combinatorial model categories and let $F: \mathcal{M} \rightarrow \mathcal{N}$ be a left Brown functor. Then by forming left derived functors we obtain a morphism of derivators $\mathbb{L}F: \mathbb{D}_{\mathcal{M}} \rightarrow \mathbb{D}_{\mathcal{N}}$. In particular, this is the case for left Quillen functors.*

Proof. Let J be a category and let us consider the induced functor $F: \mathcal{M}^J \rightarrow \mathcal{N}^J$. With respect to the injective model structures, this is again a left Brown functor. Hence, given a functor $u: J \rightarrow K$ we have the following commutative diagram of left Brown functors:

$$\begin{array}{ccc} \mathcal{M}_{\text{inj}}^K & \xrightarrow{F} & \mathcal{N}_{\text{inj}}^K \\ u^* \downarrow & & \downarrow u^* \\ \mathcal{M}_{\text{inj}}^J & \xrightarrow{F} & \mathcal{N}_{\text{inj}}^J \end{array}$$

Passing to left derived functors for the horizontal arrows and to the induced functors on the localizations for the vertical arrows gives us the following diagram which commutes up to a canonical natural isomorphism γ_u :

$$\begin{array}{ccc} \mathbb{D}_{\mathcal{M}}(K) & \xrightarrow{\mathbb{L}F} & \mathbb{D}_{\mathcal{N}}(K) \\ u^* \downarrow & \not\cong & \downarrow u^* \\ \mathbb{D}_{\mathcal{M}}(J) & \xrightarrow{\mathbb{L}F} & \mathbb{D}_{\mathcal{N}}(J) \end{array}$$

It is easy to check that these natural isomorphisms γ_u , $u: J \rightarrow K$, endow the functors $\mathbb{L}F$ with the structure of a morphism of derivators. \square

Since adjunctions and equivalences of derivators are detected levelwise we immediately obtain the following corollary.

Corollary 1.17. *Let $(F, U): \mathcal{M} \rightarrow \mathcal{N}$ be a Quillen adjunction of combinatorial model categories. Then we obtain a derived adjunction $(\mathbb{L}F, \mathbb{R}U): \mathbb{D}_{\mathcal{M}} \rightarrow \mathbb{D}_{\mathcal{N}}$. If (F, U) is a Quillen equivalence then $(\mathbb{L}F, \mathbb{R}U)$ is an equivalence of derivators.*

There is a further important class of Brown functors, namely the Quillen bifunctors. These are central to many notions of homotopical algebra.

Definition 1.18. Let \mathcal{M} , \mathcal{N} , and \mathcal{P} be model categories. A functor $\otimes: \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{P}$ is a *left Quillen bifunctor* if it preserves colimits separately in each variable and has the following property: For every cofibration $f: X_1 \rightarrow X_2$ in \mathcal{M} and every cofibration $g: Y_1 \rightarrow Y_2$ in \mathcal{N} the pushout-product map

$$f \square g = (X_2 \otimes g) \amalg (f \otimes Y_2): X_2 \otimes Y_1 \amalg_{X_1 \otimes Y_1} X_1 \otimes Y_2 \rightarrow X_2 \otimes Y_2$$

is a cofibration which is acyclic if in addition f or g is acyclic.

There is the dual notion of a right Quillen bifunctor $\text{Hom}: \mathcal{M}^{\text{op}} \times \mathcal{N} \rightarrow \mathcal{P}$. In that case one considers the induced maps

$$\text{Hom}_{\square}(f, g): \text{Hom}(X_2, Y_1) \rightarrow \text{Hom}(X_1, Y_1) \times_{\text{Hom}(X_1, Y_2)} \text{Hom}(X_2, Y_2).$$

The following is immediate.

Lemma 1.19. *Let $\otimes: \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{P}$ be a left Quillen bifunctor and let $X \in \mathcal{M}$ resp. $Y \in \mathcal{N}$ be cofibrant objects. The functors $X \otimes -: \mathcal{N} \rightarrow \mathcal{P}$ and $- \otimes Y: \mathcal{M} \rightarrow \mathcal{P}$ are then left Quillen functors. In particular, $\otimes: \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{P}$ is a left Brown functor when we endow $\mathcal{M} \times \mathcal{N}$ with the product model structure.*

Thus Proposition 1.16 can be applied to Quillen bifunctors. Under the canonical isomorphism $\mathbb{D}_{\mathcal{M} \times \mathcal{N}} \cong \mathbb{D}_{\mathcal{M}} \times \mathbb{D}_{\mathcal{N}}$ we obtain that a Quillen bifunctor $\otimes: \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{P}$ induces a morphism of derivators $\mathbb{D}_{\mathcal{M}} \times \mathbb{D}_{\mathcal{N}} \rightarrow \mathbb{D}_{\mathcal{P}}$. Let us not distinguish notationally between this morphism and the associated bimorphism (cf. Proposition 1.5) and let us denote both by

$$\overset{\mathbb{L}}{\otimes}: \mathbb{D}_{\mathcal{M}} \times \mathbb{D}_{\mathcal{N}} \rightarrow \mathbb{D}_{\mathcal{P}} \quad \text{and} \quad \overset{\mathbb{L}}{\otimes}: (\mathbb{D}_{\mathcal{M}}, \mathbb{D}_{\mathcal{N}}) \rightarrow \mathbb{D}_{\mathcal{P}}.$$

The bimorphism can also be obtained without invoking Proposition 1.5. The bifunctor \otimes induces a strict bimorphism of represented derivators $\otimes: (\mathcal{M}, \mathcal{N}) \rightarrow \mathcal{P}$. For each morphism of pairs $(u_1, u_2): (J_1, J_2) \rightarrow (K_1, K_2)$ we have a commutative diagram of left Brown functors as follows if all model categories are endowed with the injective model structures:

$$\begin{array}{ccc} \mathcal{M}^{K_1} \times \mathcal{N}^{K_2} & \xrightarrow{\otimes} & \mathcal{P}^{K_1 \times K_2} \\ u_1^* \times u_2^* \downarrow & & \downarrow (u_1 \times u_2)^* \\ \mathcal{M}^{J_1} \times \mathcal{N}^{J_2} & \xrightarrow{\otimes} & \mathcal{P}^{J_1 \times J_2} \end{array}$$

Forming derived functors at the different levels and taking the natural isomorphisms induced by these diagrams we obtain again the bimorphism $(\mathbb{D}_{\mathcal{M}}, \mathbb{D}_{\mathcal{N}}) \rightarrow \mathbb{D}_{\mathcal{P}}$.

In the context of combinatorial model categories, we get a stronger statement. Recall that the adjoint functor theorem of Freyd takes the following form in the context of presentable categories: a functor between presentable categories is a left adjoint if and only if it preserves colimits. For example, in the context of combinatorial model categories a monoidal structure which preserves colimits in each variable is always a biclosed monoidal structure, i.e., we have an adjunction of two variables $(\otimes, \text{Hom}_l, \text{Hom}_r)$.

Now, let \mathcal{M} , \mathcal{N} , and \mathcal{P} be combinatorial model categories. Then given a left Quillen bifunctor $\otimes: \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{P}$ we obtain an adjunction of two variables $(\otimes, \text{Hom}_l, \text{Hom}_r)$. This adjunction is expressed by natural isomorphisms

$$\text{hom}_{\mathcal{P}}(X \otimes Y, Z) \cong \text{hom}_{\mathcal{M}}(X, \text{Hom}_r(Y, Z)) \cong \text{hom}_{\mathcal{N}}(Y, \text{Hom}_l(X, Z))$$

for certain functors

$$\text{Hom}_l(-, -): \mathcal{M}^{\text{op}} \times \mathcal{P} \rightarrow \mathcal{N} \quad \text{and} \quad \text{Hom}_r(-, -): \mathcal{N}^{\text{op}} \times \mathcal{P} \rightarrow \mathcal{M}.$$

Lemma 1.20. *Let \mathcal{M} , \mathcal{N} , and \mathcal{P} be model categories and let $(\otimes, \text{Hom}_l, \text{Hom}_r): \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{P}$ be an adjunction of two variables. If we endow \mathcal{M}^{op} resp. \mathcal{N}^{op} with the dual model structures we have the following equivalent statements: \otimes is a left Quillen bifunctor if and only if Hom_l is a right Quillen bifunctor if and only if Hom_r is a right Quillen bifunctor.*

By the above discussion, we know that a left Quillen bifunctor $\otimes: \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{P}$ between combinatorial model categories extends to an adjunction of two variables. By Proposition 1.12 or again by the special adjoint functor theorem, we deduce that this adjunction induces adjunctions of two

variables between represented derivators $\otimes: (\mathcal{M}, \mathcal{N}) \longrightarrow \mathcal{P}$. By the last lemma, we have thus adjunctions of two variables consisting of Quillen bifunctors which induce derived adjunctions of two variables $\mathbb{D}_{\mathcal{M}}(J_1) \times \mathbb{D}_{\mathcal{N}}(J_2) \longrightarrow \mathbb{D}_{\mathcal{P}}(J_1 \times J_2)$. This shows that the morphism

$$\overset{\mathbb{L}}{\otimes}: (\mathbb{D}_{\mathcal{M}}, \mathbb{D}_{\mathcal{N}}) \longrightarrow \mathbb{D}_{\mathcal{P}}$$

is a left adjoint of two variables. We have thus established the following result.

Corollary 1.21. *Let \mathcal{M} , \mathcal{N} , and \mathcal{P} be combinatorial model categories and let $\otimes: \mathcal{M} \times \mathcal{N} \longrightarrow \mathcal{P}$ be a left Quillen bifunctor. Then, by forming derived functors, we obtain an adjunction of two variables at the level of associated derivators:*

$$(\overset{\mathbb{L}}{\otimes}, \mathbb{R}\mathrm{Hom}_l, \mathbb{R}\mathrm{Hom}_r): (\mathbb{D}_{\mathcal{M}}, \mathbb{D}_{\mathcal{N}}) \dashv \mathbb{D}_{\mathcal{P}}$$

For later reference let us quickly introduce the notion of Quillen homotopies.

Definition 1.22. Let $F, G: \mathcal{M} \longrightarrow \mathcal{N}$ be left Brown functors. A natural transformation $\tau: F \longrightarrow G$ is called a (left) *Quillen homotopy* if the components τ_X are weak equivalences for all cofibrant objects X .

Lemma 1.23. *Let $F, G: \mathcal{M} \longrightarrow \mathcal{N}$ be left Brown functors between combinatorial model categories and let $\tau: F \longrightarrow G$ be a left Quillen homotopy. Then we obtain a natural isomorphism*

$$\mathbb{L}\tau: \mathbb{L}F \xrightarrow{\cong} \mathbb{L}G$$

of induced morphisms $\mathbb{L}F, \mathbb{L}G: \mathbb{D}_{\mathcal{M}} \longrightarrow \mathbb{D}_{\mathcal{N}}$.

With these preparations we can now turn to monoidal model categories. We use the following definition of a monoidal model category, which is close to the original one in [Hov99].

Definition 1.24. A *monoidal model category* is a model category \mathcal{M} endowed with a monoidal structure such that the monoidal pairing $\otimes: \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ is a Quillen bifunctor and such that a (and hence any) cofibrant replacement $Q\mathbb{S} \longrightarrow \mathbb{S}$ of the monoidal unit has the property that the induced natural transformations $Q\mathbb{S} \otimes - \longrightarrow \mathbb{S} \otimes -$ and $- \otimes Q\mathbb{S} \longrightarrow - \otimes \mathbb{S}$ are Quillen homotopies.

Theorem 1.25. *Let \mathcal{M} be a combinatorial monoidal model category. The associated derivator $\mathbb{D}_{\mathcal{M}}$ inherits canonically the structure of a biclosed monoidal derivator. If the monoidal structure on \mathcal{M} is symmetric, then this is also the case for the induced structure on $\mathbb{D}_{\mathcal{M}}$.*

Proof. We only have to put the above results together and care about the unit. The injective model structures on the diagram categories $\mathcal{M}_{\mathrm{inj}}^J$ have the property that the natural transformations $Q\mathbb{S} \otimes - \longrightarrow \mathbb{S} \otimes -$ and $- \otimes Q\mathbb{S} \longrightarrow - \otimes \mathbb{S}$ are again Quillen homotopies since everything is defined levelwise. Thus, at each stage we can apply the corresponding result of [Hov99] to obtain a monoidal structure on $\mathrm{Ho}(\mathcal{M}^J)$. Moreover, by Corollary 1.21 these fit together to define a biclosed monoidal structure on $\mathbb{D}_{\mathcal{M}}$ since the left Quillen bifunctor \otimes induces a derived adjunction of two variables. \square

There is a similar result for monoidal left Quillen functors. Recall from [Hov99] that a monoidal left Quillen functor is a left Quillen functor which is strong monoidal and satisfies an additional unitality condition. This extra condition ensures that the derived functor will respect the monoidal unit at the level of homotopy categories. We omit the proof that such a monoidal left Quillen functor between combinatorial model categories induces a monoidal morphism of associated derivators. After having given the following central examples we will shortly consider the situation of weakly monoidal Quillen adjunctions.

Example 1.26. Let k be a commutative ring and let $\mathbf{Ch}(k)$ be the category of unbounded chain complexes over k . This category can be equipped with the combinatorial (so-called projective) model structure where the weak equivalences are the quasi-isomorphisms and the fibrations are the surjections ([Hov99]). The tensor product of chain complexes endows this category with the structure of a closed monoidal model category. The unit object is given by $k[0]$ which denotes the chain complex concentrated in degree zero where it takes the value k . Thus, the associated stable *derivator of chain complexes*

$$\mathbb{D}_k = \mathbb{D}_{\mathbf{Ch}(k)}$$

is a closed monoidal derivator. More generally, let C be a commutative monoid in $\mathbf{Ch}(k)$, i.e., let C be a commutative differential-graded algebra. Then, the category $C\text{-Mod}$ of differential-graded left C -modules inherits a stable, combinatorial model structure ([SS00]). Moreover, forming the tensor product over C endows $C\text{-Mod}$ with the structure of a closed monoidal model category. We deduce that the associated stable *derivator of differential-graded C -modules*

$$\mathbb{D}_C = \mathbb{D}_{C\text{-Mod}}$$

is also closed monoidal.

Example 1.27. Let \mathbf{Set}_Δ denote the presentable category of simplicial sets. If we endow it with the homotopy-theoretic Kan model structure ([Qui67], [GJ99, Chapter 1]) we obtain a Cartesian closed monoidal model category $\mathbf{Set}_\Delta^{\text{Kan}}$. Recall that the cofibrations are the monomorphisms, the weak equivalences are the maps which become homotopy equivalences after geometric realization and the fibrations are the Kan fibrations. Since this model structure is combinatorial, we obtain a closed monoidal *derivator of simplicial sets*:

$$\mathbb{D}_{\mathbf{Set}_\Delta} = \mathbb{D}_{\mathbf{Set}_\Delta^{\text{Kan}}}$$

But, there is also the Joyal model structure on the category of simplicial sets (see for example [Joy08], [Lur09], and also [Gro10b]). This cofibrantly generated model structure is Cartesian so that we again have a Cartesian closed monoidal model category $\mathbf{Set}_\Delta^{\text{Joyal}}$ where the underlying model category is combinatorial. Thus, we obtain a further closed monoidal derivator, the *derivator of ∞ -categories*:

$$\mathbb{D}_{\infty\text{-Cat}} = \mathbb{D}_{\mathbf{Set}_\Delta^{\text{Joyal}}}$$

Example 1.28. Let \mathbf{Sp}^Σ be the category of symmetric spectra based on simplicial sets as introduced in [HSS00]. This presentable category carries a symmetric monoidal structure given by the smash product \wedge where the monoidal unit is given by the sphere spectrum \mathbb{S} . It is shown in [HSS00] that \mathbf{Sp}^Σ endowed with the stable model structure is a cofibrantly generated, stable, symmetric monoidal model category in which the unit object is cofibrant. We obtain hence an associated stable, closed monoidal *derivator of spectra*:

$$\mathbb{D}_{\mathbf{Sp}} = \mathbb{D}_{\mathbf{Sp}^\Sigma}$$

Moreover, let us denote by $E\text{-Mod}$ the category of left E -module spectra for a commutative symmetric ring spectrum $E \in \mathbf{Sp}^\Sigma$. The category $E\text{-Mod}$ can be endowed with the projective model structure, i.e., the weak equivalences and the fibrations are reflected by the forgetful functor $E\text{-Mod} \rightarrow \mathbf{Sp}^\Sigma$. This model category is a combinatorial monoidal model category when endowed with the smash product over E and hence gives rise to the stable, closed monoidal derivator of E -module spectra:

$$\mathbb{D}_E = \mathbb{D}_{E\text{-Mod}}$$

We will now consider weakly monoidal Quillen adjunctions as introduced by Schwede and Shipley in [SS03a] and illustrate them by an example. This example will also reveal a technical advantage derivators do have when compared to model categories. Before we get to that let us give the following result (cf. [Kel74]). Let us consider an adjunction $(L, R): \mathcal{C} \rightarrow \mathcal{D}$ where both categories \mathcal{C} and \mathcal{D} are monoidal. Moreover, let us assume that we are given a lax monoidal structure on the right adjoint:

$$m: RX \otimes RY \longrightarrow R(X \otimes Y) \quad \text{and} \quad u: \mathbb{S} \longrightarrow R\mathbb{S}$$

The map u is adjoint to a map $u': L\mathbb{S} \longrightarrow \mathbb{S}$ while we can define $m': L(X \otimes Y) \longrightarrow LX \otimes LY$ to be the map adjoint to

$$X \otimes Y \xrightarrow{\eta \otimes \eta} RLX \otimes RLY \xrightarrow{m} R(LX \otimes LY).$$

It is now a lengthy formal calculation to show that the pair (m', u') defines a lax comonoidal structure on L . Similarly, if we start with a lax comonoidal structure on L given by

$$m': L(X \otimes Y) \longrightarrow L(X \otimes Y) \quad \text{and} \quad u': L\mathbb{S} \longrightarrow \mathbb{S},$$

we can consider the map $u: \mathbb{S} \longrightarrow R\mathbb{S}$ which is adjoint to u' . Moreover, let $m: RX \otimes RY \longrightarrow R(X \otimes Y)$ be the map adjoint to

$$L(RX \otimes RY) \xrightarrow{m'} LRX \otimes LRY \xrightarrow{\epsilon \otimes \epsilon} X \otimes Y.$$

This will then define a lax monoidal structure on the right adjoint R . With these preparations we can formulate the next lemma.

Lemma 1.29. *Let \mathcal{C} and \mathcal{D} be monoidal categories and let $(L, R): \mathcal{C} \rightarrow \mathcal{D}$ be an adjunction. The above constructions define a bijection between lax monoidal structures on R and lax comonoidal structures on L . Moreover, if (L, R) is an equivalence then we have a bijection between strong monoidal structures on L and strong monoidal structures on R .*

Proof. We have to show that the two constructions are inverse to each other. Let us consider the case where we start with a lax monoidal structure (m, u) on R . From this we can form the lax comonoidal structure (m', u') on L and again a lax monoidal structure (m'', u'') on R . It is immediate that we have $u = u''$, so it remains to show that we also have $m = m''$. By the definition of the morphisms m' and m'' , we have the following commutative diagram:

$$\begin{array}{ccccc} RX \otimes RY & \xrightarrow{\eta} & RL(RX \otimes RY) & \xrightarrow{\eta \otimes \eta} & RL(RLRX \otimes RLRY) \\ m'' \downarrow & & m' \downarrow & & \downarrow m \\ R(X \otimes Y) & \xleftarrow{\epsilon \otimes \epsilon} & R(LRX \otimes LRY) & \xleftarrow{\epsilon} & RLR(LRX \otimes LRY) \end{array}$$

Using $(\eta \otimes \eta) \circ \eta = \eta \circ (\eta \otimes \eta)$, we can deduce the following commutative diagram

$$\begin{array}{ccccc}
RX \otimes RY & \xrightarrow{\eta \otimes \eta} & RLRX \otimes RLRY & \xrightarrow{\eta} & RL(RLRX \otimes RLRY) \\
\downarrow m'' & & \downarrow m & & \downarrow m \\
& & R(LRX \otimes LRY) & \xrightarrow{\eta} & RLR(LRX \otimes LRY) \\
& & \downarrow = & & \downarrow = \\
R(X \otimes Y) & \xleftarrow{\epsilon \otimes \epsilon} & R(LRX \otimes LRY) & \xleftarrow{\epsilon} & RLR(LRX \otimes LRY)
\end{array}$$

where the lower right square commutes by a triangular identity. Using the triangular identity again, we can conclude by the following calculation:

$$m'' = (\epsilon \otimes \epsilon) \circ m \circ (\eta \otimes \eta) = m \circ (\epsilon \otimes \epsilon) \circ (\eta \otimes \eta) = m$$

The second statement for the case of an equivalence of monoidal categories is immediate since in that case the adjunction unit and counit are natural isomorphisms. \square

Let us now recall the following definition of [SS03a].

Definition 1.30. Let \mathcal{M} and \mathcal{N} be monoidal model categories. A *weak monoidal Quillen adjunction* $\mathcal{M} \rightarrow \mathcal{N}$ is a Quillen adjunction (F, U) together with a lax monoidal structure (m, u) on the right adjoint U such that the following two properties are satisfied:

i) The natural transformation $m': F \circ \otimes \rightarrow \otimes \circ (F \times F)$ which is part of the induced lax comonoidal structure on F is a left Quillen homotopy.

ii) For any cofibrant replacement $QS \rightarrow S$ of the monoidal unit S of \mathcal{M} the map $FQS \rightarrow FS \xrightarrow{u'} S$ is a weak equivalence.

We call such a datum a *weak monoidal Quillen equivalence* if the underlying Quillen adjunction (F, U) is a Quillen equivalence.

In the context of combinatorial monoidal model categories one checks that weak monoidal Quillen adjunctions (resp. equivalences) can be extended to weak monoidal Quillen adjunctions (resp. equivalences) at the level of diagram categories with respect to the injective model structures.

Proposition 1.31. *Let $(F, U): \mathcal{M} \rightarrow \mathcal{N}$ be a weak monoidal Quillen adjunction between combinatorial model categories. Then the left derived morphism $\mathbb{L}F: \mathbb{D}_{\mathcal{M}} \rightarrow \mathbb{D}_{\mathcal{N}}$ carries canonically the structure of a strong monoidal morphism while $\mathbb{R}U: \mathbb{D}_{\mathcal{N}} \rightarrow \mathbb{D}_{\mathcal{M}}$ is canonically lax monoidal. If (F, U) is a weak monoidal Quillen equivalence then both $\mathbb{L}F$ and $\mathbb{R}U$ carry canonically a strong monoidal structure.*

Proof. By our assumption the natural transformation $m': F \circ \otimes \rightarrow \otimes \circ (F \times F)$ is a Quillen homotopy. By the additional compatibility assumption of the induced map $u': FS \rightarrow S$ we can use m' and u' in order to obtain a strong comonoidal structure on $\mathbb{L}F: \mathbb{D}_{\mathcal{M}} \rightarrow \mathbb{D}_{\mathcal{N}}$. Since there is an obvious bijection between strong comonoidal and strong monoidal structures, we end up with a strong monoidal structure on $\mathbb{L}F$. If (F, U) is actually a weak monoidal Quillen equivalence, we can apply a variant of Lemma 1.29 for derivators to also construct a strong monoidal structure on $\mathbb{R}U$. \square

Corollary 1.32. *Let \mathcal{M}, \mathcal{N} be combinatorial monoidal model categories which are Quillen equivalent through a zigzag of weakly monoidal Quillen equivalences between combinatorial monoidal model categories. Then we obtain a strongly monoidal equivalence of derivators $\mathbb{D}_{\mathcal{M}} \xrightarrow{\cong} \mathbb{D}_{\mathcal{N}}$.*

As an illustration we want to apply this to the situation described in [Shi07]. In that paper, Shipley constructs a zigzag of three weak monoidal Quillen equivalences between the category of unbounded chain complexes of abelian groups and the category of $H\mathbb{Z}$ -module spectra. To be more specific, the monoidal model for spectra is chosen to be the category of symmetric spectra ([HSS00]) and $H\mathbb{Z}$ denotes the integral Eilenberg-MacLane spectrum. The chain of weak monoidal Quillen equivalence passes through the following intermediate model categories

$$H\mathbb{Z} - \text{Mod} \simeq_Q \text{Sp}^{\Sigma}(\mathfrak{sAb}) \simeq_Q \text{Sp}^{\Sigma}(\text{Ch}^+) \simeq_Q \text{Ch}.$$

Here, Ch^+ is the category of non-negatively graded chain complexes of abelian groups, \mathfrak{sAb} is the category of simplicial abelian groups and $\text{Sp}^{\Sigma}(-)$ denotes Hovey's stabilization process by forming symmetric spectra internal to a sufficiently nice model category ([Hov01]). There is a similar such chain of weak monoidal Quillen equivalences if we replace the integers by an arbitrary commutative ground ring k . Since all the four model categories occurring in that chain are combinatorial we can apply the last corollary in order to obtain the following example.

Example 1.33. For a commutative ring k let us denote by Hk the symmetric Eilenberg-MacLane ring spectrum. Then we have a strong monoidal equivalence of derivators

$$\mathbb{D}_k \simeq \mathbb{D}_{Hk}.$$

1.5. Additive derivators, the center of a derivator, and linear structures. For a derivator \mathbb{D} and a category J it is immediate that $\mathbb{D}(J)$ has initial and final objects as well as finite coproducts and finite products (cf. Subsection 1.1 of [Gro10a]). A pointed derivator is a derivator such that every initial object of the underlying category $\mathbb{D}(e)$ is also final. It follows then that all values $\mathbb{D}(J)$ are pointed. For additive derivators, it also suffices to impose the additivity assumption on the underlying category. For us the notion of an additive category does *not* include an enrichment in abelian groups. The additional *structure* given by the enrichment in abelian groups can be uniquely reconstructed using the exactness *properties* of an additive category. Thus, the category $\mathbb{D}(e)$ is assumed to be pointed and the canonical map from the coproduct of two objects to the product of them is to be an isomorphism. Moreover, for every object there is a self-map which ‘behaves as an additive inverse of the identity’. For a precise formulation of this axiom, compare to Definition 8.2.8 of [KS06]. Alternatively, one can demand the shear map

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}: X \sqcup X \longrightarrow X \times X$$

to be an isomorphism for each object X .

Definition 1.34. A derivator \mathbb{D} is *additive* if the underlying category $\mathbb{D}(e)$ is additive.

Proposition 1.35. *If a derivator \mathbb{D} is additive, then all categories $\mathbb{D}(J)$ are additive and for any functor $u: J \longrightarrow K$ the induced functors u^* , $u_!$, and u_* are additive.*

Proof. Let us assume \mathbb{D} to be additive and let us consider an arbitrary category J . We already know that $\mathbb{D}(J)$ is pointed. Since isomorphisms in $\mathbb{D}(J)$ can be tested pointwise and since the evaluation functors have adjoints on both sides it is easy to see that finite coproducts and finite products in $\mathbb{D}(J)$ are canonically isomorphic. Similarly, let $X \in \mathbb{D}(J)$ be an arbitrary object and let us consider the shear map $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}: X \sqcup X \longrightarrow X \times X$. This map is an isomorphism if and only if this is the case when evaluated at all objects $j \in J$. But $j^*\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ can be canonically identified with the shear map of $j^*X \in \mathbb{D}(e)$ which is an isomorphism by assumption. Finally, given a functor $u: J \longrightarrow K$, the

induced functors u^* , $u_!$, and u_* are all additive since each of them has an adjoint on at least one side. \square

In contrast to the above definition, let us call a prederivator additive if all values and all precomposition functors are additive.

Example 1.36. i) Let \mathcal{C} be a category. Then the prederivator \mathbb{D} represented by \mathcal{C} is additive if and only if the category \mathcal{C} is additive.

ii) Let \mathbb{D} be a stable derivator. Then we showed in Section 4 of [Gro10a] that \mathbb{D} is also an additive derivator. So, this is, in particular, the case for derivators associated to stable (combinatorial) model categories.

Definition 1.37. Let \mathbb{D} be a prederivator. The *center* $Z(\mathbb{D})$ of \mathbb{D} is the set of natural transformations

$$Z(\mathbb{D}) = \text{nat}(\text{id}_{\mathbb{D}}, \text{id}_{\mathbb{D}}).$$

Thus, an element of $Z(\mathbb{D})$ is a natural transformation $\tau: \text{id}_{\mathbb{D}} \rightarrow \text{id}_{\mathbb{D}}$, i.e., a family of natural transformations $\tau_J: \text{id}_{\mathbb{D}(J)} \rightarrow \text{id}_{\mathbb{D}(J)}$ which behave well with the precomposition functors u^* . The composition of natural transformations endows $Z(\mathbb{D})$ with the structure of a (commutative) monoid.

Lemma 1.38. Let \mathbb{D} be an additive derivator. The center $Z(\mathbb{D})$ of \mathbb{D} is then a commutative ring.

Proof. The multiplication on $Z(\mathbb{D})$ is given by composition. For two elements $\tau, \sigma \in Z(\mathbb{D})$, a category K and an element $X \in \mathbb{D}(K)$, by naturality we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{(\tau_K)_X} & X \\ (\sigma_K)_X \downarrow & & \downarrow (\sigma_K)_X \\ X & \xrightarrow{(\tau_K)_X} & X \end{array}$$

Thus we have $\sigma\tau = \tau\sigma$, i.e., the multiplication is commutative. Since the precomposition functors u^* are additive, the sum $\tau + \sigma$ of two elements $\tau, \sigma \in Z(\mathbb{D})$ lies again in the center. Finally, the biadditivity of the composition in the additive situation concludes the proof. \square

This commutative ring $Z(\mathbb{D})$ can be used to endow an additive derivator with k -linear structures as follows.

Definition 1.39. Let \mathbb{D} be an additive derivator and let k be a commutative ring. A *k -linear structure on \mathbb{D}* is a ring homomorphism

$$\sigma: k \rightarrow Z(\mathbb{D}).$$

A pair (\mathbb{D}, σ) consisting of an additive derivator \mathbb{D} and a k -linear structure σ on \mathbb{D} is a *k -linear derivator*.

As emphasized in the definition, k -linearity of an additive derivator is additional *structure* (contrary to the additivity of an additive derivator which is a *property*). Nevertheless, we will drop σ from notation and speak of a k -linear additive derivator \mathbb{D} . Every additive derivator is canonically endowed with a \mathbb{Z} -linear structure.

Now, let \mathbb{D} be an additive derivator. Evaluation at a category J induces a ring homomorphism $Z(\mathbb{D}) \rightarrow Z(\mathbb{D}(J))$, where $Z(\mathbb{D}(J))$ denotes the usual center of the additive category $\mathbb{D}(J)$, i.e., the commutative ring of natural transformations $\text{id}_{\mathbb{D}(J)} \rightarrow \text{id}_{\mathbb{D}(J)}$. Thus, a k -linear structure on an

additive derivator induces k -linear structures on all its values. Moreover, these k -linear structures are preserved by the precomposition functors. Recall for example from [KS06] that for a morphism $f: X \rightarrow Y$ in $\mathbb{D}(K)$ and a ring element $s \in k$ the morphism $sf: X \rightarrow Y$ is given by the diagonal in the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{s} & X \\ f \downarrow & & \downarrow f \\ Y & \xrightarrow{s} & Y \end{array}$$

Here, we simplified notation by writing s for $(\sigma(s)_K)_X$ resp. $(\sigma(s)_K)_Y$. Now, since $\sigma(s) \in Z(\mathbb{D})$ we have an equality of natural transformations $u^*s = su^*: u^* \rightarrow u^*$ for an arbitrary functor $u: J \rightarrow K$. For a morphism $f: X \rightarrow Y$ in $\mathbb{D}(K)$ this equality implies $su^*(f) = u^*(sf)$, i.e., the k -linearity of u^* . Conversely, k -linear structures on the values of an additive derivator such that the precomposition functors are k -linear give a k -linear structure on the additive derivator. This gives the first part of the following proposition.

Proposition 1.40. *Let \mathbb{D} be an additive derivator. A k -linear structure on \mathbb{D} is equivalently given by a k -linear structure on $\mathbb{D}(J)$ for each category J such that the precomposition functors are k -linear. Moreover, in that case also the homotopy Kan extension functors $u_!, u_*: \mathbb{D}(J) \rightarrow \mathbb{D}(K)$ associated to an arbitrary functor $u: J \rightarrow K$ are k -linear.*

Proof. It remains to give a proof of the second statement and, by duality, it suffices to treat the case of homotopy left Kan extensions. Let X, Y be objects of $\mathbb{D}(J)$ and let $s \in k$. Let us consider the following commutative diagram in which the horizontal isomorphisms are the adjunction isomorphisms:

$$\begin{array}{ccc} \mathrm{hom}_{\mathbb{D}(K)}(u_!X, u_!Y) & \xrightarrow{\cong} & \mathrm{hom}_{\mathbb{D}(J)}(X, u^*u_!Y) \\ s_* \downarrow & & \downarrow (u^*(s))_* \\ \mathrm{hom}_{\mathbb{D}(K)}(u_!X, u_!Y) & \xleftarrow{\cong} & \mathrm{hom}_{\mathbb{D}(J)}(X, u^*u_!Y) \end{array}$$

The vertical map on the left sends $u_!(f): u_!X \rightarrow u_!Y$ to $su_!(f)$. So let us calculate the image of $u_!(f)$ under the composition of the three maps. Let us remark first that $(u^*(s))_* = s_*$ since u^* is k -linear. Thus, the image of $u_!(f)$ under the composition of the first two maps is the composition of the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\eta} & u^*u_!X & \xrightarrow{f} & u^*u_!Y \\ s \downarrow & & s \downarrow & & \downarrow s \\ X & \xrightarrow{\eta} & u^*u_!X & \xrightarrow{f} & u^*u_!Y \end{array}$$

But, using the triangular identities, this composition is sent by the second adjunction isomorphism to $u_!(f)u_!(s) = u_!(sf)$. Hence, we obtain the intended relation $su_!(f) = u_!(sf)$ expressing the k -linearity of $u_!$. \square

We finish by giving the notion of k -linear morphisms of k -linear derivators. Let us note that an additive morphism $F: \mathbb{D} \rightarrow \mathbb{D}'$ of additive derivators induces ring maps $F_*: Z(\mathbb{D}) \rightarrow \mathrm{nat}(F, F)$ and $F^*: Z(\mathbb{D}') \rightarrow \mathrm{nat}(F, F)$.

Definition 1.41. Let \mathbb{D} and \mathbb{D}' be k -linear derivators with respective k -linear structures σ and σ' . An additive morphism $F: \mathbb{D} \rightarrow \mathbb{D}'$ is k -linear if $F_* \circ \sigma = F^* \circ \sigma': k \rightarrow \mathbf{nat}(F, F)$. With all natural transformations as 2-morphisms we thus obtain the 2-category $\mathbf{Der}^{\mathbf{add}, k}$ of k -linear derivators.

In particular, we have $\mathbf{Der}^{\mathbf{add}, \mathbb{Z}} = \mathbf{Der}^{\mathbf{add}}$. It is easy to see that an additive morphism $F: \mathbb{D} \rightarrow \mathbb{D}'$ of k -linear derivators is k -linear if and only if all components $F_K: \mathbb{D}(K) \rightarrow \mathbb{D}'(K)$ are k -linear functors. Thus we obtain the following example – more specific examples of linear structures will be given at the end of this subsection.

Example 1.42. Let \mathbb{D} and \mathbb{D}' be additive derivators. Then a \mathbb{Z} -linear morphism $F: \mathbb{D} \rightarrow \mathbb{D}'$ is the same as a coproduct-preserving morphism. In particular, all exact morphisms between stable derivators are \mathbb{Z} -linear. This is, for example, the case for all morphisms $u^*: \mathbb{D}_K \rightarrow \mathbb{D}_J$ induced by the precomposition functors of a stable derivator \mathbb{D} . Recall that \mathbb{D}_J is the derivator which sends a category L to $\mathbb{D}(J \times L)$.

In the case of a stable derivator \mathbb{D} there is the following graded variant of the center. Recall from Section 4 of [Gro10a] that the suspension functor $\Sigma: \mathbb{D}(J) \rightarrow \mathbb{D}(J)$ is defined as the following composition:

$$\Sigma: \mathbb{D}(J) \xrightarrow{(0,0)^*} \mathbb{D}(J \times \ulcorner) \xrightarrow{i_{\ulcorner}} \mathbb{D}(J \times \square) \xrightarrow{(1,1)^*} \mathbb{D}(J)$$

Since the morphisms of derivators $u^*: \mathbb{D}_K \rightarrow \mathbb{D}_J$ preserve homotopy left and homotopy right Kan extensions, the above suspension functors can be taken together to define a self-equivalence $\Sigma: \mathbb{D} \rightarrow \mathbb{D}$ of the derivator, the *suspension morphism*. More precisely, we use Lemma 2.9 twice which states that the homotopy Kan extensions at the different levels assemble into a morphism of derivators.

Let us consider the values of a stable derivator as graded categories in the following way. For a category J and two objects $X, Y \in \mathbb{D}(J)$, the graded abelian groups $\mathbf{hom}_{\mathbb{D}(J)}(X, Y)_\bullet$ and $\mathbf{hom}_{\mathbb{D}(J)}(X, Y)^\bullet$ are defined to be

$$\mathbf{hom}_{\mathbb{D}(J)}(X, Y)_n = \mathbf{hom}_{\mathbb{D}(J)}(X, Y)^{-n} = \mathbf{hom}_{\mathbb{D}(J)}(\Sigma^n X, Y), \quad n \in \mathbb{Z}.$$

Here, we used that the suspension is invertible in the stable situation in order to define the \mathbb{Z} -graded abelian groups.

Example 1.43. For a commutative ring k and k -modules M and N we have the following identification:

$$\mathbf{hom}_{\mathbb{D}_k(e)}(\Sigma^{-n} M, N) = \mathbf{Ext}_k^n(M, N).$$

For a functor $u: J \rightarrow K$ the induced functors u^* , $u_!$, and u_* are graded since they are exact with respect to the canonical triangulated structures [Gro10a]. Let us now come to a graded-commutative variant of the center for stable derivators.

Definition 1.44. Let \mathbb{D} be a stable derivator and let $\Sigma: \mathbb{D} \rightarrow \mathbb{D}$ be the suspension morphism. Then the *graded center* $\mathbf{Z}_\bullet(\mathbb{D})$ of \mathbb{D} is the \mathbb{Z} -graded abelian group which in degree n is the subgroup $\mathbf{Z}_n(\mathbb{D}) = \mathbf{Z}^{-n}(\mathbb{D})$ of $\mathbf{nat}(\Sigma^n, \mathrm{id}_{\mathbb{D}})$ given by the natural transformations τ that commute with the suspension up to a sign, i.e., satisfy $\Sigma\tau = (-1)^n \tau\Sigma: \Sigma^{n+1} \rightarrow \Sigma$.

It is immediate to see that the composition of elements of the center endows $\mathbf{Z}_\bullet(\mathbb{D})$ with the structure of a graded-commutative ring. Similarly to the unstable case, we can now talk about graded-linear structures. A *graded-linear structure on a stable derivator* is a map $\sigma: R_\bullet \rightarrow \mathbf{Z}_\bullet(\mathbb{D})$ of graded rings. Similarly to the ungraded case, it follows that also the homotopy Kan extensions are linear over R_\bullet .

Lemma 1.45. *Let \mathbb{D} be a stable derivator endowed with a linear structure over the graded ring R_\bullet and let $u: J \rightarrow K$ be a functor. The graded category $\mathbb{D}(K)$ is then canonically R_\bullet -linear. Moreover, the induced graded functors $u^*: \mathbb{D}(K) \rightarrow \mathbb{D}(J)$ and $u_!, u_*: \mathbb{D}(J) \rightarrow \mathbb{D}(K)$ are linear over R_\bullet .*

Let us now turn towards the linear structures which are canonically available for suitable additive, monoidal derivators. The 2-categorical Yoneda lemma gives us for every monoidal prederivator \mathbb{D} the strict morphism $\kappa_{\mathbb{S}_e}: e \rightarrow \mathbb{D}$ corresponding to the monoidal unit \mathbb{S}_e of the underlying monoidal category $\mathbb{D}(e)$.

Lemma 1.46. *Let \mathbb{D} be a monoidal prederivator. Then the unit morphism $\mathbb{S}: e \rightarrow \mathbb{D}$ and the strict morphism $\kappa_{\mathbb{S}_e}: e \rightarrow \mathbb{D}$ are naturally isomorphic.*

Proof. Recall from the proof of the 2-Yoneda lemma that the value of $\kappa_{\mathbb{S}_e}$ at a category K is just the element $p_K^*(\mathbb{S}_e)$ where $p_K: K \rightarrow e$ is the unique functor to the terminal category e . Moreover, for a functor $u: J \rightarrow K$ the induced functors $u^*: \mathbb{D}(K) \rightarrow \mathbb{D}(J)$ are canonically monoidal functors. In particular, there is a canonical isomorphism $u^*(\mathbb{S}_K) \rightarrow \mathbb{S}_J$ which is, as we saw in the first subsection, the structure isomorphism γ_u belonging to the morphism of prederivators $\mathbb{S}: e \rightarrow \mathbb{D}$. Applied to the canonical functor p_K , this gives us an isomorphism

$$\tau_K = \gamma_{p_K}: (\kappa_{\mathbb{S}_e})_K = p_K^*(\mathbb{S}_e) \rightarrow \mathbb{S}_K.$$

These τ_K assemble to a natural isomorphism $\tau: \kappa_{\mathbb{S}_e} \rightarrow \mathbb{S}$. In fact, we just have to check that the following diagram commutes:

$$\begin{array}{ccc} u^* p_K^*(\mathbb{S}_e) & \xrightarrow{u^* \tau_K = u^* \gamma_{p_K}} & u^* \mathbb{S}_K \\ \parallel & & \downarrow \gamma_u \\ p_J^*(\mathbb{S}_e) & \xrightarrow{\tau_J u^* = \gamma_{p_J} u^*} & \mathbb{S}_J \end{array}$$

But this is just a special case of the coherence properties of the isomorphisms belonging to the morphisms of prederivators $\mathbb{S}: e \rightarrow \mathbb{D}$. \square

We can also give a more conceptual proof of this lemma. For this purpose, let us recall the *bicategorical* Yoneda lemma. For a general introduction to the theory of bicategories cf. [Bén67]. Although we are only concerned with 2-categories, let us quickly mention that the basic idea with bicategories is that one wants to relax the notion of 2-categories in the sense that one only asks for a composition law which is unital and associative up to specified natural coherent isomorphisms. Given two 2-categories \mathcal{C} and \mathcal{D} and two parallel 2-functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$ one can now consider the category $\text{PsNat}(F, G)$ of pseudo-natural transformations where the morphisms are given by the modifications. As a special case, let us take $\mathcal{D} = \text{CAT}$, let us fix an object $X \in \mathcal{C}$ and let us consider the corepresented 2-functor $y(X) = \text{Hom}(X, -): \mathcal{C} \rightarrow \text{CAT}$. If we are given in addition a CAT -valued 2-functor $F: \mathcal{C} \rightarrow \text{CAT}$ then we can consider the category $\text{PsNat}(y(X), F)$. The bicategorical Yoneda lemma states that the evaluation at the identity of X induces a natural *equivalence of categories*:

$$Y: \text{PsNat}(y(X), F) \xrightarrow{\cong} F(X)$$

The bicategorical Yoneda lemma in the more general situation of homomorphisms of bicategories can be found in [Str80].

Now, given two prederivators \mathbb{D} and \mathbb{D}' the category of morphisms from \mathbb{D} to \mathbb{D}' is given by $\mathbf{Hom}(\mathbb{D}, \mathbb{D}') = \mathbf{PsNat}(\mathbb{D}, \mathbb{D}')$. In the situation of the last lemma, the bicategorical Yoneda lemma hence gives us an equivalence of categories

$$Y: \mathbf{Hom}(e, \mathbb{D}) = \mathbf{Hom}(y(e), \mathbb{D}) \xrightarrow{\cong} \mathbb{D}(e).$$

Both morphisms \mathbb{S} and $\kappa_{\mathbb{S}_e}$ are mapped to \mathbb{S}_e under Y showing that they must be isomorphic.

Let now \mathbb{D} be a monoidal, additive derivator and let us assume that the monoidal structure preserves coproducts. The natural isomorphism of the last lemma induces a ring map

$$\mathbf{hom}_{\mathbb{D}(e)}(\mathbb{S}_e, \mathbb{S}_e) \longrightarrow \mathbf{nat}(\kappa_{\mathbb{S}_e} \otimes -, \kappa_{\mathbb{S}_e} \otimes -) \xrightarrow{\cong} \mathbf{nat}(\mathbb{S} \otimes -, \mathbb{S} \otimes -).$$

A final conjugation with the coherence isomorphism $l: \mathbb{S} \otimes - \cong \text{id}$ thus gives us a ring map

$$\mathbf{hom}_{\mathbb{D}(e)}(\mathbb{S}_e, \mathbb{S}_e) \longrightarrow \mathbf{Z}(\mathbb{D}),$$

i.e., the derivator \mathbb{D} is endowed with a linear structure over the endomorphisms of \mathbb{S} . Thus, we have proved the following result.

Corollary 1.47. *Let \mathbb{D} be an additive, monoidal derivator with an additive monoidal structure. Then \mathbb{D} is canonically endowed with a linear structure over $\mathbf{hom}_{\mathbb{D}(e)}(\mathbb{S}_e, \mathbb{S}_e)$. In particular, let \mathcal{M} be a combinatorial, closed monoidal model category with unit object \mathbb{S} such that the associated derivator is additive. The derivator $\mathbb{D}_{\mathcal{M}}$ is then canonically endowed with a linear structure over $\mathbf{hom}_{\mathbf{Ho}(\mathcal{M})}(\mathbb{S}, \mathbb{S})$.*

Note that there is a certain asymmetry in the construction of the linear structures. We only used the coherence isomorphism $\mathbb{S} \otimes - \cong \text{id}$. As a consequence, a similar result concerning the existence of linear structures can also be established for suitably left-tensored derivators (cf. Section 2). Furthermore, there is a graded variant of this result for stable, monoidal derivators if the monoidal structure has certain exactness properties. But before we come to that we want to mention that the existence of these linear structures is only the shadow of a much more structured result. In the next section we will see that a derivator has an associated derivator of endomorphisms denoted $\mathbf{END}(\mathbb{D})$. Using Theorem B.11 of Appendix B we deduce that $\mathbf{END}(\mathbb{D})$ is canonically monoidal and that associated to a monoidal derivator \mathbb{D} there is a monoidal morphism $\mathbb{D} \rightarrow \mathbf{END}(\mathbb{D})$. In the additive context, the ring map of the last result is just a shadow of this monoidal morphism. We will prove such a result in the more general context of tensored derivators in Section 2.

Recall, e.g. from [HPS97, Definition A.2.1] and [May01, Section 4], that there are notions of when a closed monoidal structure on a triangulated category is *compatible* with the triangulation. In the context of stable derivators the ‘triangulation’ is not an additional structure (cf. [Gro10a]) but we nevertheless want to introduce a similar notion here. Before we introduce it, let us assume we were given a stable, monoidal derivator \mathbb{D} such that the monoidal structure \otimes commutes with the suspension in both variables. Then, for arbitrary s, t we can consider the following possibly non-commutative diagram in which the left vertical arrow is given by the symmetry constraint:

$$\begin{array}{ccc} \Sigma^s \mathbb{S} \otimes \Sigma^t \mathbb{S} & \xrightarrow{\cong} & \Sigma^{s+t} \mathbb{S} \\ \cong \downarrow & & \downarrow (-1)^{st} \\ \Sigma^t \mathbb{S} \otimes \Sigma^s \mathbb{S} & \xrightarrow{\cong} & \Sigma^{t+s} \mathbb{S} \end{array}$$

Definition 1.48. A derivator \mathbb{D} is *compatibly stable and closed monoidal* if \mathbb{D} is stable, closed monoidal, and if the above diagram commutes for all r and s .

We could have given the same definition in the more general context of a stable derivator with a monoidal structure which commutes with the suspension and is additive in both variables. However, to be closer to the situations as considered in [HPS97, May01] we assumed the derivator to be closed monoidal which is anyhow fulfilled by all examples we are considering here.

Let \mathbb{D} be a compatibly stable and closed monoidal derivator and let \mathbb{S}_e be the monoidal unit of the underlying monoidal category $\mathbb{D}(e)$. It follows that the graded abelian group of self-maps $\mathbf{hom}_{\mathbb{D}(e)}(\mathbb{S}_e, \mathbb{S}_e)_\bullet$ is a graded-commutative ring. In fact, as a special case of the composition in the graded category $\mathbb{D}(e)$, the composition of $g: \Sigma^n \mathbb{S}_e \rightarrow \mathbb{S}_e$ and $f: \Sigma^m \mathbb{S}_e \rightarrow \mathbb{S}_e$ is given by:

$$g \circ f: \Sigma^{n+m} \mathbb{S}_e \xrightarrow{\Sigma^n f} \Sigma^n \mathbb{S}_e \xrightarrow{g} \mathbb{S}_e$$

The graded-commutativity of this composition follows now from the following diagram which uses the fact that we are given a *compatibly* stable and closed monoidal derivator:

$$\begin{array}{ccccccc}
 \Sigma^m \mathbb{S}_e & \xleftarrow{\Sigma^m g} & \Sigma^{m+n} \mathbb{S}_e & \xrightarrow{(-1)^{mn}} & \Sigma^{n+m} \mathbb{S}_e & \xrightarrow{\Sigma^n f} & \Sigma^n \mathbb{S}_e \\
 \uparrow \cong & & \uparrow \cong & & \uparrow \cong & & \uparrow \cong \\
 \Sigma^m \mathbb{S}_e \otimes \mathbb{S}_e & \xleftarrow{\text{id} \otimes g} & \Sigma^m \mathbb{S}_e \otimes \Sigma^n \mathbb{S}_e & \xrightarrow{\cong} & \Sigma^n \mathbb{S}_e \otimes \Sigma^m \mathbb{S}_e & \xrightarrow{\text{id} \otimes f} & \Sigma^n \mathbb{S}_e \otimes \mathbb{S}_e \\
 \uparrow f & & \downarrow f \otimes \text{id} & & \downarrow g \otimes \text{id} & & \uparrow g \\
 \mathbb{S}_e & \xrightarrow{f \otimes \text{id}} & \mathbb{S}_e \otimes \Sigma^n \mathbb{S}_e & & \mathbb{S}_e \otimes \Sigma^m \mathbb{S}_e & & \mathbb{S}_e \\
 \downarrow \cong & & \downarrow \text{id} \otimes g & & \downarrow \text{id} \otimes f & & \downarrow \cong \\
 \mathbb{S}_e & \xrightarrow{\cong} & \mathbb{S}_e \otimes \mathbb{S}_e & \xrightarrow{\cong} & \mathbb{S}_e \otimes \mathbb{S}_e & \xrightarrow{\cong} & \mathbb{S}_e
 \end{array}$$

Here, the composition of the bottom line just gives $\text{id}_{\mathbb{S}_e}$ by one of the coherence axioms for a symmetric monoidal category.

Proposition 1.49. *A compatibly stable and closed monoidal derivator \mathbb{D} is canonically endowed with a linear structure over the graded-commutative ring $\mathbf{hom}_{\mathbb{D}(e)}(\mathbb{S}_e, \mathbb{S}_e)_\bullet$, i.e., we have a morphism of graded rings*

$$\mathbf{hom}_{\mathbb{D}(e)}(\mathbb{S}_e, \mathbb{S}_e)_\bullet \longrightarrow \mathbf{Z}_\bullet(\mathbb{D}).$$

Proof. We only give a sketch of the proof. Using the same notation as in the unstable case, we obtain a map $\mathbf{hom}_{\mathbb{D}(e)}(\mathbb{S}_e, \mathbb{S}_e)_n = \mathbf{hom}_{\mathbb{D}(e)}(\Sigma^n \mathbb{S}_e, \mathbb{S}_e) \rightarrow \mathbf{nat}(\kappa_{\Sigma^n \mathbb{S}_e} \otimes -, \kappa_{\mathbb{S}_e} \otimes -)$ which can be composed with the following chain of identifications:

$$\begin{aligned}
 \mathbf{nat}(\kappa_{\Sigma^n \mathbb{S}_e} \otimes -, \kappa_{\mathbb{S}_e} \otimes -) &\cong \mathbf{nat}(\Sigma^n \circ (\kappa_{\mathbb{S}_e} \otimes -), \kappa_{\mathbb{S}_e} \otimes -) \\
 &\cong \mathbf{nat}(\Sigma^n \circ (\mathbb{S} \otimes -), \mathbb{S} \otimes -) \\
 &\cong \mathbf{nat}(\Sigma^n, \text{id}_{\mathbb{D}}) \\
 &= \mathbf{Z}_n(\mathbb{D})
 \end{aligned}$$

These assemble together to define the intended map of graded rings $\mathbf{hom}_{\mathbb{D}(e)}(\mathbb{S}_e, \mathbb{S}_e)_\bullet \longrightarrow \mathbf{Z}_\bullet(\mathbb{D})$. \square

This can be applied to interesting derivators which are associated to certain combinatorial, stable, monoidal model categories. We take up again two of the examples of Subsection 1.4 which give rise to compatibly stable and closed monoidal derivators. In both contexts, the differential-graded and the spectral one, it is well-known that they satisfy the compatibility condition.

Example 1.50. Let us consider the projective model structure on the category $\text{Ch}(k)$ of chain complexes over k . The ring of endomorphisms of the monoidal unit $k[0]$ in the homotopy category, i.e., in the derived category $D(k)$ of the ring k , is just the ground ring, i.e., we have $\text{hom}_{D(k)}(k[0], k[0]) \cong k$. Thus, the derivator \mathbb{D}_k is canonically endowed with a k -linear structure. Furthermore, the projective model structure on unbounded chain complexes is a stable model structure so that we obtain even a linear structure over a graded ring by the last corollary. But, since the graded ring of endomorphisms $\text{hom}_{D(k)}(k[0], k[0])_\bullet$ is concentrated in degree zero, we gain no additional structure by considering the graded ring map

$$\text{hom}_{D(k)}(k[0], k[0])_\bullet \longrightarrow \mathbf{Z}_\bullet(\mathbb{D}_k).$$

But, if we consider a commutative differential-graded algebra C over k we have the associated closed monoidal derivator \mathbb{D}_C of C -modules. The monoidal unit in this case is C itself and the ring of graded self-maps in $\mathbb{D}_C(e) = \text{Ho}(\text{Mod } -C)$ is canonically isomorphic to the homology $H_\bullet(C)$. Thus, \mathbb{D}_C is endowed with a linear structure over the graded ring $H_\bullet(C)$ via a map of graded rings

$$H_\bullet(C) \longrightarrow \mathbf{Z}_\bullet(\mathbb{D}_C).$$

Example 1.51. Let us consider the absolute projective stable model structure on the category Sp^Σ . The endomorphisms of the sphere spectrum in the homotopy category, i.e., in *the stable homotopy category* SHC , are the integers, i.e., we have $\text{hom}_{\text{SHC}}(\mathbb{S}, \mathbb{S}) \cong \mathbb{Z}$. Thus, the derivator \mathbb{D}_{Sp} is endowed with a \mathbb{Z} -linear structure what we already knew since \mathbb{D}_{Sp} is stable. But there is even more structure in this case: the graded self-maps of the sphere spectrum in SHC form the graded ring $\pi_\bullet^{\mathbb{S}}$ given by the stable homotopy groups of spheres. Thus, the derivator \mathbb{D}_{Sp} is endowed with a $\pi_\bullet^{\mathbb{S}}$ -linear structure, i.e., we have a map of graded rings

$$\pi_\bullet^{\mathbb{S}} \longrightarrow \mathbf{Z}_\bullet(\mathbb{D}_{\text{Sp}}).$$

In particular, all categories $\mathbb{D}_{\text{Sp}}(K)$ are $\pi_\bullet^{\mathbb{S}}$ -linear categories and all induced functors u^* , $u_!$, and u_* preserve these linear structures. Similarly, if E is a commutative ring spectrum, then the derivator \mathbb{D}_E of right E -module spectra is canonically endowed with a linear structure over the graded ring of self-maps of E in the homotopy category $\text{Ho}(\text{Mod } -E)$. Thus, we obtain a canonical morphism of graded rings

$$\pi_\bullet(E) \longrightarrow \mathbf{Z}_\bullet(\mathbb{D}_E)$$

where $\pi_\bullet(E)$ denotes the graded-commutative ring of homotopy groups of E .

2. DERIVATORS TENSORED OR COTENSORED OVER A MONOIDAL DERIVATOR

2.1. The 2-Grothendieck fibration of tensored categories. Let us motivate the construction of this subsection by an analogy with algebra. For a ring R , we denote by $R - \text{Mod}$ the category of left R -modules. Moreover, given a ring homomorphism $f: R \rightarrow S$ we denote the associated restriction of scalar functor by $f^*: S - \text{Mod} \rightarrow R - \text{Mod}$. Since restricting scalars is functorial we can use the two assignments $R \mapsto R - \text{Mod}$ and $f \mapsto f^*$ to obtain a functor

$$(-) - \text{Mod}: \text{Ring}^{\text{op}} \rightarrow \text{CAT}.$$

In Appendix A, we recall that in the context of a category-valued functor there is the so-called Grothendieck construction [Bor94b, Vis05]: it turns such a functor into a Grothendieck fibration or a Grothendieck opfibration depending on its variance over the domain of the original functor. The basic idea behind this construction is to glue the different values together in order to obtain a single category which memorizes for each object that it lived in the image category of a certain object. Applied to our situation of the functor $\text{Ring}^{\text{op}} \rightarrow \text{CAT}$, the Grothendieck construction gives us the category Mod of modules. An object in this category is a pair (R, M) consisting of a ring R and an R -module M . A morphism $(R, M) \rightarrow (S, N)$ is a pair (f, h) consisting of a map of rings $f: R \rightarrow S$ and a map of R -modules $h: M \rightarrow f^*N$. This category is endowed with the Grothendieck fibration

$$p: \text{Mod} \rightarrow \text{Ring}$$

which projects an object (R, M) resp. a morphism (f, h) onto the first component. There is a further canonical functor associated to Mod , namely the functor $U: \text{Mod} \rightarrow \text{Ab}$ which sends a module to the underlying abelian group. Using the restriction of scalars, this functor sends (R, M) to c_R^*M where $c_R: \mathbb{Z} \rightarrow R$ is the characteristic of the ring. We thus have the following diagram in which the vertical arrow is a Grothendieck fibration:

$$\begin{array}{ccc} \text{Mod} & \xrightarrow{U} & \text{Ab} \\ p \downarrow & & \\ \text{Ring} & & \end{array}$$

For a ring R the fiber of p over R , i.e., the left pullback below, is canonically isomorphic to the category $R - \text{Mod}$. Given an abelian group A , we can consider the category $\text{Mod}(A) = U^{-1}(A)$ of *module structures on A* which is defined to be the pullback on the right-hand-side:

$$\begin{array}{ccc} R - \text{Mod} & \longrightarrow & \text{Mod} \\ \downarrow & & \downarrow p \\ e & \xrightarrow{R} & \text{Ring} \end{array} \qquad \begin{array}{ccc} \text{Mod}(A) & \longrightarrow & \text{Mod} \\ \downarrow & & \downarrow U \\ e & \xrightarrow{A} & \text{Ab} \end{array}$$

The universal example of a ring acting on A is given by the ring $\text{end}(A) = \text{hom}_{\mathbb{Z}}(A, A)$ of \mathbb{Z} -linear endomorphisms together with the action by evaluation. The universal property of this action is precisely the fact that the pair consisting of the ring $\text{end}(A)$ and this action is a terminal object of the category $\text{Mod}(A)$.

In Subsection 2.3, we want to redo the same reasoning where we replace the closed monoidal category of abelian groups by the Cartesian closed monoidal 2-category Der of derivators. In particular, given a derivator \mathbb{D} we want to construct a derivator $\text{END}(\mathbb{D})$ of endomorphisms and show that it satisfies the 2-categorical version of this universal property.

Recall that we have defined a monoidal derivator \mathbb{E} as a monoidal object in Der . We have then observed that such an \mathbb{E} is, in particular, a 2-functor $\mathbb{E}: \text{Cat}^{\text{op}} \rightarrow \text{CAT}$ which factors over the 2-category MonCAT of monoidal categories. To have a similar ‘pointwise description’ for derivators which are tensored over a monoidal derivator, it is convenient to consider the 2-category ModCAT of tensored categories. This 2-category comes up naturally as a 2-categorical Grothendieck construction as we describe it now. In fact, this is just a special case of results from Appendix B applied to the Cartesian monoidal 2-category CAT . Since the details for the case of an arbitrary monoidal 2-category are given in that appendix we allow ourselves to be sketchy here.

Let us recall that a monoidal category \mathcal{C} is just a monoidal object in the Cartesian monoidal 2-category CAT . Associated to such a monoidal category \mathcal{C} there is the 2-category $\mathcal{C} - \text{Mod}^{\text{lax}}$ of categories which are left \mathcal{C} -modules. An object of this 2-category is a category \mathcal{D} endowed with a left action $\otimes: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D}$ and certain specified coherence isomorphisms expressing adequate multiplicativity and unitality conditions. There are the notions of lax \mathcal{C} -module morphisms and \mathcal{C} -module transformations between two such so that we indeed obtain a 2-category $\mathcal{C} - \text{Mod}^{\text{lax}}$.

Moreover, given a monoidal functor $f: \mathcal{C}_1 \rightarrow \mathcal{C}_2$, we obtain an induced restriction of scalars functor $f^*: \mathcal{C}_2 - \text{Mod}^{\text{lax}} \rightarrow \mathcal{C}_1 - \text{Mod}^{\text{lax}}$ and similar observations can be made for monoidal transformations between monoidal functors. Thus, summarizing these associations, we obtain a 2-functor

$$(-) - \text{Mod}^{\text{lax}}: \text{MonCAT}^{\text{op}} \rightarrow 2\text{-CAT}.$$

Here, 2-CAT denotes the 2-category of large 2-categories. An application of the 2-categorical Grothendieck construction (cf. Appendix A) gives us the 2-Grothendieck fibration of tensored categories $p: \text{ModCAT}^{\text{lax}} \rightarrow \text{MonCAT}$. Here, $\text{ModCAT}^{\text{lax}}$ is the 2-category where the objects are pairs $(\mathcal{C}, \mathcal{D})$ consisting of a monoidal category \mathcal{C} and a category \mathcal{D} which is left-tensored over \mathcal{C} . A morphism $(\mathcal{C}_1, \mathcal{D}_1) \rightarrow (\mathcal{C}_2, \mathcal{D}_2)$ is a pair (f, h) where $f: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is a monoidal functor and $h: \mathcal{D}_1 \rightarrow f^*\mathcal{D}_2$ is a lax morphism of \mathcal{C}_1 -modules. We will not make the 2-morphisms explicit here since this is done in more generality in Appendix A. Let us form the 2-subcategory $\text{ModCAT} \subseteq \text{ModCAT}^{\text{lax}}$ given by all objects, the strong module morphisms and all 2-cells. Thus, the morphisms are expected to be multiplicative up to specified natural isomorphism. The projection on the second component defines a 2-functor $U: \text{ModCAT} \rightarrow \text{CAT}$ which will be used in the next subsection to express the universal property of the (pre)derivator of endomorphisms. Thus, as an upshot we obtain the following diagram of 2-categories in which the vertical arrow is again called the 2-Grothendieck fibration of tensored categories:

$$\begin{array}{ccc} \text{ModCAT} & \xrightarrow{U} & \text{CAT} \\ p \downarrow & & \\ \text{MonCAT} & & \end{array}$$

2.2. Tensors and cotensors on derivators. In this subsection, we want to formalize actions of monoidal derivators on other derivators. Recall that a monoidal derivator is just a monoidal object in the Cartesian 2-category Der . As it is the case for every monoidal 2-category, there is thus the derived notion of a module over a monoidal derivator. Since the coherence conditions are harder to find in the literature we include them in Appendix B. In contrast, we allow ourselves to be a bit sketchy here in setting up the 2-category of left module derivators over a monoidal derivator.

So, let $(\mathbb{E}, \otimes, \mathbb{S})$ be a monoidal derivator and let \mathbb{D} be a derivator. A (left) \mathbb{E} -module structure on \mathbb{D} is a triple (\otimes, m, u) consisting of a morphism of derivators $\otimes: \mathbb{E} \times \mathbb{D} \rightarrow \mathbb{D}$ together with

natural isomorphisms m and u as indicated in the following diagrams:

$$\begin{array}{ccc} \mathbb{E} \times \mathbb{E} \times \mathbb{D} & \xrightarrow{\text{id} \times \otimes} & \mathbb{E} \times \mathbb{D} \\ \otimes \times \text{id} \downarrow & \Downarrow & \downarrow \otimes \\ \mathbb{E} \times \mathbb{D} & \xrightarrow{\otimes} & \mathbb{D} \end{array} \qquad \begin{array}{ccc} e \times \mathbb{D} & \xrightarrow{\mathbb{S}} & \mathbb{E} \times \mathbb{D} \\ \cong \searrow & \nearrow & \downarrow \otimes \\ & & \mathbb{D} \end{array}$$

These natural isomorphisms expressing the multiplicativity and unitality of the action are subject to certain coherence axioms. Given two \mathbb{E} -modules $(\mathbb{D}_i, \otimes, m_i, u_i)$, $i = 1, 2$, a *lax* \mathbb{E} -module morphism $\mathbb{D}_1 \rightarrow \mathbb{D}_2$ is a morphism $F: \mathbb{D}_1 \rightarrow \mathbb{D}_2$ of derivators together with a natural transformation m_F

$$\begin{array}{ccc} \mathbb{E} \times \mathbb{D}_1 & \xrightarrow{\otimes} & \mathbb{D}_1 \\ F \downarrow & \nearrow & \downarrow F \\ \mathbb{E} \times \mathbb{D}_2 & \xrightarrow{\otimes} & \mathbb{D}_2 \end{array}$$

which again have to satisfy certain coherence conditions. If the 2-cell m_F belonging to such a morphism is invertible we speak of a *strong morphism* or simply of a *morphism of \mathbb{E} -modules*. Finally, given two \mathbb{E} -module morphisms (F, m_F) and (G, m_G) , a natural transformation $\phi: F \rightarrow G$ is an \mathbb{E} -module transformation if the following diagram commutes:

$$\begin{array}{ccc} - \otimes F(-) & \xrightarrow{\phi} & - \otimes G(-) \\ m_F \downarrow & & \downarrow m_G \\ F(- \otimes -) & \xrightarrow{\phi} & G(- \otimes -) \end{array}$$

With these notions, we obtain the 2-category $\mathbb{E}\text{-Mod}^{\text{lax}}$ of \mathbb{E} -modules, lax \mathbb{E} -module morphisms and \mathbb{E} -module transformations. There is a similar 2-category $\mathbb{E}\text{-Mod}$ if one only takes the strong \mathbb{E} -module morphisms. Moreover, there are different flavors of actions like exact actions in the stable case, colimit-preserving actions and so on. We do not give explicit definitions for all of these but, nevertheless, allow ourselves to use these notions. Moreover, using the Cartesian 2-category PDer instead of Der we obtain corresponding notions for prederivators. Since the dual of a monoidal derivator is again a monoidal derivator we can make the following definition.

Definition 2.1. Let \mathbb{E} be a monoidal derivator. A derivator is *tensoried over* \mathbb{E} if it is a left module over \mathbb{E} and is *cotensoried over* \mathbb{E} if it is a right module over \mathbb{E}^{op} . A left \mathbb{E} -module \mathbb{D} is called a *closed module* if the action map $\otimes: \mathbb{E} \times \mathbb{D} \rightarrow \mathbb{D}$ is a left adjoint of two variables.

Given a monoidal derivator \mathbb{E} we just constructed the 2-category $\mathbb{E}\text{-Mod}^{\text{lax}}$ of \mathbb{E} -modules. We leave it to the reader to check that a monoidal morphism of derivators $F: \mathbb{E}_1 \rightarrow \mathbb{E}_2$ induces a restriction of scalars 2-functor $F^*: \mathbb{E}_2\text{-Mod}^{\text{lax}} \rightarrow \mathbb{E}_1\text{-Mod}^{\text{lax}}$. The assignment $F \mapsto F^*$ is functorial and there is a similar observation for monoidal transformations so that we end up with a 2-functor

$$(-)\text{-Mod}^{\text{lax}}: \text{MonDer}^{\text{op}} \rightarrow 2\text{-CAT}.$$

Thus, we can again apply the 2-categorical Grothendieck construction of Appendix A in order to obtain the 2-Grothendieck fibration of tensoried derivators $p: \text{ModDer}^{\text{lax}} \rightarrow \text{MonDer}$. If we form the 2-subcategory consisting of all objects, the strong module morphisms only and all 2-cells

then we obtain the 2-category \mathbf{ModDer} . Moreover, it is easy to verify that there is a 2-functor $U: \mathbf{ModDer} \rightarrow \mathbf{Der}$ which sends an object, i.e., a pair consisting of a monoidal derivator and a module over it to the derivator underlying the module derivator. Thus we are in the situation of the following diagram:

$$\begin{array}{ccc} \mathbf{ModDer} & \xrightarrow{U} & \mathbf{Der} \\ p \downarrow & & \\ \mathbf{MonDer} & & \end{array}$$

Given a derivator \mathbb{D} let us call the 2-category $\mathbf{Mod}(\mathbb{D}) = U^{-1}(\mathbb{D})$ the *2-category of module structures on \mathbb{D}* .

Before we give some immediate examples let us mention the ‘pointwise description’ of tensored derivators. Let \mathbb{D} be an \mathbb{E} -module derivator and let J be a category. Then it is immediate that $\mathbb{D}(J)$ is canonically an $\mathbb{E}(J)$ -module. Moreover, let us consider a functor $u: J \rightarrow K$. By the reasoning in Section 1 the functor $u_{\mathbb{E}}^*: \mathbb{E}(K) \rightarrow \mathbb{E}(J)$ is canonically endowed with a strong monoidal structure. Thus, we have the induced restriction of scalars functor $\mathbb{E}(u)^*: \mathbb{E}(J)\text{-Mod} \rightarrow \mathbb{E}(K)\text{-Mod}$. In particular, we can consider the $\mathbb{E}(K)$ -module $\mathbb{E}(u)^*\mathbb{D}(J)$ for which the action map is given by

$$\mathbb{E}(K) \times \mathbb{D}(J) \xrightarrow{u_{\mathbb{E}}^* \times \text{id}} \mathbb{E}(J) \times \mathbb{D}(J) \xrightarrow{\otimes} \mathbb{D}(J).$$

Since $\otimes: \mathbb{E} \times \mathbb{D} \rightarrow \mathbb{D}$ is a morphism of derivators we have a natural isomorphism γ_u^{\otimes} which can be rewritten as:

$$\begin{array}{ccc} \mathbb{E}(K) \times \mathbb{D}(K) & \xrightarrow{\otimes} & \mathbb{D}(K) \\ \text{id} \times u_{\mathbb{D}}^* \downarrow & \Rightarrow & \downarrow u_{\mathbb{D}}^* \\ \mathbb{E}(K) \times \mathbb{D}(J) & \xrightarrow{u_{\mathbb{E}}^* \times \text{id}} \mathbb{E}(J) \times \mathbb{D}(J) \xrightarrow{\otimes} & \mathbb{D}(J) \end{array}$$

Similarly to the case of a monoidal derivator one can check that the pair $(u_{\mathbb{D}}^*, \gamma_u^{\otimes})$ defines a morphism $\mathbb{D}(K) \rightarrow \mathbb{E}(u)^*\mathbb{D}(J)$ in $\mathbb{E}(K)\text{-Mod}$. Said differently, we have a morphism

$$u^* = (u_{\mathbb{E}}^*, (u_{\mathbb{D}}^*, \gamma_u^{\otimes})): (\mathbb{E}(K), \mathbb{D}(K)) \rightarrow (\mathbb{E}(J), \mathbb{D}(J))$$

in \mathbf{ModCAT} and one can make similar observations for natural transformations. Using the 2-Grothendieck fibration of left-tensored categories and the corresponding forgetful functor as indicated in

$$\begin{array}{ccc} \mathbf{ModCAT} & \xrightarrow{U} & \mathbf{CAT} \\ p \downarrow & & \\ \mathbf{MonCAT} & & \end{array}$$

we can hence give the following ‘pointwise description’. A left-tensored prederivator is a 2-functor $\mathbb{D}: \mathbf{Cat}^{\text{op}} \rightarrow \mathbf{ModCAT}$. Such a 2-functor has an underlying prederivator $U \circ \mathbb{D}: \mathbf{Cat}^{\text{op}} \rightarrow \mathbf{CAT}$ and this prederivator is then left-tensored over the monoidal prederivator $\mathbb{E} = p \circ \mathbb{D}: \mathbf{Cat}^{\text{op}} \rightarrow \mathbf{MonCAT}$. A left-tensored derivator is a left-tensored prederivator such that the underlying prederivator is a derivator.

Example 2.2. The 2-functor $y: \mathbf{CAT} \rightarrow \mathbf{PDer}: \mathcal{C} \mapsto y(\mathcal{C})$ sending a category to the represented prederivator preserves 2-products and hence monoidal objects and modules. It follows that with

a monoidal category \mathcal{C} also the represented prederivator \mathcal{C} is canonically monoidal and there is a similar remark for \mathcal{C} -modules. Thus, we have induced 2-functors

$$y: \mathcal{C} - \text{Mod} \longrightarrow y(\mathcal{C}) - \text{Mod} \quad \text{and} \quad y: \text{ModCAT} \longrightarrow \text{ModPDer}.$$

Example 2.3. Every monoidal derivator is canonically a left and a right module over itself. In particular, this is the case for the monoidal derivators associated to combinatorial monoidal model categories.

In the last section we proved that an additive, monoidal derivator with an additive monoidal structure is canonically endowed with a linear structure over the ring of endomorphisms of the monoidal unit of the underlying monoidal category. Recall from the proof of that result that we only used the fact that the monoidal derivator is left-tensored over itself. Thus, we obtain immediately the following more general result.

Corollary 2.4. *Let \mathbb{E} be an additive, monoidal derivator with an additive monoidal structure. Any additive left \mathbb{E} -module \mathbb{D} is canonically endowed with a linear structure over $\text{hom}_{\mathbb{E}(e)}(\mathbb{S}_e, \mathbb{S}_e)$.*

In this context, an additive \mathbb{E} -module \mathbb{D} is of course an \mathbb{E} -module \mathbb{D} such that the underlying derivator and the action are additive. We give more specific examples in Subsection 2.4 where we also apply this last corollary. But before that let us develop a bit more of the general theory and show that the Cartesian monoidal 2-categories PDer and Der are closed. This will also put into perspective the above corollary in that the ring map giving the linear structure is only a shadow of the fact that there is monoidal morphism of derivators in the background.

2.3. The closedness of the Cartesian monoidal 2-categories PDer and Der . Recall from classical category theory that given a category \mathcal{D} there is the monoidal category of endomorphisms of \mathcal{D} . This is the universal example of a monoidal category acting from the left on \mathcal{D} . The corresponding result is also true in the world of ∞ -categories as is shown by Lurie in Chapter 6 of [Lur11]. If one wants to give a corresponding result in the world of prederivators one should at first show that the 2-category PDer is Cartesian closed in a sense which is to be specified.

For this purpose, let us recall from [Gro10a] that PDer is right-tensored over Cat^{op} . In fact, for every prederivator \mathbb{D} and every small category J we have the prederivator

$$\mathbb{D}_J = \mathbb{D}(J \times -): \text{Cat}^{\text{op}} \xrightarrow{J \times -} \text{Cat}^{\text{op}} \xrightarrow{\mathbb{D}} \text{CAT}.$$

This gives us an induced 2-functor $(-)_(-): \text{PDer} \times \text{Cat}^{\text{op}} \longrightarrow \text{PDer}$ which turns PDer into a right Cat^{op} -module.

The aim is now to show that the Cartesian monoidal 2-category PDer is closed in the *bicategorical* sense. Thus, given three prederivators \mathbb{D} , \mathbb{D}' , and \mathbb{D}'' we want to construct a prederivator $\text{HOM}(\mathbb{D}', \mathbb{D}'')$ of morphisms and a natural equivalence of categories

$$\text{Hom}(\mathbb{D} \times \mathbb{D}', \mathbb{D}'') \xrightarrow{\cong} \text{Hom}(\mathbb{D}, \text{HOM}(\mathbb{D}', \mathbb{D}'')).$$

In more formal terms, we are looking for a *biadjunction* (see [Gra74], [Fio06, Chapter 9] or Appendix B.2). Note that we have $\text{Hom}(-, -) = \text{PsNat}(-, -)$ in our situation. For a category J let us again denote the represented prederivator by $y(J)$. If we now assume that we were given such a construction of an internal hom $\text{HOM}(-, -)$ then for an arbitrary category J we would deduce the

following chain of natural equivalences of categories:

$$\begin{aligned} \mathrm{HOM}(\mathbb{D}, \mathbb{D}')(J) &\simeq \mathrm{PsNat}(y(J), \mathrm{HOM}(\mathbb{D}, \mathbb{D}')) \\ &\simeq \mathrm{PsNat}(y(J) \times \mathbb{D}, \mathbb{D}') \\ &\simeq \mathrm{PsNat}(\mathbb{D}, \mathrm{HOM}(y(J), \mathbb{D}')) \end{aligned}$$

The equivalences are given by the bicategorical Yoneda lemma and the assumed closedness property. So, we have reduced the problem to giving an identification of $\mathrm{HOM}(y(J), \mathbb{D}')$ for a category J and a prederivator \mathbb{D}' . By similar arguments and for a category K we obtain natural equivalences of categories as follows:

$$\begin{aligned} \mathrm{HOM}(y(J), \mathbb{D}')(K) &\simeq \mathrm{PsNat}(y(K), \mathrm{HOM}(y(J), \mathbb{D}')) \\ &\simeq \mathrm{PsNat}(y(K) \times y(J), \mathbb{D}') \\ &\simeq \mathrm{PsNat}(y(J \times K), \mathbb{D}') \\ &\simeq \mathbb{D}'(J \times K) = \mathbb{D}'_J(K) \end{aligned}$$

Putting these chains of natural equivalences together we would obtain as an upshot the following equivalence which motivates the next definition

$$\mathrm{HOM}(\mathbb{D}, \mathbb{D}')(J) \simeq \mathrm{PsNat}(\mathbb{D}, \mathbb{D}'_J) = \mathrm{Hom}(\mathbb{D}, \mathbb{D}'_J).$$

Definition 2.5. The *prederivator hom* HOM is the 2-functor $\mathrm{HOM}: \mathrm{PDer}^{\mathrm{op}} \times \mathrm{PDer} \rightarrow \mathrm{PDer}$ which is adjoint to

$$\mathrm{PDer}^{\mathrm{op}} \times \mathrm{PDer} \times \mathrm{Cat}^{\mathrm{op}} \xrightarrow{\mathrm{id} \times (-)_{(-)}} \mathrm{PDer}^{\mathrm{op}} \times \mathrm{PDer} \xrightarrow{\mathrm{Hom}} \mathrm{CAT}.$$

Given two prederivators \mathbb{D} and \mathbb{D}' the prederivator $\mathrm{HOM}(\mathbb{D}, \mathbb{D}')$ is called the *prederivator of morphisms from \mathbb{D} to \mathbb{D}'* . Moreover, for a single prederivator \mathbb{D} we set

$$\mathrm{END}(\mathbb{D}) = \mathrm{HOM}(\mathbb{D}, \mathbb{D}) \in \mathrm{PDer}$$

and call this the *prederivator of endomorphisms of \mathbb{D}* . For derivators \mathbb{D} and \mathbb{D}' we define $\mathrm{HOM}(\mathbb{D}, \mathbb{D}')$ and $\mathrm{END}(\mathbb{D})$ using the underlying prederivators.

More explicitly, for two prederivators \mathbb{D}, \mathbb{D}' , and a small category J we have thus

$$\mathrm{HOM}(\mathbb{D}, \mathbb{D}')(J) = \mathrm{Hom}(\mathbb{D}, \mathbb{D}'_J) \in \mathrm{CAT}.$$

Our next aim is to show that the bifunctor HOM defines an internal hom in the bicategorical sense (cf. Appendix B.2) for the Cartesian monoidal 2-category PDer . As a preparation for that result let us construct pseudo-natural transformations which will be used in order to define the adjunction.

Lemma 2.6. For $\mathbb{D}, \mathbb{D}' \in \mathrm{PDer}$ there are canonical morphisms $\eta: \mathbb{D} \rightarrow \mathrm{HOM}(\mathbb{D}', \mathbb{D} \times \mathbb{D}')$ and $\epsilon: \mathrm{HOM}(\mathbb{D}, \mathbb{D}') \times \mathbb{D} \rightarrow \mathbb{D}'$. Moreover, η resp. ϵ is pseudo-natural in \mathbb{D} resp. \mathbb{D}' .

Proof. Let us begin with the construction of $\eta: \mathbb{D} \rightarrow \mathrm{HOM}(\mathbb{D}', \mathbb{D} \times \mathbb{D}')$. For a category K we thus have to construct a functor $\eta_K: \mathbb{D}(K) \rightarrow \mathrm{Hom}(\mathbb{D}', (\mathbb{D} \times \mathbb{D}')_K)$. For an arbitrary category J , let us define the component $\eta_K(-)_J: \mathbb{D}(K) \rightarrow \mathrm{Fun}(\mathbb{D}'(J), \mathbb{D}(K \times J) \times \mathbb{D}'(K \times J))$ to be adjoint to the functor

$$(\mathrm{pr}_1^*, \mathrm{pr}_2^*): \mathbb{D}(K) \times \mathbb{D}'(J) \rightarrow \mathbb{D}(K \times J) \times \mathbb{D}'(K \times J),$$

i.e., we set $\eta_K(X)_J(Y) = (\text{pr}_1^*(X), \text{pr}_2^*(Y))$ for $X \in \mathbb{D}(K)$ and $Y \in \mathbb{D}'(J)$. For a functor $u: J_1 \rightarrow J_2$ we can use the diagram

$$\begin{array}{ccccc} K & \xleftarrow{\text{pr}_1} & K \times J_1 & \xrightarrow{\text{pr}_2} & J_1 \\ = \downarrow & & \downarrow \text{id} \times u & & \downarrow u \\ K & \xleftarrow{\text{pr}_1} & K \times J_2 & \xrightarrow{\text{pr}_2} & J_2 \end{array}$$

to convince ourselves that the maps $\{\eta_K(-)_J\}_J$ assemble to define a strict morphism of prederivators $\eta_K(-): \mathbb{D}' \rightarrow (\mathbb{D} \times \mathbb{D}')_K$. A similar reasoning shows that also the $\{\eta_K\}_K$ together define a morphism of prederivators $\eta: \mathbb{D} \rightarrow \text{HOM}(\mathbb{D}', \mathbb{D} \times \mathbb{D}')$. Let us show now that η is pseudo-natural in \mathbb{D} . For this purpose, let $F: \mathbb{D}_1 \rightarrow \mathbb{D}_2$ be a morphism of prederivators, let K be a category and let us consider the following non-commutative diagram:

$$\begin{array}{ccc} \mathbb{D}_1(K) & \xrightarrow{\eta_K} & \text{Hom}(\mathbb{D}', (\mathbb{D}_1 \times \mathbb{D}')_K) \\ F_K \downarrow & & \downarrow (F \times \text{id})_{K*} \\ \mathbb{D}_2(K) & \xrightarrow{\eta_K} & \text{Hom}(\mathbb{D}', (\mathbb{D}_2 \times \mathbb{D}')_K) \end{array}$$

For $X \in \mathbb{D}_1(K)$ and $Y \in \mathbb{D}'(J)$ we can use the natural isomorphisms belonging to the morphism F to deduce the following one:

$$\begin{aligned} ((F \times \text{id})_{K*} \circ \eta_K)(X)_J(Y) &= (F \times \text{id})_K \circ (\text{pr}_1^*(X), \text{pr}_2^*(Y)) \\ &= (F_K \times_J \text{pr}_1^*(X), \text{pr}_2^*(Y)) \\ &\cong (\text{pr}_1^* F_K(X), \text{pr}_2^*(Y)) \\ &= (\eta_K \circ F_K)(X)_J(Y) \end{aligned}$$

One checks that these isomorphisms can be used to obtain a pseudo-natural transformation η as intended.

Let us now construct a morphism $\epsilon: \text{HOM}(\mathbb{D}, \mathbb{D}') \times \mathbb{D} \rightarrow \mathbb{D}'$. Thus, for a category K we have to define a functor $\epsilon_K: \text{Hom}(\mathbb{D}, \mathbb{D}'_K) \times \mathbb{D}(K) \rightarrow \mathbb{D}'(K)$. This is defined to be the following composition

$$\begin{array}{ccc} \text{Hom}(\mathbb{D}, \mathbb{D}'_K) \times \mathbb{D}(K) & \xrightarrow{\epsilon_K} & \mathbb{D}'(K) \\ \text{pr} \times \text{id} \downarrow & & \uparrow \Delta_K^* \\ \text{Fun}(\mathbb{D}(K), \mathbb{D}'_K(K)) \times \mathbb{D}(K) & \xrightarrow{\text{ev}} & \mathbb{D}'_K(K) = \mathbb{D}'(K \times K) \end{array}$$

i.e., for $F \in \text{Hom}(\mathbb{D}, \mathbb{D}'_K)$ and $X \in \mathbb{D}(K)$ we set $\epsilon_K(F, X) = \Delta_K^* F_K(X)$. To see that these ϵ_K assemble to define a morphism of prederivators let us consider a functor $u: J \rightarrow K$ and let us check that there is a canonical natural isomorphism γ_u^ϵ as in:

$$\begin{array}{ccc} \text{Hom}(\mathbb{D}, \mathbb{D}'_K) \times \mathbb{D}(K) & \xrightarrow{\epsilon_K} & \mathbb{D}'(K) \\ \mathbb{D}'_{u*} \times \mathbb{D}(u) \downarrow & \cong & \downarrow \mathbb{D}(u) \\ \text{Hom}(\mathbb{D}, \mathbb{D}'_J) \times \mathbb{D}(J) & \xrightarrow{\epsilon_J} & \mathbb{D}'(J) \end{array}$$

But evaluated at $F \in \mathbf{Hom}(\mathbb{D}, \mathbb{D}'_K)$ and $X \in \mathbb{D}(K)$ we can use the natural isomorphisms belonging to F to obtain

$$\begin{aligned}
\mathbb{D}(u) \circ \epsilon_K(F, X) &= u^* \Delta_K^* F_K(X) \\
&= \Delta_J^*(u \times \text{id})^*(\text{id} \times u)^* F_K(X) \\
&\cong \Delta_J^*(u \times \text{id})^* F_J(u^* X) \\
&= \Delta_J^*(\mathbb{D}'_{u^*} F)_J(u^* X) \\
&= \epsilon_J \circ (\mathbb{D}'_{u^*} \times \mathbb{D}(u))(F, X).
\end{aligned}$$

Thus, slightly sloppy we have $(\gamma_u^\epsilon)_{F, X} = (\gamma_u^F)_X$. Again one checks that these isomorphisms assemble together to define a morphism of prederivators $\epsilon: \mathbf{HOM}(\mathbb{D}, \mathbb{D}') \times \mathbb{D} \rightarrow \mathbb{D}'$. In order to show that ϵ is pseudo-natural let us consider a morphism of prederivators $G: \mathbb{D}' \rightarrow \mathbb{D}''$ and let us construct a natural isomorphism as in:

$$\begin{array}{ccc}
\mathbf{HOM}(\mathbb{D}, \mathbb{D}') \times \mathbb{D} & \xrightarrow{\epsilon} & \mathbb{D}' \\
G_* \times \text{id} \downarrow & \cong & \downarrow G \\
\mathbf{HOM}(\mathbb{D}, \mathbb{D}'') \times \mathbb{D} & \xrightarrow{\epsilon} & \mathbb{D}''
\end{array}$$

Using the natural isomorphisms belonging to G and a similar calculation as above we obtain for $F \in \mathbf{Hom}(\mathbb{D}, \mathbb{D}'_K)$ and $X \in \mathbb{D}(K)$ the following isomorphism:

$$\begin{aligned}
(\epsilon \circ (G_* \times \text{id}))_K(F, X) &= \Delta_K^* G_{K \times K} F_K(X) \\
&\cong G_K \Delta_K^* F_K(X) \\
&= (G \circ \epsilon)_K(F, X)
\end{aligned}$$

These isomorphisms give us the desired natural isomorphisms turning ϵ into a pseudo-natural transformation which concludes the proof. \square

With this preparation we can now give the following desired result.

Proposition 2.7. *The prederivator of morphisms defines an internal hom in the Cartesian monoidal 2-category \mathbf{PDer} , i.e., for three prederivators \mathbb{D} , \mathbb{D}' , and \mathbb{D}'' we have pseudo-natural equivalences of categories:*

$$\mathbf{Hom}_{\mathbf{PDer}}(\mathbb{D} \times \mathbb{D}', \mathbb{D}'') \simeq \mathbf{Hom}_{\mathbf{PDer}}(\mathbb{D}, \mathbf{HOM}(\mathbb{D}', \mathbb{D}''))$$

Proof. We use the pseudo-natural transformations of the last lemma to define functors l and r by $l = \eta^* \circ \mathbf{HOM}(\mathbb{D}', -)$ and $r = \epsilon_* \circ (- \times \mathbb{D}')$ as depicted in:

$$\begin{array}{ccc}
\mathbf{Hom}(\mathbb{D} \times \mathbb{D}', \mathbb{D}'') & \xrightarrow{\mathbf{HOM}(\mathbb{D}', -)} & \mathbf{Hom}(\mathbf{HOM}(\mathbb{D}', \mathbb{D} \times \mathbb{D}'), \mathbf{HOM}(\mathbb{D}', \mathbb{D}'')) \\
\epsilon_* \uparrow & & \downarrow \eta^* \\
\mathbf{Hom}(\mathbb{D} \times \mathbb{D}', \mathbf{HOM}(\mathbb{D}', \mathbb{D}'') \times \mathbb{D}') & \xleftarrow{- \times \mathbb{D}'} & \mathbf{Hom}(\mathbb{D}, \mathbf{HOM}(\mathbb{D}', \mathbb{D}''))
\end{array}$$

Let us check that these are inverse equivalences of categories and let us begin by showing that we have a natural isomorphism $r \circ l \cong \text{id}$. For this purpose, let us consider a morphism $F: \mathbb{D} \times \mathbb{D}' \rightarrow \mathbb{D}''$.

We claim that we have the following diagram which commutes up to a natural isomorphism:

$$\begin{array}{ccccc}
\mathbb{D} \times \mathbb{D}' & \xrightarrow{\eta \times 1} & \mathbf{HOM}(\mathbb{D}', \mathbb{D} \times \mathbb{D}') \times \mathbb{D}' & \xrightarrow{F_* \times \text{id}} & \mathbf{HOM}(\mathbb{D}', \mathbb{D}'') \times \mathbb{D}' \\
& \searrow & \downarrow \epsilon & \cong & \downarrow \epsilon \\
& & \mathbb{D} \times \mathbb{D}' & \xrightarrow{F} & \mathbb{D}'' \\
& \text{id} & & &
\end{array}$$

By the lemma we only have to check that the triangle commutes. But using the explicit formulas of the last proof we can calculate for $X \in \mathbb{D}(K)$ and $Y \in \mathbb{D}'(K)$ the following:

$$(\epsilon \circ (\eta \times 1))(X, Y) = \epsilon(\eta(X), Y) = \Delta_K^* \eta_K(X)_K(Y) = \Delta_K^*(\text{pr}_1^* X, \text{pr}_2^* Y) = (X, Y)$$

Since the longer boundary path from $\mathbb{D} \times \mathbb{D}'$ to \mathbb{D}'' calculates $r \circ l(F)$ we conclude $r \circ l \cong \text{id}$.

Let us show that we also have $l \circ r \cong \text{id}$. Thus, let us consider a morphism $G: \mathbb{D} \rightarrow \mathbf{HOM}(\mathbb{D}', \mathbb{D}'')$ and let us show that the following diagram commutes up to natural isomorphisms:

$$\begin{array}{ccccc}
\mathbf{HOM}(\mathbb{D}', \mathbb{D} \times \mathbb{D}') & \xrightarrow{\mathbf{HOM}(\mathbb{D}', G \times 1)} & \mathbf{HOM}(\mathbb{D}', \mathbf{HOM}(\mathbb{D}', \mathbb{D}'') \times \mathbb{D}') & \xrightarrow{\mathbf{HOM}(\mathbb{D}', \epsilon)} & \mathbf{HOM}(\mathbb{D}', \mathbb{D}'') \\
\uparrow \eta & \cong & \uparrow \eta & \cong & \nearrow \text{id} \\
\mathbb{D} & \xrightarrow{G} & \mathbf{HOM}(\mathbb{D}', \mathbb{D}'') & &
\end{array}$$

By the last lemma it remains to show that the triangle commutes up to a natural isomorphism. But for $F \in \mathbf{Hom}(\mathbb{D}', \mathbb{D}''_K)$ and $Y \in \mathbb{D}''_J$ we can again use the formulas of the last proof to make the following calculation:

$$\begin{aligned}
((\epsilon_*)_K \circ \eta_K)(F)_J(Y) &= \epsilon_{K \times J}(\text{pr}_1^* F, \text{pr}_2^* Y) \\
&= \Delta_{K \times J}^*(\text{pr}_1^*)_{K \times J} F_{K \times J} \text{pr}_2^* Y \\
&\cong \Delta_{K \times J}^*(\text{pr}_1 \times 1 \times 1)^*(1 \times \text{pr}_2)^* F_J(Y) \\
&= F_J(Y).
\end{aligned}$$

Here, we used the natural isomorphism belonging to F and the commutativity of the following diagram in which the composition of the bottom row is just the identity:

$$\begin{array}{ccccccc}
\mathbb{D}''_K(J) & \xrightarrow{\text{pr}_2^*} & \mathbb{D}''_K(K \times J) & \xrightarrow{(\text{pr}_1^*)_{(K \times J)}} & \mathbb{D}''_{K \times J}(K \times J) & & \\
= \downarrow & & = \downarrow & & \downarrow = & & \\
\mathbb{D}''(K \times J) & \xrightarrow{(1 \times \text{pr}_2)^*} & \mathbb{D}''(K \times K \times J) & \xrightarrow{(\text{pr}_1 \times 1 \times 1)^*} & \mathbb{D}''(K \times J \times K \times J) & \xrightarrow{\Delta_{K \times J}^*} & \mathbb{D}''(K \times J)
\end{array}$$

It follows that the triangle in the previous diagram also commutes up to natural isomorphism. Again, the longer path passing through the boundary from \mathbb{D} to $\mathbf{HOM}(\mathbb{D}', \mathbb{D}'')$ is $l \circ r(G)$ and we can thus deduce that we have a natural isomorphism $l \circ r \cong \text{id}$. This concludes the proof of the proposition. \square

From classical category theory we know that a functor category $\mathbf{Fun}(J, \mathcal{C})$ is (co)complete as soon as this is the case for the target category \mathcal{C} . The corresponding result for derivators also holds true as we will show now.

Proposition 2.8. *If \mathbb{D} is a prederivator and \mathbb{D}' a derivator then the prederivator $\mathbf{HOM}(\mathbb{D}, \mathbb{D}')$ is a derivator. If \mathbb{D}' is in addition pointed resp. additive then this is also the case for $\mathbf{HOM}(\mathbb{D}, \mathbb{D}')$. Thus, the 2-categories \mathbf{Der} , \mathbf{Der}^* , and $\mathbf{Der}^{\text{add}}$ are Cartesian closed 2-categories.*

Proof. The axiom (Der1) is immediate. For axiom (Der2), let us consider a map $\phi: F \rightarrow G$ in $\mathbf{HOM}(\mathbb{D}, \mathbb{D}')(K)$. Then ϕ is an isomorphism if and only if $\phi_J: F_J \rightarrow G_J$ is an isomorphism in $\mathbf{nat}(\mathbb{D}(J), \mathbb{D}'(K \times J))$ for all categories J . The fact that isomorphisms in \mathbb{D}' are detected pointwise shows that this is equivalent to all $(\phi_J)_k = (\phi_k)_J$ being isomorphisms. Thus, ϕ is an isomorphism if and only if all ϕ_k are isomorphisms. For axiom (Der3), let us consider a functor $u: J \rightarrow K$. We will prove in Lemma 2.9 that u induces an adjunction $(u_!, u^*): \mathbb{D}'_J \rightarrow \mathbb{D}'_K$ of derivators. Then, since $\mathbf{Hom}(\mathbb{D}, -)$ preserves adjunctions, we obtain the intended adjunction $(u_!, u^*): \mathbf{HOM}(\mathbb{D}, \mathbb{D}')(J) \rightarrow \mathbf{HOM}(\mathbb{D}, \mathbb{D}')(K)$. One proceeds similarly for homotopy right Kan extensions. For the base change axiom, let $u: J \rightarrow K$ be a functor and $k \in K$ an object. Then, we have to show that the base change morphism in the square on the right-hand-side induced by the natural transformation on the left-hand-side is an isomorphism:

$$\begin{array}{ccc} J/k & \longrightarrow & J \\ \downarrow & \searrow & \downarrow \\ e & \longrightarrow & K \end{array} \qquad \begin{array}{ccc} \mathbf{Hom}(\mathbb{D}, \mathbb{D}'_{J/k}) & \longleftarrow & \mathbf{Hom}(\mathbb{D}, \mathbb{D}'_J) \\ \downarrow & \searrow & \downarrow \\ \mathbf{Hom}(\mathbb{D}, \mathbb{D}'_e) & \longleftarrow & \mathbf{Hom}(\mathbb{D}, \mathbb{D}'_K) \end{array}$$

Evaluation of this base change morphism is just given by postcomposition with the base change morphism belonging to \mathbb{D}' . But this one is an isomorphism because \mathbb{D}' is a derivator by assumption. Thus, this together with a dual reasoning for homotopy right Kan extensions implies (Der4) for $\mathbf{HOM}(\mathbb{D}, \mathbb{D}')$. Since homotopy Kan extensions are calculated pointwise it follows that $\mathbf{HOM}(\mathbb{D}, \mathbb{D}')$ is pointed resp. additive if this is the case for \mathbb{D}' . \square

Lemma 2.9. *Let \mathbb{D} be a derivator and let $u: J \rightarrow K$ be a functor. Then we obtain an induced adjunction of derivators $(u_!, u^*): \mathbb{D}_J \rightarrow \mathbb{D}_K$.*

Proof. The morphism $u^*: \mathbb{D}_K \rightarrow \mathbb{D}_J$ has an adjoint at least levelwise: for a category M , an adjoint to $(u^*)_M$ is given by $(u_!)_M = (u \times \text{id}_M)_!: \mathbb{D}(J \times M) \rightarrow \mathbb{D}(K \times M)$. We thus have to check that these can be canonically assembled into a morphism of derivators. So, let $f: M \rightarrow N$ be a functor and let us consider the following diagram:

$$\begin{array}{ccc} \mathbb{D}(J \times N) & \xrightarrow{(\text{id} \times f)^*} & \mathbb{D}(J \times M) \\ (u \times \text{id})_! \downarrow & & \downarrow (u \times \text{id})_! \\ \mathbb{D}(K \times N) & \xrightarrow{(\text{id} \times f)^*} & \mathbb{D}(K \times M) \end{array}$$

But, $f^*: \mathbb{D}(- \times N) \rightarrow \mathbb{D}(- \times M)$ preserves homotopy Kan extensions by Proposition 2.8 of [Gro10a] from where we obtain that the base change morphism β_f given by

$$\begin{array}{ccc} u_! f^* & \xrightarrow{\beta_f} & f^* u_! \\ \eta \downarrow & & \uparrow \epsilon \\ u_! f^* u^* u_! & \xrightarrow{=} & u_! u^* f^* u_! \end{array}$$

have the additional property that it gives us a Cartesian closed 2-category. Once we have this good notion of stable derivators it would then correct two more of the typical drawbacks of the theory of triangulated categories, namely the absence of both products and functor categories inside the world of triangulated categories. The related notion of stable ∞ -categories as developed by Joyal and Lurie has all these nice properties. An exposition of that theory can be found e.g. in [Lur11], while an introduction is given in [Gro10b].

By Proposition 2.7 and Proposition 2.8 the 2-categories \mathbf{PDer} and \mathbf{Der} are Cartesian closed monoidal 2-categories. We are thus in the context of Appendix B and can, in particular, apply Theorem B.11. This gives us the following result which we formulate for the 2-category of derivators. Recall also Definition B.9 of terminal objects in 2-categories from that appendix.

Theorem 2.11. *For a derivator \mathbb{D} the derivator $\mathbf{END}(\mathbb{D})$ of endomorphisms can be canonically endowed with a monoidal structure. Moreover, \mathbb{D} can canonically be turned into a left $\mathbf{END}(\mathbb{D})$ -module and this module structure defines a terminal object in the 2-category $\mathbf{Mod}(\mathbb{D})$ of module structures on \mathbb{D} .*

In fact, the action map belonging to the module structure is just given by the pseudo-natural transformation $\mathbf{ev}: \mathbf{END}(\mathbb{D}) \times \mathbb{D} \rightarrow \mathbb{D}$ of Lemma 2.6. The monoidal structure on $\mathbf{END}(\mathbb{D})$ is derived from this map using the biadjunction. For the details of this structure see the constructive proof of Theorem B.11.

We can use this theorem to put Corollary 2.4 into perspective. Namely, let \mathbb{E} be a monoidal derivator and let $(\mathbb{D}, a: \mathbb{E} \times \mathbb{D} \rightarrow \mathbb{D})$ be an \mathbb{E} -module. By the theorem, there is an essentially unique morphism $(\mathbb{E}, a) \rightarrow (\mathbb{D}, \mathbf{ev})$ in $\mathbf{Mod}(\mathbb{D})$. Thus we obtain a pair consisting of a monoidal morphism $\mathbb{E} \rightarrow \mathbf{END}(\mathbb{D})$ and a natural isomorphism as in:

$$\begin{array}{ccc} \mathbb{E} \times \mathbb{D} & & \\ \downarrow & \searrow^a & \\ \mathbf{END}(\mathbb{D}) \times \mathbb{D} & \xrightarrow{\mathbf{ev}} & \mathbb{D} \end{array}$$

Now, a monoidal morphism of derivators induces monoidal functors at all values so that we obtain, in particular, a monoidal functor $\mathbb{E}(e) \rightarrow \mathbf{END}(\mathbb{D})(e) = \mathbf{Hom}(\mathbb{D}, \mathbb{D})$. If we consider from this functor only the induced map between the endomorphisms of the respective monoidal units then we obtain a map of monoids

$$\mathbf{hom}_{\mathbb{E}(e)}(\mathbb{S}_e, \mathbb{S}_e) \rightarrow \mathbf{nat}(\mathbf{id}, \mathbf{id}) = \mathbf{Z}(\mathbb{D}).$$

In the context of an additive action this gives us back the ring map of Corollary 2.4. Thus, the canonical linear structure over the ring of self-maps of the monoidal unit of the underlying monoidal category is only the shadow of the fact that we have a monoidal morphism $\mathbb{E} \rightarrow \mathbf{END}(\mathbb{D})$ of monoidal derivators.

2.4. Examples coming from model categories.

Definition 2.12. Let \mathcal{M} be a monoidal model category. A model category \mathcal{N} is a *left \mathcal{M} -module as a model category* if \mathcal{N} is a left \mathcal{M} -module via a Quillen bifunctor $\otimes: \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{N}$ which has the following additional property: For any cofibrant replacement $Q\mathbb{S} \rightarrow \mathbb{S}$ of the monoidal unit of \mathcal{M} the induced natural transformation $Q\mathbb{S} \otimes - \rightarrow \mathbb{S} \otimes -$ is a Quillen homotopy.

We want to emphasize that there is automatically more structure available in the setting of combinatorial model categories, i.e., it will follow that \mathcal{N} is a left \mathcal{M} -module as a model category if

and only if it is an \mathcal{M} -model category in the sense of [DS07a]. The first part of this observation is a purely categorical one. So, let \mathcal{C} and \mathcal{D} be presentable categories, such that \mathcal{C} is monoidal and \mathcal{D} is a left \mathcal{C} -module via an action $\otimes: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D}$ which preserves colimits separately in each variable. Then, using the special form of the Freyd adjoint functor theorem for presentable categories, we obtain an adjunction of two variables

$$(\otimes, \text{Hom}_l, \text{Hom}_r): \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D}$$

for certain functors

$$\text{Hom}_l(-, -): \mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \mathcal{D} \quad \text{and} \quad \text{Hom}_r(-, -): \mathcal{D}^{\text{op}} \times \mathcal{D} \rightarrow \mathcal{C}.$$

In order to better distinguish these functors notationally let us write from now on $X^K = \text{Hom}_l(K, X)$ for $X \in \mathcal{D}$, $K \in \mathcal{C}$, and $\text{Hom}(X, Y) = \text{Hom}_r(X, Y)$ for $X, Y \in \mathcal{D}$. With these notations the adjunction of two variables takes the familiar form

$$\text{hom}_{\mathcal{D}}(X, Y^K) \cong \text{hom}_{\mathcal{D}}(K \otimes X, Y) \cong \text{hom}_{\mathcal{C}}(K, \text{Hom}(X, Y)).$$

Once one has this adjunction of two variables it can be used to endow \mathcal{D} with an enrichment over \mathcal{C} . This works in full generality, i.e., without any presentability assumption on the categories involved. The enriched mapping object for two objects $X, Y \in \mathcal{D}$ is of course $\text{Hom}(X, Y) \in \mathcal{C}$. The enriched identity $i_X: \mathbb{S} \rightarrow \text{Hom}(X, X)$ is the map which is adjoint to the left unitality constraint $\lambda: \mathbb{S} \otimes X \rightarrow X$. Finally, we only need to specify a composition law. This is constructed in two steps. First, for objects $X, Y \in \mathcal{D}$ we obtain an evaluation map $\text{ev}_{X, Y}: \text{Hom}(X, Y) \otimes X \rightarrow Y$ given by the counit of the adjunction $(-\otimes X, \text{Hom}(X, -)): \mathcal{C} \rightarrow \mathcal{C}$. Thus, we set $\text{ev}_{X, Y} = (\epsilon_X)_Y$ if ϵ_X is the adjunction counit. With these evaluation maps we can construct a composition map associated to three objects X, Y , and $Z \in \mathcal{D}$. The composition \circ is defined to be the map which is adjoint to the following composition:

$$\begin{array}{ccc} (\text{Hom}(Y, Z) \otimes \text{Hom}(X, Y)) \otimes X & \xrightarrow{\alpha} & \text{Hom}(Y, Z) \otimes (\text{Hom}(X, Y) \otimes X) \\ \downarrow & & \downarrow \text{ev} \\ Z & \xleftarrow{\text{ev}} & \text{Hom}(Y, Z) \otimes Y \end{array}$$

It is straightforward but lengthy to see that this defines an enrichment in \mathcal{C} . Thus, in the context of presentable categories, one only has to specify a colimit-preserving action $\otimes: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D}$ in order to obtain that \mathcal{D} is also canonically cotensored and enriched over \mathcal{C} .

Let us now switch back to the homotopical setting. So, let us assume \mathcal{M} to be a combinatorial monoidal model category and \mathcal{N} a combinatorial model category which is a left \mathcal{M} -module as a model category. The adjunction of two variables

$$(\otimes, (-)^{(-)}, \text{Hom}): \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{N}$$

has then the additional property that the involved functors are Quillen bifunctors. By the results of Section 1 (in particular, Corollary 1.21) we thus obtain the following.

Theorem 2.13. *Let \mathcal{M} be a combinatorial monoidal model category and let \mathcal{N} be a combinatorial model category which is a left \mathcal{M} -module as a model category. Then we have an adjunction of two variables at the level of derivators*

$$(\overset{\mathbb{L}}{\otimes}, \mathbb{R}(-)^{(-)}, \mathbb{R}\text{Hom}): \mathbb{D}_{\mathcal{M}} \times \mathbb{D}_{\mathcal{N}} \rightarrow \mathbb{D}_{\mathcal{N}}$$

exhibiting $\mathbb{D}_{\mathcal{N}}$ as a closed $\mathbb{D}_{\mathcal{M}}$ -module. In particular, $\mathbb{D}_{\mathcal{N}}$ is tensored and cotensored over $\mathbb{D}_{\mathcal{M}}$.

It follows, in particular, that for each category K the value $\mathbb{D}_{\mathcal{N}}(K)$ is canonically tensored, cotensored, and enriched over $\mathbb{D}_{\mathcal{M}}(K)$. We come back to this enrichment issue in Section 3. As a special case we can apply this to the case of a combinatorial monoidal model category \mathcal{M} which is in a canonical way a left \mathcal{M} -module as a model category. This reproves then the corresponding results of the last section. But, in addition, we see that there are the cotensors and that at each level we have a canonical enrichment of $\mathbb{D}_{\mathcal{M}}(K)$ over itself.

In the additive context, there is moreover the following result about canonical linear structures.

Corollary 2.14. *Let \mathcal{M} be a combinatorial monoidal model category and let \mathcal{N} be a combinatorial model category with additive associated derivators. If \mathcal{N} is a left \mathcal{M} -module as a model category then the associated derivator $\mathbb{D}_{\mathcal{N}}$ is canonically endowed with a linear structure over the ring $\mathrm{hom}_{\mathrm{Ho}(\mathcal{M})}(\mathbb{S}, \mathbb{S})$.*

If both \mathcal{M} and \mathcal{N} are, in addition, stable and if the induced derivator $\mathbb{D}_{\mathcal{M}}$ is compatibly stable and closed monoidal, we obtain a linear structure over the graded-commutative ring $\mathrm{hom}_{\mathrm{Ho}(\mathcal{M})}(\mathbb{S}, \mathbb{S})_{\bullet}$. Thus depending on the context we have a canonical map of (graded) rings

$$\mathrm{hom}_{\mathrm{Ho}(\mathcal{M})}(\mathbb{S}, \mathbb{S}) \longrightarrow \mathbf{Z}(\mathbb{D}_{\mathcal{N}}) \quad \text{resp.} \quad \mathrm{hom}_{\mathrm{Ho}(\mathcal{M})}(\mathbb{S}, \mathbb{S})_{\bullet} \longrightarrow \mathbf{Z}_{\bullet}(\mathbb{D}_{\mathcal{N}}).$$

Let us now give three important classes of situations to which these results can be applied. Note that the category Set_{Δ} of simplicial sets, the category $\mathrm{Ch}(k)$ of chain complexes over some commutative ground ring k , and the category Sp^{Σ} of symmetric spectra based on simplicial sets are all examples of presentable categories. Moreover, the model structures mentioned in the last section have the property that the respective monoidal units (i.e., the zero-simplex Δ^0 , the ground ring $k[0]$, and the sphere spectrum \mathbb{S} respectively) are cofibrant. Thus, the unit condition for modules over these model categories is for free. The class of model categories for which the usual definition of a simplicial, spectral, or differential-graded model category can be simplified is a bit larger than the class of combinatorial ones: it suffices that the underlying category is presentable. In particular, this is the case for the class of presentable model categories in the sense of Dugger [Dug06] (which turns out to be the closure of the class of combinatorial model categories under Quillen equivalences).

Proposition 2.15. *Let \mathcal{N} be a model category with an underlying presentable category.*

- i) *The model category \mathcal{N} is a simplicial model category if and only if it is a left Set_{Δ} -module via a Quillen bifunctor $\otimes: \mathrm{Set}_{\Delta} \times \mathcal{N} \longrightarrow \mathcal{N}$.*
- ii) *The model category \mathcal{N} is a spectral model category if and only if it is a left Sp^{Σ} -module via a Quillen bifunctor $\otimes: \mathrm{Sp}^{\Sigma} \times \mathcal{N} \longrightarrow \mathcal{N}$.*
- iii) *The model category \mathcal{N} is a dg model category if and only if it is a left $\mathrm{Ch}(k)$ -module for some commutative ground ring k via a Quillen bifunctor $\otimes: \mathrm{Ch}(k) \times \mathcal{N} \longrightarrow \mathcal{N}$.*

Theorem 2.13 applied to a combinatorial simplicial, spectral, resp. differential-graded model category \mathcal{N} thus gives us that the associated derivator $\mathbb{D}_{\mathcal{N}}$ is canonically tensored and cotensored over the derivator of simplicial sets, spectra, resp. chain complexes. Moreover, the category $\mathbb{D}_{\mathcal{N}}(K)$ is canonically enriched over the category $\mathrm{Ho}(\mathrm{Set}_{\Delta}^K)$, $\mathrm{Ho}((\mathrm{Sp}^{\Sigma})^K)$, resp. $\mathrm{Ho}(\mathrm{Ch}(k)^K)$. To illustrate the applicability of this result let us recall from [Dug06] that every stable combinatorial model category is Quillen equivalent to a spectral model category. An alternative set of sufficient conditions for this conclusion can be found in [SS03b, Theorem 3.8.2]. Let us now give more specific examples for the differential-graded and the spectral setting.

Let A be a differential-graded algebra over a ground ring k . As we recalled already in the context of a commutative differential-graded algebra the category $\mathrm{Mod} - A$ of right modules over A can be

endowed with the projective model structure. A map in this category is a weak equivalence resp. a fibration if and only if the induced map of underlying chain complexes is a quasi-isomorphism resp. an epimorphism. This model structure is stable and combinatorial so that we can consider the associated stable derivator $\mathbb{D}_{A^{\text{op}}} = \mathbb{D}_{\text{Mod}-A}$. Moreover, the usual tensor product $\otimes_k: \mathbf{Ch}(k) \times \text{Mod}-A \rightarrow \text{Mod}-A$ turns $\text{Mod}-A$ into a differential-graded model category.

Example 2.16. For a differential-graded algebra A we have an adjunction of two variables

$$\otimes_k^{\mathbb{L}}: \mathbb{D}_k \otimes \mathbb{D}_{A^{\text{op}}} \rightarrow \mathbb{D}_{A^{\text{op}}}$$

exhibiting $\mathbb{D}_{A^{\text{op}}}$, in particular, as an additive left \mathbb{D}_k -module. Thus, the derivator $\mathbb{D}_{A^{\text{op}}}$ is canonically endowed with a k -linear structure induced by a ring map

$$k = \text{hom}_{D(k)}(k, k) \rightarrow Z(\mathbb{D}_{A^{\text{op}}}).$$

A bit more general, let us consider three differential-graded algebras A , B , and C . For more details about module categories in this one object case and also in the more object case we refer to [Hei07, Appendix A]. The tensor product over B gives us a functor

$$\otimes_B: (A - \text{Mod}-B) \times (B - \text{Mod}-C) \rightarrow A - \text{Mod}-C.$$

Here, $A - \text{Mod}-B$ denotes the category of left A -, right B -modules, i.e., of left $A \otimes B^{\text{op}}$ -modules. The functors \otimes_B are coherently associative and unital in the obvious sense. Moreover, each of them is part of an adjunction of two variables. In fact, the adjunctions look like

$$(\otimes_B, \text{Hom}_C, \text{Hom}_A): (A - \text{Mod}-B) \times (B - \text{Mod}-C) \rightarrow A - \text{Mod}-C.$$

If we now endow the bimodule categories with the projective model structures then one checks that \otimes_B is a left Brown functor as soon as the underlying chain complex of B is cofibrant in $\mathbf{Ch}(k)$. Thus, in that case we obtain an adjunction of two variables

$$\mathbb{D}_{A \otimes B^{\text{op}}} \times \mathbb{D}_{B \otimes C^{\text{op}}} \rightarrow \mathbb{D}_{A \otimes C^{\text{op}}}$$

which by the closedness of Der induces a morphism of derivators $\mathbb{D}_{A \otimes B^{\text{op}}} \rightarrow \text{HOM}(\mathbb{D}_{B \otimes C^{\text{op}}}, \mathbb{D}_{A \otimes C^{\text{op}}})$. In case we take C to be the monoidal unit $k[0]$ we get a map

$$\mathbb{D}_{A \otimes B^{\text{op}}} \rightarrow \text{HOM}(\mathbb{D}_B, \mathbb{D}_A)$$

from the derivator of bimodules to the derivator of morphisms. Specializing further to the situation of $A = B$ we obtain an action of the monoidal derivator $\mathbb{D}_{A \otimes A^{\text{op}}}$ on \mathbb{D}_A . By Theorem 2.11, this action induces a *monoidal* morphism of derivators

$$\mathbb{D}_{A \otimes A^{\text{op}}} \rightarrow \text{HOM}(\mathbb{D}_A, \mathbb{D}_A).$$

Example 2.17. Let A be a differential-graded algebra over k which is cofibrant as an object of $\mathbf{Ch}(k)$. Then we have an adjunction of two variables

$$\otimes_A^{\mathbb{L}}: \mathbb{D}_{A \otimes A^{\text{op}}} \times \mathbb{D}_A \rightarrow \mathbb{D}_A$$

exhibiting \mathbb{D}_A as an additive left $\mathbb{D}_{A \otimes A^{\text{op}}}$ -module. In particular, the derivator \mathbb{D}_A can canonically be endowed with a linear structure over the Ext-algebra of A via a ring map

$$\text{hom}_{D(A \otimes A^{\text{op}})}^{\bullet}(A, A) \rightarrow Z^{\bullet}(\mathbb{D}_A).$$

Under our cofibrancy condition (cf. [Kel94, Example 6.6]) one can identify this Ext-algebra with $HH^{\bullet}(A, A)$, the Hochschild cohomology of A . Thus, in that case we obtain that the derivator \mathbb{D}_A is canonically linear over the Hochschild cohomology of A .

This example can still be generalized if one sticks to the ‘many objects versions’ ([Mit72]) of differential-graded algebras, i.e., to small differential-graded categories. What we are about to do can be done axiomatically with $\mathbf{Ch}(k)$ replaced by a sufficiently nice closed monoidal model category but we prefer to give some details in the case of $\mathbf{Ch}(k)$. In the corresponding examples where the role of $\mathbf{Ch}(k)$ is taken by the category \mathbf{Sp}^Σ of symmetric spectra (based on simplicial sets) we will be much shorter.

Recall that every biclosed monoidal category is canonically enriched over itself (this is a special case of the result that an adjunction of two variables such that the action is part of a module structure induces a canonical enrichment on the module). In particular, the category $\mathbf{Ch}(k)$ is canonically enriched over itself. Thus, given a small dg-category J it makes sense to consider the dg-functors from J to $\mathbf{Ch}(k)$. Spelling out this definition, such a dg-functor X associates to each object $j \in J$ a chain complex $X(j) \in \mathbf{Ch}(k)$ together with action maps

$$\mathrm{Hom}_J(j_1, j_2) \otimes_k X(j_1) \longrightarrow X(j_2).$$

These maps are supposed to be coherently associative and unital. Taking as morphisms of such dg-functors the dg-natural transformations we obtain the category $J - \mathbf{Mod}$ of J -modules. The category $A - \mathbf{Mod}$ can be considered as a special case of this situation: Given a differential-graded algebra A we can associate a dg-category J_A with one object and A as endomorphism object. It is easy to see that in that case $J_A - \mathbf{Mod}$ is canonically isomorphic to $A - \mathbf{Mod}$. As a special case of [SS03a, Theorem 6.1] we deduce that $J - \mathbf{Mod}$ can be endowed with the projective model structure. Moreover, that same theorem guarantees that the model structure is cofibrantly generated and it is also true that the category $J - \mathbf{Mod}$ is presentable. Thus, we have the combinatorial model category $J - \mathbf{Mod}$ and can consequently consider the associated derivator

$$\mathbb{D}_J = \mathbb{D}(J - \mathbf{Mod}).$$

The passage to bimodules involves the following additional bit of enriched category theory which can be done as soon as the enrichment level is symmetric. Given two dg-categories J and K we can form the dg-category $J \otimes K$ as follows. The objects are given by pairs (j, k) consisting of an object $j \in J$ and an object $k \in K$. Given two such pairs, the chain complex of morphisms is defined to be

$$\mathrm{Hom}_{J \otimes K}((j_1, k_1), (j_2, k_2)) = \mathrm{Hom}_J(j_1, j_2) \otimes \mathrm{Hom}_K(k_1, k_2).$$

The symmetry constraint allows one to define a composition law. Similarly, one can use the symmetry constraint in order to define the opposite of a dg-category. The opposite J^{op} of J has the same objects as J but the morphism complexes are defined by

$$\mathrm{Hom}_{J^{\mathrm{op}}}(j_1, j_2) = \mathrm{Hom}_J(j_2, j_1).$$

Given two small dg-categories J and K , we can now combine these two constructions and define the category of $J - K$ -bimodules as

$$J - \mathbf{Mod} - K = (J \otimes K^{\mathrm{op}}) - \mathbf{Mod}.$$

The next step is to give a generalization of the tensor product over a dga. So, let us consider three small dg-categories J , K , and L and two bimodules $X \in J - \mathbf{Mod} - K$, $Y \in K - \mathbf{Mod} - L$. Then we define $X \otimes_K Y \in J - \mathbf{Mod} - L$ by the following enriched coend construction (cf. [Dub70, Section 1.3]). For $j \in J$ and $l \in L$ we define $X \otimes_K Y$ evaluated at (j, l) to be the coequalizer of

$$\bigoplus_{k_1, k_2 \in K} X(j, k_2) \otimes K(k_1, k_2) \otimes Y(k_1, l) \begin{array}{c} \xrightarrow{\lambda} \\ \xrightarrow{\rho} \end{array} \bigoplus_{k \in K} X(j, k) \otimes Y(k, l).$$

Here, the two morphisms are induced by the action maps given by the K -module structures. This tensor product \otimes_K is part of an adjunction of two variables

$$\otimes_K: J - \text{Mod} - K \times K - \text{Mod} - L \rightarrow J - \text{Mod} - L.$$

It can be shown ([Hei07, Appendix A]) that \otimes_K is a left Brown functor as soon as K is locally cofibrant, i.e., if all mapping objects of K are cofibrant. If we assume this we can specialize as in the one object case and obtain a morphism of derivators

$$\mathbb{D}_{J \otimes_K \text{op}} \rightarrow \text{HOM}(\mathbb{D}_K, \mathbb{D}_J).$$

In the case $J = K$ this morphism is monoidal and we thus obtain the next example. Recall that the monoidal unit in the category $K - \text{Mod} - K$ of bimodules is given by the hom-functor K itself.

Example 2.18. Let K be a small dg-category which is locally cofibrant then we have an adjunction of two variables

$$\otimes_K^{\mathbb{L}}: \mathbb{D}_{K \otimes_K \text{op}} \times \mathbb{D}_K \rightarrow \mathbb{D}_K$$

exhibiting \mathbb{D}_K as a left $\mathbb{D}_{K \otimes_K \text{op}}$ -module. Thus, \mathbb{D}_K is canonically endowed with a linear structure over $\text{hom}_{D(K \otimes_K \text{op})}^{\bullet}(K, K)$. Again, under our cofibrancy condition this can be identified with the *Hochschild-Mitchell cohomology* $HH^{\bullet}(K, K)$ of the small dg-category K .

Similar examples are obtained if we replace chain complexes by symmetric spectra. For this purpose let us endow Sp^{Σ} with the absolute projective stable model structure which interacts nicely with the smash product. Then, for a ring spectrum R the category $\text{Mod} - R$ of right R -modules is a stable, combinatorial model category when endowed with the projective model structure giving rise to the stable derivator $\mathbb{D}_{R \text{op}}$. Moreover, the smash product

$$\wedge: \text{Sp} \times \text{Mod} - R \rightarrow \text{Mod} - R$$

turns $\text{Mod} - R$ into a spectral model category.

Example 2.19. For a symmetric ring spectrum R , we have an adjunction of two variables

$$\mathbb{D}_{\text{Sp}} \times \mathbb{D}_{R \text{op}} \rightarrow \mathbb{D}_{R \text{op}}$$

turning $\mathbb{D}_{R \text{op}}$ into an additive left \mathbb{D}_{Sp} -module. In particular, $\mathbb{D}_{R \text{op}}$ is canonically endowed with a graded linear structure over the stable homotopy groups of spheres

$$\pi_{\bullet}^S \rightarrow \mathbf{Z}_{\bullet}(\mathbb{D}_{R \text{op}}).$$

The same reasoning as in the differential-graded context leads to results about bimodules. Given two symmetric ring spectra R and S such that the underlying spectrum of S is cofibrant we obtain a morphism of derivators

$$\mathbb{D}_{R \wedge S \text{op}} \rightarrow \text{HOM}(\mathbb{D}_S, \mathbb{D}_R).$$

Let us emphasize that we are working with the *flat* stable model structure so that this cofibrancy condition is not an empty condition. In the case of $R = S$ we can again apply Theorem 2.11 in order to obtain a monoidal morphism of derivators $\mathbb{D}_{R \wedge R \text{op}} \rightarrow \text{END}(\mathbb{D}_R)$. This can be specialized to the following result.

Example 2.20. Let R be a symmetric ring spectrum such that the underlying spectrum is cofibrant. Then we have an adjunction of two variables

$$\wedge_R^{\mathbb{L}}: \mathbb{D}_{R \wedge R \text{op}} \times \mathbb{D}_R \rightarrow \mathbb{D}_R$$

which turns \mathbb{D}_R into an additive left $\mathbb{D}_{R \wedge R \text{op}}$ -module. In particular, \mathbb{D}_R is canonically endowed with a linear structure over $\text{hom}_{D(R \wedge R \text{op})}^{\bullet}(R, R)$. By [DS07b, 4.4] this graded ring can be identified

with the graded homotopy groups of the Topological Hochschild cohomology spectrum $THH(R, R)$ of R .

For completeness, let us quickly mention the many object variant thereof. Given small spectral categories J and K such that K is locally cofibrant, the smash product over K induces a morphism of derivators

$$\mathbb{D}_{J \wedge K^{\text{op}}} \longrightarrow \mathbf{HOM}(\mathbb{D}_K, \mathbb{D}_J).$$

Specializing to $J = K$ we finally get

Example 2.21. Let J be a locally cofibrant spectral category. We then have an adjunction of two variables

$$\wedge_J^{\mathbb{L}}: \mathbb{D}_{J \wedge J^{\text{op}}} \times \mathbb{D}_J \rightarrow \mathbb{D}_J$$

endowing \mathbb{D}_J with the structure of an additive left $\mathbb{D}_{J \wedge J^{\text{op}}}$ -module. In particular, this induces a graded linear structure on \mathbb{D}_J :

$$\mathbf{hom}_{\mathbb{D}(J \wedge J^{\text{op}})}^{\bullet}(J, J) \longrightarrow Z^{\bullet}(\mathbb{D}_J)$$

3. ENRICHED DERIVATORS

3.1. The 2-Grothendieck opfibration of enriched categories. To motivate the construction of this subsection let us quickly recall the following. In Section 1 we defined a monoidal prederivator as a monoidal object in \mathbf{PDer} . In Section 2 we defined a left-tensored prederivator as a module object in the same 2-category. In both cases we saw that the respective notion can be equivalently defined as a 2-functor

$$\mathbb{D}: \mathbf{Cat}^{\text{op}} \longrightarrow \mathbf{MonCAT} \quad \text{resp.} \quad \mathbb{D}: \mathbf{Cat}^{\text{op}} \longrightarrow \mathbf{ModCAT}.$$

In this subsection we want to construct a target 2-category \mathbf{ECAT} of enriched categories which will then be used in the definition of an enriched prederivator as a 2-functor $\mathbb{D}: \mathbf{Cat}^{\text{op}} \longrightarrow \mathbf{ECAT}$.

Recall that given a monoidal category \mathcal{M} we have the notion of categories enriched over \mathcal{M} . An \mathcal{M} -enriched category \mathcal{C} consists of the following. First, we are given a class of objects \mathcal{C}_0 and for two such objects $X, Y \in \mathcal{C}_0$ we have a mapping object $\mathbf{Hom}_{\mathcal{C}}(X, Y) \in \mathcal{M}$. Moreover, for each object X there is a ‘unit map’ specified by a morphism $\mathbb{S} \longrightarrow \mathbf{Hom}_{\mathcal{C}}(X, X)$ and for each triple X, Y , and Z of objects we have a composition morphism $\mathbf{Hom}_{\mathcal{C}}(Y, Z) \otimes \mathbf{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathbf{Hom}_{\mathcal{C}}(X, Z)$. These data are subject to the expected unitality and associativity conditions. For details see for example [Kel05a, Bor94b]. There is also a notion of enriched functors and enriched natural transformations over a fixed monoidal category \mathcal{M} so that we have in fact the 2-category $\mathcal{M} - \mathbf{CAT}$ of \mathcal{M} -enriched categories. In the special case where the monoidal category is given by the Cartesian monoidal category \mathbf{Set} of sets enriched category theory reduces to classical category theory.

Enriched category theory has the nice feature that given a monoidal functor $F: \mathcal{M} \longrightarrow \mathcal{N}$ we obtain an induced *base change 2-functor*

$$F_*: \mathcal{M} - \mathbf{CAT} \longrightarrow \mathcal{N} - \mathbf{CAT}.$$

For convenience let us quickly recall the construction at least on objects. Given an \mathcal{M} -enriched category \mathcal{C} then the \mathcal{N} -enriched category $F_*\mathcal{C}$ is defined to have the same class of objects. The mapping objects are given by $\mathbf{Hom}_{F_*\mathcal{C}}(X, Y) = F(\mathbf{Hom}_{\mathcal{C}}(X, Y))$. Since F is a monoidal functor we obtain unit maps in $F_*\mathcal{C}$ by taking

$$\mathbb{S}_{\mathcal{N}} \longrightarrow F(\mathbb{S}_{\mathcal{M}}) \longrightarrow F(\mathbf{Hom}_{\mathcal{C}}(X, X)).$$

The first map is given by the monoidal structure on F while the second one is the image under F of the unit map of X in the \mathcal{M} -enriched category \mathcal{C} . Similarly, using the other part of the monoidal structure on F one defines a composition law in $F_*\mathcal{C}$ and it is straightforward to check that this defines an \mathcal{N} -enriched category $F_*\mathcal{C}$.

Example 3.1. Let \mathcal{M} be a monoidal category and \mathbb{S} the unit object. The *functor of elements* $\mathcal{M} \longrightarrow \mathbf{Set}: M \longmapsto \mathbf{hom}_{\mathcal{M}}(\mathbb{S}, M)$ can be canonically endowed with the structure of a monoidal functor. Thus, we obtain an induced 2-functor

$$U = U_{\mathcal{M}}: \mathcal{M} - \mathbf{CAT} \longrightarrow \mathbf{CAT}$$

which sends an \mathcal{M} -enriched category to its *underlying category*.

Let us also give the following well-known more specific examples.

Example 3.2. i) Let k be a commutative ring and let us consider the categories $\mathbf{Mod}(k)$, $\mathbf{grMod}(k)$, and $\mathbf{Ch}(k)$ of k -modules, \mathbb{Z} -graded k -modules, and unbounded chain complexes over k respectively. The homology functor $H_*: \mathbf{Ch}(k) \longrightarrow \mathbf{grMod}(k)$ and the evaluation at zero functor $\mathbf{grMod}(k) \longrightarrow \mathbf{Mod}(k)$ are canonically monoidal functors. Thus, for every dg-category \mathcal{C} [Kel06, Toë07] we have two associated homology categories, namely the graded one $H_*\mathcal{C}$ and the ungraded one $H_0\mathcal{C}$.

ii) The functor $\pi_0: \mathbf{Set}_\Delta \rightarrow \mathbf{Set}$ sending a simplicial set to its set of path components preserves products. We thus have a base change functor which sends a simplicial category \mathcal{C} to its ('naive') homotopy category $\pi_0\mathcal{C}$. For a more concrete example, let \mathcal{C} be a simplicial model category and let us denote by \mathcal{C}_{cf} the full simplicial subcategory of \mathcal{C} spanned by the bifibrant objects. Since for maps between bifibrant objects the left homotopy, the right homotopy, and the simplicial homotopy relations coincide we have that $\pi_0\mathcal{C}_{cf}$ is the classical homotopy category of \mathcal{C} as described in [DS95].

Note that there is also a similar base change construction for monoidal transformations of monoidal functors. So, let $F, G: \mathcal{M} \rightarrow \mathcal{N}$ be monoidal functors such that we have induced base change 2-functors $F_*, G_*: \mathcal{M} - \mathbf{CAT} \rightarrow \mathcal{N} - \mathbf{CAT}$. If we have in addition a monoidal transformation $\beta: F \rightarrow G$ we can construct a 2-natural transformation $\beta_*: F_* \rightarrow G_*$ as follows. For an \mathcal{M} -enriched category \mathcal{C} we obtain an \mathcal{N} -enriched functor $\beta_*: F_*\mathcal{C} \rightarrow G_*\mathcal{C}$ which is the identity on objects by setting

$$(\beta_*)_{X,Y} = \beta_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}: F \mathrm{Hom}_{\mathcal{C}}(X,Y) \rightarrow G \mathrm{Hom}_{\mathcal{C}}(X,Y).$$

The coherence conditions imposed on a monoidal natural transformation guarantee that β_* is in fact an \mathcal{N} -enriched functor. For example, the compatibility with the composition is ensured by the following commutative diagram in which the left square commutes since β is monoidal:

$$\begin{array}{ccccc} F \mathrm{Hom}_{\mathcal{C}}(Y,Z) \otimes F \mathrm{Hom}_{\mathcal{C}}(X,Y) & \longrightarrow & F(\mathrm{Hom}_{\mathcal{C}}(Y,Z) \otimes \mathrm{Hom}_{\mathcal{C}}(X,Y)) & \longrightarrow & F \mathrm{Hom}_{\mathcal{C}}(X,Z) \\ \beta \otimes \beta \downarrow & & \downarrow \beta & & \downarrow \beta \\ G \mathrm{Hom}_{\mathcal{C}}(Y,Z) \otimes G \mathrm{Hom}_{\mathcal{C}}(X,Y) & \longrightarrow & G(\mathrm{Hom}_{\mathcal{C}}(Y,Z) \otimes \mathrm{Hom}_{\mathcal{C}}(X,Y)) & \longrightarrow & G \mathrm{Hom}_{\mathcal{C}}(X,Z) \end{array}$$

These constructions taken together give the following result.

Proposition 3.3. *The assignments $\mathcal{M} \mapsto \mathcal{M} - \mathbf{CAT}$, $F \mapsto F_*$, and $\beta \mapsto \beta_*$ define a 2-functor*

$$(-) - \mathbf{CAT}: \mathbf{MonCAT} \rightarrow 2\text{-}\mathbf{CAT}.$$

Thus, the 2-categorical Grothendieck construction of Appendix A can be applied to this 2-functor and yields a single 2-category \mathbf{ECAT} of enriched categories together with a projection functor $p: \mathbf{ECAT} \rightarrow \mathbf{MonCAT}$. Let us call p the *2-Grothendieck opfibration of enriched categories*.

Let us describe \mathbf{ECAT} in some more detail since this will be helpful in the remainder of this section. The objects of \mathbf{ECAT} are given by pairs $(\mathcal{M}, \mathcal{C})$ where \mathcal{M} is a monoidal category and \mathcal{C} is an \mathcal{M} -enriched category. Given two such objects $(\mathcal{M}, \mathcal{C})$ and $(\mathcal{N}, \mathcal{D})$, a morphism $(\mathcal{M}, \mathcal{C}) \rightarrow (\mathcal{N}, \mathcal{D})$ is a pair (u, f) where $u: \mathcal{M} \rightarrow \mathcal{N}$ is a monoidal functor and $f: u_*\mathcal{C} \rightarrow \mathcal{D}$ is an \mathcal{N} -enriched functor. The first component of the composition of two such composable morphisms $(u, f): (\mathcal{M}, \mathcal{C}) \rightarrow (\mathcal{N}, \mathcal{D})$ and $(v, g): (\mathcal{N}, \mathcal{D}) \rightarrow (\mathcal{P}, \mathcal{E})$ is $vu: \mathcal{M} \rightarrow \mathcal{P}$. The second component is given by the \mathcal{P} -enriched functor

$$v_*u_*\mathcal{C} \xrightarrow{v_*f} v_*\mathcal{D} \xrightarrow{g} \mathcal{E}.$$

It is obvious that the identity of an object $(\mathcal{M}, \mathcal{C})$ with respect to this composition is given by the morphism $(\mathrm{id}_{\mathcal{M}}, \mathrm{id}_{\mathcal{C}})$. Now, let us turn to 2-morphisms. So let us assume (u, f) and (v, g) to be a pair of parallel morphisms $(\mathcal{M}, \mathcal{C}) \rightarrow (\mathcal{N}, \mathcal{D})$. A 2-morphism $(u, f) \rightarrow (v, g)$ is a pair (β, α) where $\beta: u \rightarrow v$ is a monoidal natural transformation and $\alpha: f \rightarrow g \circ (\beta_*)_{\mathcal{C}}$ is an \mathcal{N} -enriched natural

transformation as indicated in:

$$\begin{array}{ccc}
 u_*\mathcal{C} & & \\
 \beta_* \downarrow & \searrow f & \\
 v_*\mathcal{C} & \xrightarrow{g} & \mathcal{D}
 \end{array}$$

Note that we have just seen that β_* is an \mathcal{N} -enriched functor so that this definition makes sense. Moreover, given an object $(\mathcal{M}, \mathcal{C})$, a morphism (u, f) , or a 2-morphism (β, α) in ECAT we can project onto the first component \mathcal{M} , u , or β in order to obtain a monoidal category, a monoidal functor or a monoidal natural transformation respectively. This describes the 2-Grothendieck opfibration $p: \text{ECAT} \rightarrow \text{MonCAT}$.

The remaining aim of this subsection is to show that we can elaborate on Example 3.1 in a way that the formation of underlying categories defines a 2-functor $U: \text{ECAT} \rightarrow \text{CAT}$. To begin with let $(\mathcal{M}, \mathcal{C})$ be an object of ECAT then we associate to it the underlying category $U(\mathcal{M}, \mathcal{C}) = U_{\mathcal{M}}(\mathcal{C})$. Now, let $(u, f): (\mathcal{M}, \mathcal{C}) \rightarrow (\mathcal{N}, \mathcal{D})$ be a morphism in ECAT . Then, we obtain a functor

$$U(u, f): U_{\mathcal{M}}\mathcal{C} \rightarrow U_{\mathcal{N}}\mathcal{D}$$

as the composition of the following two (unenriched) functors

$$U_{\mathcal{M}}\mathcal{C} \longrightarrow U_{\mathcal{N}}(u_*\mathcal{C}) \xrightarrow{U_{\mathcal{N}}f} U_{\mathcal{N}}\mathcal{D}.$$

Here, the first arrow is the functor which is the identity on objects and is given on morphism sets by the composition

$$\text{hom}_{\mathcal{M}}(\mathbb{S}, \text{Hom}_{\mathcal{C}}(X, Y)) \xrightarrow{u} \text{hom}_{\mathcal{N}}(u\mathbb{S}, \text{Hom}_{u_*\mathcal{C}}(X, Y)) \xrightarrow{\cong} \text{hom}_{\mathcal{N}}(\mathbb{S}, \text{Hom}_{u_*\mathcal{C}}(X, Y)).$$

Using the monoidal structure on u one checks that this indeed defines a functor. The isomorphism in this composition is of course given by the monoidal structure on u . It remains only to define the value of U on 2-morphisms in ECAT . So, let (u, f) and (v, g) be parallel morphisms $(\mathcal{M}, \mathcal{C}) \rightarrow (\mathcal{N}, \mathcal{D})$ and let $(\beta, \alpha): (u, f) \rightarrow (v, g)$ be such a 2-morphism. Recall that we hence have, in particular, an \mathcal{N} -natural transformation $\alpha: f \rightarrow g \circ \beta_*$. Let us consider the following diagram in which the rows are given by the value of U at (u, f) resp. (v, g) and where the 2-morphism is given by $U_{\mathcal{N}}\alpha$:

$$\begin{array}{ccccc}
 U_{\mathcal{M}}\mathcal{C} & \longrightarrow & U_{\mathcal{N}}(u_*\mathcal{C}) & \xrightarrow{U_{\mathcal{N}}f} & U_{\mathcal{N}}\mathcal{D} \\
 \parallel & & U_{\mathcal{N}}\beta_* \downarrow & \not\parallel & \parallel \\
 U_{\mathcal{M}}\mathcal{C} & \longrightarrow & U_{\mathcal{N}}(v_*\mathcal{C}) & \xrightarrow{U_{\mathcal{N}}g} & U_{\mathcal{N}}\mathcal{D}
 \end{array}$$

Using the fact that β is a monoidal transformation it follows that the left square commutes. So, let us define the value of $U: \text{ECAT} \rightarrow \text{CAT}$ on the 2-morphism (β, α) to be the composite natural transformation of this diagram.

Proposition 3.4. *The above constructions define a 2-functor $U: \text{ECAT} \rightarrow \text{CAT}$.*

Proof. We will not give the details of the proof since it is quite lengthy but essentially straightforward. Let us only show that U preserves the composition of morphisms. For that purpose let $(\mathcal{M}, \mathcal{C}) \xrightarrow{(u, f)} (\mathcal{N}, \mathcal{D}) \xrightarrow{(v, g)} (\mathcal{P}, \mathcal{E})$ be a pair of composable morphisms in ECAT . By definition of the composition of morphisms in ECAT we have $(v, g) \circ (u, f) = (vu, g \circ v_*f)$. Both functors $U(v, g) \circ U(u, f)$ and $U((v, g) \circ (u, f))$ send an object $X \in U(\mathcal{M}, \mathcal{C})$ to $gf(X) \in U(\mathcal{P}, \mathcal{E})$.

Thus, it remains to show that both functors have the same behavior on morphisms. The functor $U((v, g) \circ (u, f))$ sends a morphism $\phi: \mathbb{S} \rightarrow \text{Hom}_e(X, Y)$ in $U_{\mathcal{M}}\mathcal{C}$ to the composition

$$\mathbb{S} \cong v(\mathbb{S}) \cong vu(\mathbb{S}) \xrightarrow{vu\phi} vu \text{Hom}_e(X, Y) \xrightarrow{vf} v \text{Hom}_{\mathcal{D}}(fX, fY) \xrightarrow{g} \text{Hom}_e(gfX, gfY).$$

On the other hand, $U(u, f)$ maps such a morphism ϕ to

$$\mathbb{S} \cong u(\mathbb{S}) \xrightarrow{u\phi} u \text{Hom}_e(X, Y) \xrightarrow{f} \text{Hom}_{\mathcal{D}}(fX, fY)$$

which is then sent to $U((v, g) \circ (u, f))\phi$ by $U(v, g)$. \square

Thus, the upshot of this subsection is that we have constructed the 2-category ECAT of enriched categories together with two 2-functors:

$$\begin{array}{ccc} \text{ECAT} & \xrightarrow{U} & \text{CAT} \\ p \downarrow & & \\ \text{MonCAT} & & \end{array}$$

This will allow us to give compact definitions of enriched (pre)derivators in the next subsection.

3.2. Enriched derivators. After the preparations of the last subsection we can immediately give the following definition.

Definition 3.5. An *enriched prederivator* \mathbb{D} is a 2-functor $\mathbb{D}: \text{Cat}^{\text{op}} \rightarrow \text{ECAT}$. Given such a \mathbb{D} it is said to be *enriched over the monoidal prederivator* $\mathbb{E} = p \circ \mathbb{D}$ while the prederivator $U \circ \mathbb{D}$ is called the *underlying prederivator*.

Note that by the very definition an enriched prederivator is not a prederivator but –parallel to classical enriched category theory– it canonically has an underlying prederivator.

Let us unravel this definition a bit. A prederivator \mathbb{D} enriched over a monoidal prederivator \mathbb{E} gives us for each category $K \in \text{Cat}$ a category $\mathbb{D}(K)$ enriched over $\mathbb{E}(K)$. Moreover, for a functor $u: J \rightarrow K$, the monoidal prederivator \mathbb{E} induces a monoidal functor $\mathbb{E}(u): \mathbb{E}(K) \rightarrow \mathbb{E}(J)$ which has an associated base change 2-functor. Then, the enriched prederivator assigns to the functor u a morphism

$$(\mathbb{E}(u), \mathbb{D}(u)): (\mathbb{E}(K), \mathbb{D}(K)) \rightarrow (\mathbb{E}(J), \mathbb{D}(J))$$

in ECAT . Thus, we have an $\mathbb{E}(J)$ -enriched functor $\mathbb{D}(u): \mathbb{E}(u)_* \mathbb{D}(K) \rightarrow \mathbb{D}(J)$. There are similar assignments for natural transformations and these satisfy certain coherence relations and all this is nicely hidden by the construction of ECAT and the associated 2-functors.

From now on, given a prederivator \mathbb{D} enriched over a monoidal prederivator \mathbb{E} we will commit a slight abuse of notation and write u^* for both $\mathbb{E}(u)$ and $\mathbb{D}(u)$ and similarly for natural transformations. It will always be clear from the context which one of the two is meant.

There is also the notion of an *enriched derivator* which is an enriched prederivator such that the underlying prederivator is a derivator. Similarly, an *enrichment of a derivator* \mathbb{D} over a monoidal prederivator \mathbb{E} is given by an \mathbb{E} -enriched derivator \mathbb{D}' and an isomorphism $U \circ \mathbb{D}' \cong \mathbb{D}$.

Let us give an immediate example. Given a commutative ring k the monoidal category $k\text{-Mod}$ of k -modules gives us the constant monoidal prederivator $\text{Cat}^{\text{op}} \rightarrow e \xrightarrow{k\text{-Mod}} \text{CAT}$.

Example 3.6. Let \mathbb{D} be an additive derivator (e.g. a stable derivator). Then there is a canonical enrichment of \mathbb{D} over the monoidal prederivator with constant value $\mathbb{Z} - \text{Mod}$. Similarly, let \mathbb{D} be an additive derivator and let $\sigma: k \rightarrow \mathbb{Z}(\mathbb{D})$ be a k -linear structure on it. Then there is a canonical enrichment of \mathbb{D} over the monoidal prederivator with constant value $k - \text{Mod}$.

Example 3.7. Let \mathcal{M} be a bicomplete monoidal closed category and let us also denote by \mathcal{M} the associated constant monoidal derivator. Moreover, let \mathcal{C} be a category enriched over \mathcal{M} . Recall that for a small category J , the ordinary functor category $\text{Fun}(J, \mathcal{C})$ can be enriched over \mathcal{M} . Given two functors $F, G: J \rightarrow \mathcal{C}$ there is an object $\text{Nat}(F, G) \in \mathcal{M}$ of natural transformations defined by the following end formula:

$$\text{Nat}(F, G) = \int_J \text{Hom}(Fj, Gj)$$

In fact, since we assumed \mathcal{M} to have coproducts we can consider the free \mathcal{M} -enriched category $\mathbb{S}J$ on the ordinary category J . Then this construction is just a special case of an \mathcal{M} -enriched category of \mathcal{M} -enriched functors. These \mathcal{M} -enriched functor categories assemble to define an \mathcal{M} -enriched prederivator which provides us with an enrichment of the prederivator represented by the underlying category of \mathcal{C} . For the corresponding statement in enriched category theory cf. [Kel05a, Section 2.5].

Definition 3.8. Let \mathbb{E} be a monoidal prederivator and let \mathbb{D} and \mathbb{D}' be \mathbb{E} -enriched prederivators. A *morphism of \mathbb{E} -enriched prederivators* $\mathbb{D} \rightarrow \mathbb{D}'$ is a pseudo-natural transformation $F: \mathbb{D} \rightarrow \mathbb{D}'$ of 2-functors $\text{Cat}^{\text{op}} \rightarrow \text{ECAT}$ such that $p \circ F = \text{id}_{\mathbb{E}}$.

Unraveling the definition such a morphism consists of an $\mathbb{E}(K)$ -enriched functor $\mathbb{D}(K) \rightarrow \mathbb{D}'(K)$ for each category K and an $\mathbb{E}(J)$ -natural isomorphism as indicated in

$$\begin{array}{ccc} (u^*)_* \mathbb{D}(K) & \longrightarrow & (u^*)_* \mathbb{D}'(K) \\ \downarrow & \Downarrow & \downarrow \\ \mathbb{D}(J) & \longrightarrow & \mathbb{D}'(J) \end{array}$$

for each functor $u: J \rightarrow K$. Here, the vertical morphisms are the structure morphisms of the enriched prederivators while the horizontal ones belong to the morphism of enriched prederivators. These data have to satisfy certain coherence properties which are precisely the same as in the case of a morphism of unenriched prederivators. As in the unenriched case the direction of the above natural isomorphism is not important since we can always pass to its inverse. With a similar notion of \mathbb{E} -enriched natural transformations we obtain thus the 2-category of \mathbb{E} -enriched prederivators and the full 2-subcategory spanned by the \mathbb{E} -enriched derivators which are denoted by

$$\mathbb{E} - \text{PDer} \quad \text{resp.} \quad \mathbb{E} - \text{Der}.$$

We now give an analog in the theory of derivators of the following result from category theory. Let us consider an adjunction of two variables

$$(\otimes, \text{Hom}_l, \text{Hom}_r): \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D}$$

exhibiting \mathcal{D} as a closed \mathcal{C} -module. Then, there is a canonical enrichment of \mathcal{D} over \mathcal{C} where the mapping objects are given by Hom_r . We will now establish the corresponding result for derivators which will then be used to give important examples of enriched derivators. As a preparation for the proof let us give the following lemma in which we only state the results for Hom_r . Similar results are also valid for Hom_l .

Lemma 3.9. *Let \mathbb{D} , \mathbb{E} , and \mathbb{F} be prederivators and let us consider an adjunction of two variables $(\otimes, \text{Hom}_l, \text{Hom}_r): \mathbb{D} \times \mathbb{E} \rightarrow \mathbb{F}$. The adjunction units and counits at the different levels are compatible in the following sense. For a functor $u: J \rightarrow K$ and objects $X \in \mathbb{D}(K)$, $Y \in \mathbb{E}(K)$, and $Z \in \mathbb{F}(K)$ the following diagrams commute:*

$$\begin{array}{ccc}
 u^*X & \xrightarrow{u^*\eta} & u^* \text{Hom}_r(Y, X \otimes Y) \\
 \eta \downarrow & & \downarrow \gamma^{\text{Hom}_r} \\
 \text{Hom}_r(u^*Y, u^*X \otimes u^*Y) & \xrightarrow{\gamma^\otimes} & \text{Hom}_r(u^*Y, u^*(X \otimes Y)) \\
 \\
 u^* \text{Hom}_r(Y, Z) \otimes u^*Y & \xrightarrow{\gamma^{\text{Hom}_r}} & \text{Hom}_r(u^*Y, u^*Z) \otimes u^*Y \\
 \gamma^\otimes \downarrow & & \downarrow \epsilon \\
 u^*(\text{Hom}_r(Y, Z) \otimes Y) & \xrightarrow{u^*\epsilon} & u^*Z
 \end{array}$$

Proof. Let us begin with the statement about the adjunction units. Recall from Subsection 1.3 that in the context of an adjunction of two variables the adjunctions at the different levels are compatible with each other. This is expressed by the commutativity of the upper rectangle in the next diagram for the special case where we chose $Z = X \otimes Y$:

$$\begin{array}{ccc}
 \text{hom}_{\mathbb{F}(K)}(X \otimes Y, X \otimes Y) & \longrightarrow & \text{hom}_{\mathbb{D}(K)}(X, \text{Hom}_r(Y, X \otimes Y)) \\
 u^* \downarrow & & \downarrow u^* \\
 \text{hom}_{\mathbb{F}(J)}(u^*(X \otimes Y), u^*(X \otimes Y)) & & \text{hom}_{\mathbb{D}(J)}(u^*X, u^* \text{Hom}_r(Y, X \otimes Y)) \\
 \gamma^\otimes \downarrow & & \downarrow \gamma^{\text{Hom}_r} \\
 \text{hom}_{\mathbb{F}(J)}(u^*X \otimes u^*Y, u^*(X \otimes Y)) & \longrightarrow & \text{hom}_{\mathbb{D}(J)}(u^*X, \text{Hom}_r(u^*Y, u^*(X \otimes Y))) \\
 \gamma^\otimes \downarrow & & \downarrow \gamma^\otimes \\
 \text{hom}_{\mathbb{F}(J)}(u^*X \otimes u^*Y, u^*X \otimes u^*Y) & \longrightarrow & \text{hom}_{\mathbb{D}(J)}(u^*X, \text{Hom}_r(u^*Y, u^*X \otimes u^*Y))
 \end{array}$$

Starting with the identity in the upper left corner and comparing its two images in the bottom right corner we obtain the compatibility statement about the units. The corresponding result for the adjunction counits is obtained in a very similar manner. For this purpose, let us take $X = \text{Hom}_r(Y, Z)$ and let us consider the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{hom}_{\mathbb{F}(K)}(\mathrm{Hom}_r(Y, Z) \otimes Y, Z) & \longleftarrow & \mathrm{hom}_{\mathbb{D}(K)}(\mathrm{Hom}_r(Y, Z), \mathrm{Hom}_r(Y, Z)) \\
\downarrow u^* & & \downarrow u^* \\
\mathrm{hom}_{\mathbb{F}(J)}(u^*(\mathrm{Hom}_r(Y, Z) \otimes Y), u^*Z) & & \mathrm{hom}_{\mathbb{D}(J)}(u^* \mathrm{Hom}_r(Y, Z), u^* \mathrm{Hom}_r(Y, Z)) \\
\downarrow \gamma^\otimes & & \downarrow \gamma^{\mathrm{Hom}_r} \\
\mathrm{hom}_{\mathbb{F}(J)}(u^* \mathrm{Hom}_r(Y, Z) \otimes u^*Y, u^*Z) & \longleftarrow & \mathrm{hom}_{\mathbb{D}(J)}(u^* \mathrm{Hom}_r(Y, Z), \mathrm{Hom}_r(u^*Y, u^*Z)) \\
\uparrow \gamma^{\mathrm{Hom}_r} & & \uparrow \gamma^{\mathrm{Hom}_r} \\
\mathrm{hom}_{\mathbb{F}(J)}(\mathrm{Hom}_r(u^*Y, u^*Z) \otimes u^*Y, u^*Z) & \longleftarrow & \mathrm{hom}_{\mathbb{D}(J)}(\mathrm{Hom}_r(u^*Y, u^*Z), \mathrm{Hom}_r(u^*Y, u^*Z))
\end{array}$$

Since both identities—the one in the upper right and the one in the lower right corner—have the same image in $\mathrm{hom}_{\mathbb{D}(J)}(u^* \mathrm{Hom}_r(Y, Z), \mathrm{Hom}_r(u^*Y, u^*Z))$ they are both mapped onto the same element of $\mathrm{hom}_{\mathbb{F}(J)}(u^* \mathrm{Hom}_r(Y, Z) \otimes u^*Y, u^*Z)$. Using the commutativity of this diagram one sees immediately that this is precisely the compatibility statement about the adjunction counits. \square

Using the last lemma we can now establish the following theorem on the existence of enrichments in the context of adjunctions of two variables. Since this theorem is our main source for enriched derivators we will give a fairly complete proof.

Theorem 3.10. *Let \mathbb{E} be a monoidal derivator and let us consider an adjunction of two variables*

$$(\otimes, \mathrm{Hom}_l, \mathrm{Hom}_r): \mathbb{E} \times \mathbb{D} \rightarrow \mathbb{D}$$

exhibiting \mathbb{D} as a closed \mathbb{E} -module. The derivator \mathbb{D} can then be canonically enriched over \mathbb{E} and is naturally cotensored over \mathbb{E} .

Proof. Since we have an adjunction of two variables of derivators, by evaluation at a category K we obtain a corresponding adjunction of two variables which we will write as

$$(\otimes, (-)^{(-)}, \mathrm{Hom}_{\mathbb{D}(K)}): \mathbb{E}(K) \times \mathbb{D}(K) \rightarrow \mathbb{D}(K).$$

Let us agree that we use the short-hand-notation H for the functor Hom_r and also γ^{H} instead of γ^{Hom_r} .

Using the corresponding result from category theory, we obtain thus that the category $\mathbb{D}(K)$ can be canonically enriched over $\mathbb{E}(K)$ where the enrichment is given by the functor $\mathrm{H} = \mathrm{Hom}_{\mathbb{D}(K)}$. The composition law $\circ_{\mathbb{D}(K)}: \mathrm{Hom}_{\mathbb{D}(K)}(Y, Z) \otimes \mathrm{Hom}_{\mathbb{D}(K)}(X, Y) \rightarrow \mathrm{Hom}_{\mathbb{D}(K)}(X, Z)$ is given by the map which is adjoint to the following composition:

$$\begin{array}{ccc}
(\mathrm{Hom}_{\mathbb{D}(K)}(Y, Z) \otimes \mathrm{Hom}_{\mathbb{D}(K)}(X, Y)) \otimes X & \longrightarrow & Z \\
\cong \downarrow & & \uparrow \mathrm{ev} \\
\mathrm{Hom}_{\mathbb{D}(K)}(Y, Z) \otimes (\mathrm{Hom}_{\mathbb{D}(K)}(X, Y) \otimes X) & \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} & \mathrm{Hom}_{\mathbb{D}(K)}(Y, Z) \otimes Y
\end{array}$$

Here, ev is an adjunction counit, i.e., a map which is adjoint to the identity on $\mathrm{Hom}_{\mathbb{D}(K)}(X, Y)$ and similarly for Y, Z . The unit morphism for an object X is the map adjoint to the left unitality constraint $\lambda: \mathbb{S} \otimes X \rightarrow X$. We will not distinguish notationally between the ordinary category $\mathbb{D}(K)$ and the $\mathbb{E}(K)$ -enriched version.

Now, for a functor $u: J \rightarrow K$ we have to construct a morphism $(\mathbb{E}(K), \mathbb{D}(K)) \rightarrow (\mathbb{E}(J), \mathbb{D}(J))$ in $\mathbb{E}\text{CAT}$. The first component is of course given by the monoidal functor $\mathbb{E}(u): \mathbb{E}(K) \rightarrow \mathbb{E}(J)$. Let us recall from Subsection 1.2 that the monoidal structure on $\mathbb{E}(u)$ is given by the 2-cells belonging to the morphisms \otimes and \mathbb{S} . It remains hence to construct an $\mathbb{E}(J)$ -enriched functor

$$\mathbb{E}(u)_* \mathbb{D}(K) \rightarrow \mathbb{D}(J).$$

Since the base change 2-functor $\mathbb{E}(u): \mathbb{E}(K)\text{-CAT} \rightarrow \mathbb{E}(J)\text{-CAT}$ sends an $\mathbb{E}(K)$ -enriched category to an $\mathbb{E}(J)$ -enriched category with the same objects, we can define our would-be enriched functor to have the same behavior on objects as the unenriched functor $u^*: \mathbb{D}(K) \rightarrow \mathbb{D}(J)$ given by the derivator \mathbb{D} . Now, for two objects $X, Y \in \mathbb{E}(u)_* \mathbb{D}(K)$, we have to specify a map on morphism objects

$$\alpha_u: \text{Hom}_{\mathbb{E}(u)_* \mathbb{D}(K)}(X, Y) = u^* \text{Hom}_{\mathbb{D}(K)}(X, Y) \rightarrow \text{Hom}_{\mathbb{D}(J)}(u^* X, u^* Y).$$

We take α_u to be the morphisms which belong to

$$\mathbb{H} = \text{Hom}_r: \mathbb{D}^{\text{op}} \times \mathbb{D} \rightarrow \mathbb{E},$$

i.e., we set $\alpha_u = \gamma_u^{\mathbb{H}}$. Let us now check that these definitions assemble to give the intended $\mathbb{E}(J)$ -enriched functor.

We begin by the unitality condition so let us fix an object X of $\mathbb{E}(u)_* \mathbb{D}(K)$ which is hence simultaneously an object of $\mathbb{D}(K)$. For this purpose let us consider the following diagram:

$$\begin{array}{ccc}
 \mathbb{S} & \xrightarrow{\eta} & \text{Hom}(u^* X, \mathbb{S} \otimes u^* X) \\
 \downarrow & & \downarrow \\
 u^* \mathbb{S} & \xrightarrow{\eta} & \text{Hom}(u^* X, u^* \mathbb{S} \otimes u^* X) \\
 \downarrow \eta & & \downarrow \gamma^{\otimes} \\
 u^* \text{Hom}(X, \mathbb{S} \otimes X) & \xrightarrow{\gamma^{\mathbb{H}}} & \text{Hom}(u^* X, u^*(\mathbb{S} \otimes X)) \\
 \downarrow \lambda & & \downarrow \lambda \\
 u^* \text{Hom}(X, X) & \xrightarrow{\gamma^{\mathbb{H}}} & \text{Hom}(u^* X, u^* X)
 \end{array}
 \quad \begin{array}{l} \\ \\ \\ \lambda \end{array}$$

In this diagram, the identity of $X \in \mathbb{E}(u)_* \mathbb{D}(K)$ is given by the left column while the identity of $u^* X \in \mathbb{D}(J)$ is given by $\lambda \circ \eta$. The right part of the diagram commutes since u^* is a monoidal functor and that part is precisely one of the coherence conditions for a monoidal functor. Moreover, the two outer squares commute by naturality while the last one does by Lemma 3.9. Thus, the diagram commutes and we can hence conclude that our would-be enriched functor is unital.

It remains to show that the maps on morphism objects are compatible with composition. So, let us consider three objects X, Y, Z of $\mathbb{E}(u)_* \mathbb{D}(K)$. Spelling out the definition of the composition

laws we thus have to check the commutativity of the following diagram:

$$\begin{array}{ccc}
u^* H(Y, Z) \otimes u^* H(X, Y) & \xrightarrow{\gamma^H \otimes \gamma^H} & H(u^* Y, u^* Z) \otimes H(u^* X, u^* Y) \\
\downarrow \gamma^\otimes & & \downarrow \eta \\
u^*(H(Y, Z) \otimes H(X, Y)) & & H(u^* X, (H(u^* Y, u^* Z) \otimes H(u^* X, u^* Y)) \otimes u^* X) \\
\downarrow \eta & & \downarrow a \\
u^* H(X, (H(Y, Z) \otimes H(X, Y)) \otimes X) & & H(u^* X, H(u^* Y, u^* Z) \otimes (H(u^* X, u^* Y) \otimes u^* X)) \\
\downarrow a & & \downarrow \epsilon \\
u^* H(X, H(Y, Z) \otimes (H(X, Y) \otimes X)) & & H(u^* X, H(u^* Y, u^* Z) \otimes u^* Y) \\
\downarrow \epsilon & & \downarrow \epsilon \\
u^* H(X, H(Y, Z) \otimes Y) & & H(u^* X, u^* Z) \\
\downarrow \epsilon & & \downarrow = \\
u^* H(X, Z) & \xrightarrow{\gamma^H} & H(u^* X, u^* Z)
\end{array}$$

Let us consider the composition of morphisms from the upper left corner to the bottom right corner which passes through $u^* H(X, Z)$ and remark that it can be rewritten as:

$$\begin{array}{ccc}
u^* H(Y, Z) \otimes u^* H(X, Y) & \xrightarrow{\eta} & H(u^* X, (u^* H(Y, Z) \otimes u^* H(X, Y)) \otimes u^* X) \\
\downarrow \gamma^\otimes & & \downarrow \gamma^\otimes \\
u^*(H(Y, Z) \otimes H(X, Y)) & \xrightarrow{\eta} & H(u^* X, u^*(H(Y, Z) \otimes H(X, Y)) \otimes u^* X) \\
\downarrow \eta & & \downarrow \gamma^\otimes \\
u^* H(X, (H(Y, Z) \otimes H(X, Y)) \otimes X) & \xrightarrow{\gamma^H} & H(u^* X, u^*((H(Y, Z) \otimes H(X, Y)) \otimes X)) \\
\downarrow a & & \downarrow a \\
u^* H(X, H(Y, Z) \otimes (H(X, Y) \otimes X)) & \xrightarrow{\gamma^H} & H(u^* X, u^*(H(Y, Z) \otimes (H(X, Y) \otimes X))) \\
\downarrow \epsilon & & \downarrow \epsilon \\
u^* H(X, H(Y, Z) \otimes Y) & \xrightarrow{\gamma^H} & H(u^* X, u^*(H(Y, Z) \otimes Y)) \\
\downarrow \epsilon & & \downarrow \epsilon \\
u^* H(X, Z) & \xrightarrow{\gamma^H} & H(u^* X, u^* Z)
\end{array}$$

An application of the last lemma guarantees the commutativity of the second square while the remaining ones are commutative by naturality. Now the right column of this diagram itself can be

rewritten as follows:

$$\begin{array}{ccc}
\mathrm{H}(u^*X, (u^* \mathrm{H}(Y, Z) \otimes u^* \mathrm{H}(X, Y)) \otimes u^*X) & \xrightarrow{a} & \mathrm{H}(u^*X, u^* \mathrm{H}(Y, Z) \otimes (u^* \mathrm{H}(X, Y) \otimes u^*X)) \\
\downarrow \gamma^\otimes & & \downarrow \gamma^\otimes \\
\mathrm{H}(u^*X, u^*(\mathrm{H}(Y, Z) \otimes \mathrm{H}(X, Y)) \otimes u^*X) & & \\
\downarrow \gamma^\otimes & & \\
\mathrm{H}(u^*X, u^*((\mathrm{H}(Y, Z) \otimes \mathrm{H}(X, Y)) \otimes X)) & & \\
\downarrow a & & \\
\mathrm{H}(u^*X, u^*(\mathrm{H}(Y, Z) \otimes (\mathrm{H}(X, Y) \otimes X))) & \xleftarrow{\gamma^\otimes} & \mathrm{H}(u^*X, u^* \mathrm{H}(Y, Z) \otimes u^*(\mathrm{H}(X, Y) \otimes X)) \\
\downarrow \epsilon & & \downarrow \epsilon \\
\mathrm{H}(u^*X, u^*(\mathrm{H}(Y, Z) \otimes Y)) & \xleftarrow{\gamma^\otimes} & \mathrm{H}(u^*X, u^* \mathrm{H}(Y, Z) \otimes u^*Y) \\
\downarrow \epsilon & & \downarrow \gamma^H \\
\mathrm{H}(u^*X, u^*Z) & \xleftarrow{\epsilon} & \mathrm{H}(u^*X, \mathrm{H}(u^*Y, u^*Z) \otimes u^*Y)
\end{array}$$

In this diagram, the upper square commutes since for a monoidal derivator the restriction functors u^* are canonically monoidal and that square just expresses one of the coherence axioms for a monoidal functor. The second square commutes by naturality while the bottom square does by Lemma 3.9. Now, this new composition $\epsilon \circ \gamma^H \circ \epsilon \circ \gamma^\otimes \circ a$ can again be rewritten as $\epsilon \circ \epsilon \circ \gamma^\otimes \circ \gamma^H \circ a$ as depicted in the next diagram:

$$\begin{array}{ccc}
\mathrm{H}(u^*X, (u^* \mathrm{H}(Y, Z) \otimes u^* \mathrm{H}(X, Y)) \otimes u^*X) & \xrightarrow{\gamma^H \otimes \gamma^H} & \mathrm{H}(u^*X, (\mathrm{H}(u^*Y, u^*Z) \otimes \mathrm{H}(u^*X, u^*Y)) \otimes u^*X) \\
\downarrow a & & \downarrow a \\
\mathrm{H}(u^*X, u^* \mathrm{H}(Y, Z) \otimes (u^* \mathrm{H}(X, Y) \otimes u^*X)) & \xrightarrow{\gamma^H \otimes \gamma^H} & \mathrm{H}(u^*X, \mathrm{H}(u^*Y, u^*Z) \otimes (\mathrm{H}(u^*X, u^*Y) \otimes u^*X)) \\
\downarrow \gamma^H & & \downarrow = \\
\mathrm{H}(u^*X, \mathrm{H}(u^*Y, u^*Z) \otimes (u^* \mathrm{H}(X, Y) \otimes u^*X)) & \xrightarrow{\gamma^H} & \mathrm{H}(u^*X, \mathrm{H}(u^*Y, u^*Z) \otimes (\mathrm{H}(u^*X, u^*Y) \otimes u^*X)) \\
\downarrow \gamma^\otimes & & \downarrow \epsilon \\
\mathrm{H}(u^*X, \mathrm{H}(u^*Y, u^*Z) \otimes u^*(\mathrm{H}(X, Y) \otimes X)) & \xrightarrow{\epsilon} & \mathrm{H}(u^*X, \mathrm{H}(u^*Y, u^*Z) \otimes u^*Y) \\
& & \downarrow \epsilon \\
& & \mathrm{H}(u^*X, u^*Z)
\end{array}$$

In this diagram the bottom square commutes by a further application of Lemma 3.9 while the upper two squares do by naturality. Now, by the commutativity of the square

$$\begin{array}{ccc}
 u^* \mathbb{H}(Y, Z) \otimes u^* \mathbb{H}(X, Y) & \xrightarrow{\gamma^{\mathbb{H}} \otimes \gamma^{\mathbb{H}}} & \mathbb{H}(u^* Y, u^* Z) \otimes \mathbb{H}(u^* X, u^* Y) \\
 \eta \downarrow & & \downarrow \eta \\
 \mathbb{H}(u^* X, (u^* \mathbb{H}(Y, Z) \otimes u^* \mathbb{H}(X, Y))) \otimes u^* X & \xrightarrow{\gamma^{\mathbb{H}} \otimes \gamma^{\mathbb{H}}} & \mathbb{H}(u^* X, (\mathbb{H}(u^* Y, u^* Z) \otimes \mathbb{H}(u^* X, u^* Y))) \otimes u^* X
 \end{array}$$

we can conclude that we have constructed an $\mathbb{E}(J)$ -enriched functor. In fact, putting the above diagrams together we get the following chain of equalities

$$\begin{aligned}
 \gamma^{\mathbb{H}} \circ \epsilon \circ \epsilon \circ a \circ \eta \circ \gamma^{\otimes} &= \epsilon \circ \epsilon \circ a \circ \gamma^{\otimes} \circ \gamma^{\otimes} \circ \eta \\
 &= \epsilon \circ \gamma^{\mathbb{H}} \circ \epsilon \circ \gamma^{\otimes} \circ a \circ \eta \\
 &= \epsilon \circ \epsilon \circ \gamma^{\otimes} \circ \gamma^{\mathbb{H}} \circ a \circ \eta \\
 &= \epsilon \circ \epsilon \circ a \circ (\gamma^{\mathbb{H}} \otimes \gamma^{\mathbb{H}}) \circ \eta \\
 &= \epsilon \circ \epsilon \circ a \circ \eta \circ (\gamma^{\mathbb{H}} \otimes \gamma^{\mathbb{H}})
 \end{aligned}$$

expressing the compatibility of our functor with enriched composition laws. This concludes the proof that the above constructions assemble to an $\mathbb{E}(J)$ -enriched functor $\mathbb{E}(u)_* \mathbb{D}(K) \rightarrow \mathbb{D}(J)$. The assignment which sends a functor u to the enriched functors we just constructed is itself functorial by Lemma 1.11.

The final part of the proof consists of the construction of enriched natural transformations associated to 2-cells in Cat . More precisely, given a natural transformation $\alpha: u \rightarrow v$ between functors $u, v: J \rightarrow K$ we want to construct a 2-cell in ECAT between the induced morphisms $(\mathbb{E}(K), \mathbb{D}(K)) \rightarrow (\mathbb{E}(J), \mathbb{D}(J))$. Unraveling definitions we have to construct an $\mathbb{E}(J)$ -enriched natural transformation between the $\mathbb{E}(J)$ -enriched functors

$$\mathbb{E}(u)_* \mathbb{D}(K) \xrightarrow{u^*} \mathbb{D}(J) \quad \text{and} \quad \mathbb{E}(u)_* \mathbb{D}(K) \xrightarrow{\mathbb{E}(\alpha)_*} \mathbb{E}(v)_* \mathbb{D}(K) \xrightarrow{v^*} \mathbb{D}(J).$$

Evaluated at an object X we take the component $\alpha^X: \mathbb{S} \rightarrow \text{Hom}(u^* X, v^* X)$ of our would-be enriched natural transformation to be the map adjoint to $\mathbb{S} \otimes u^* X \rightarrow u^* X \rightarrow v^* X$. In order to show that this defines an enriched natural transformation we have to check that for any pair X, Y of objects the following square commutes:

$$\begin{array}{ccc}
 u^* \mathbb{H}(X, Y) & \xrightarrow{\cong} & u^* \mathbb{H}(X, Y) \otimes \mathbb{S} & \longrightarrow & v^* \mathbb{H}(X, Y) \otimes \mathbb{S} \\
 \cong \downarrow & & & & \downarrow \gamma^{\mathbb{H}} \otimes \alpha^X \\
 \mathbb{S} \otimes u^* \mathbb{H}(X, Y) & & & & \mathbb{H}(v^* X, v^* Y) \otimes \mathbb{H}(u^* X, v^* X) \\
 \alpha^Y \otimes \gamma^{\mathbb{H}} \downarrow & & & & \downarrow \circ \\
 \mathbb{H}(u^* Y, v^* Y) \otimes \mathbb{H}(u^* X, u^* Y) & \xrightarrow{\quad \circ \quad} & & & \mathbb{H}(u^* X, v^* Y)
 \end{array}$$

Let us show that the two maps $u^* \mathbb{H}(X, Y) \rightarrow \mathbb{H}(u^* X, v^* Y)$ are sent by the adjunction to the same maps $u^* \mathbb{H}(X, Y) \otimes u^* X \rightarrow v^* Y$. We begin by calculating this adjoint morphism for the map obtained by passing through the bottom left corner. By naturality of the associativity constraint

this can be written as:

$$\begin{array}{ccc}
u^* \mathbb{H}(X, Y) \otimes u^* X & \xrightarrow{\cong} & \mathbb{S} \otimes (u^* \mathbb{H}(X, Y) \otimes u^* X) \\
\downarrow & & \downarrow \alpha^Y \\
& & \mathbb{H}(u^* Y, v^* Y) \otimes (u^* \mathbb{H}(X, Y) \otimes u^* X) \\
& & \downarrow \gamma_u^{\mathbb{H}} \\
& & \mathbb{H}(u^* Y, v^* Y) \otimes (\mathbb{H}(u^* X, u^* Y) \otimes u^* X) \\
& & \downarrow \text{ev} \\
v^* Y & \xleftarrow{\text{ev}} & \mathbb{H}(u^* Y, v^* Y) \otimes u^* Y
\end{array}$$

From the construction of $\gamma_u^{\mathbb{H}}$ in Subsection 1.3 we know that the composition $\text{ev} \circ \gamma_u^{\mathbb{H}}$ is just the map

$$u^* \mathbb{H}(X, Y) \otimes u^* X \xrightarrow{\gamma^{\otimes}} u^*(\mathbb{H}(X, Y) \otimes X) \xrightarrow{\text{ev}} u^* Y.$$

Thus we can rewrite the above diagram as

$$\begin{array}{ccc}
u^* \mathbb{H}(X, Y) \otimes u^* X & \xrightarrow{\cong} & \mathbb{S} \otimes (u^* \mathbb{H}(X, Y) \otimes u^* X) \\
\downarrow \gamma^{\otimes} & & \downarrow \gamma^{\otimes} \\
u^*(\mathbb{H}(X, Y) \otimes X) & \xrightarrow{\cong} & \mathbb{S} \otimes u^*(\mathbb{H}(X, Y) \otimes X) \\
\downarrow \text{ev} & & \downarrow \text{ev} \\
u^* Y & \xrightarrow{\cong} & \mathbb{S} \otimes u^* Y \\
\downarrow \alpha^* & & \downarrow \alpha^Y \\
v^* Y & \xleftarrow{\text{ev}} & \mathbb{H}(u^* Y, v^* Y) \otimes u^* Y
\end{array}$$

and conclude that the adjoint map is just given by $\alpha^* \circ \text{ev} \circ \gamma^{\otimes}$.

Let us now calculate the morphism $u^* \mathbb{H}(X, Y) \otimes u^* X \rightarrow v^* Y$ which is adjoint to the map obtained by passing through the upper right corner. Using again the naturality of the associativity constraint we can identify the morphism as

$$\begin{array}{ccc}
(u^* \mathbb{H}(X, Y) \otimes \mathbb{S}) \otimes u^* X & \xrightarrow{\alpha^* \otimes \alpha^X} & (v^* \mathbb{H}(X, Y) \otimes \mathbb{H}(u^* X, v^* X)) \otimes u^* X \\
\uparrow \cong & & \downarrow a \\
u^* \mathbb{H}(X, Y) \otimes u^* X & & v^* \mathbb{H}(X, Y) \otimes (\mathbb{H}(u^* X, v^* X) \otimes u^* X) \\
\downarrow & & \downarrow \text{ev} \\
& & v^* \mathbb{H}(X, Y) \otimes v^* X \\
& & \downarrow \gamma_v^{\mathbb{H}} \\
v^* Y & \xleftarrow{\text{ev}} & \mathbb{H}(v^* X, v^* Y) \otimes v^* X
\end{array}$$

This description can be simplified by using again the relation $\text{ev} \circ \gamma_v^{\mathbb{H}} = \text{ev} \circ \gamma_v^{\otimes}$ and the definition of α^X as the adjoint map of $\mathbb{S} \otimes u^*X \cong u^*X \rightarrow v^*X$. So, the map under consideration is given by:

$$\begin{array}{ccc}
u^* \mathbb{H}(X, Y) \otimes (\mathbb{S} \otimes u^*X) & \xrightarrow{\alpha^X} & u^* \mathbb{H}(X, Y) \otimes (\mathbb{H}(u^*X, v^*X) \otimes u^*X) \\
\cong \uparrow & & \downarrow \text{ev} \\
u^* \mathbb{H}(X, Y) \otimes u^*X & \xrightarrow{\alpha^*} & u^* \mathbb{H}(X, Y) \otimes v^*X \\
\downarrow & & \downarrow \alpha^* \\
& & v^* \mathbb{H}(X, Y) \otimes v^*X \\
& & \downarrow \gamma_v^{\otimes} \\
v^*Y & \xleftarrow{\text{ev}} & v^*(\mathbb{H}(X, Y) \otimes X)
\end{array}$$

Hence, we have calculated the second adjoint morphism as $\text{ev} \circ \gamma_v^{\otimes} \circ (\alpha^* \otimes \alpha^*)$.

With these descriptions of the two adjoint morphisms it is easy to see that they coincide since both fit into the following commutative diagram:

$$\begin{array}{ccccc}
u^* \mathbb{H}(X, Y) \otimes u^*X & \xrightarrow{\gamma_u^{\otimes}} & u^*(\mathbb{H}(X, Y) \otimes X) & \xrightarrow{\text{ev}} & u^*Y \\
\alpha^* \otimes \alpha^* \downarrow & & \downarrow \alpha^* & & \downarrow \alpha^* \\
v^* \mathbb{H}(X, Y) \otimes v^*X & \xrightarrow{\gamma_v^{\otimes}} & v^*(\mathbb{H}(X, Y) \otimes X) & \xrightarrow{\text{ev}} & v^*Y
\end{array}$$

Here the left square commutes since $\alpha^*: u^* \rightarrow v^*$ is a monoidal transformation while the right square does by naturality. Thus, we have shown that the family $\{\alpha^X: \mathbb{S} \rightarrow \text{Hom}(u^*X, v^*X)\}_X$ defines an $\mathbb{E}(\mathcal{J})$ -enriched natural transformation as intended. We omit the details verifying that this assignment is compatible with identities and horizontal and vertical compositions, which then concludes the proof. \square

Corollary 3.11. *Let \mathbb{D} be a biclosed monoidal derivator. Then \mathbb{D} is canonically tensored, cotensored, and enriched over itself.*

The first class of examples of enriched (pre)derivators is obtained by an application of Theorem 3.10 to represented prederivators. Further examples coming from model categories will be given in the next subsection.

Example 3.12. Let \mathcal{C} be a complete monoidal category and let \mathcal{D} be a left \mathcal{C} -module such that the underlying category is complete. If the action map $\otimes: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D}$ is a left adjoint of two variables then Proposition 1.12 implies that we obtain an adjunction of two variables $\otimes: y(\mathcal{C}) \times y(\mathcal{D}) \rightarrow y(\mathcal{D})$ exhibiting $y(\mathcal{D})$ as a closed left $y(\mathcal{C})$ -module. Thus, the prederivator $y(\mathcal{D})$ is canonically enriched over $y(\mathcal{C})$.

3.3. Enriched model categories induce enriched derivators. We now only have to put together the above results in order to obtain the second important class of enriched derivators as guaranteed by the following theorem.

Theorem 3.13. *Let \mathcal{M} and \mathcal{N} be combinatorial model categories and let \mathcal{M} be in addition a monoidal model category. If \mathcal{N} is a left \mathcal{M} -module as a model category, then the derivator $\mathbb{D}_{\mathcal{N}}$ is canonically tensored, cotensored, and enriched over $\mathbb{D}_{\mathcal{M}}$.*

Proof. By the discussion preceding Theorem 2.13, we know that the action $\otimes: \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{M}$ is part of an adjunction of two variables

$$(\otimes, (-)^{(-)}, \mathbf{Hom}): \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{N}.$$

That theorem implies that this adjunction of two variables induces an adjunction of two variables at the level of the associated derivators:

$$(\overset{\mathbb{L}}{\otimes}, \mathbb{R}(-)^{(-)}, \mathbb{R}\mathbf{Hom}): \mathbb{D}_{\mathcal{M}} \times \mathbb{D}_{\mathcal{N}} \rightarrow \mathbb{D}_{\mathcal{N}}$$

Moreover, this adjunction exhibits $\mathbb{D}_{\mathcal{M}}$ as a left $\mathbb{D}_{\mathcal{N}}$ -module. Thus it suffices to apply Theorem 3.10 to deduce that $\mathbb{D}_{\mathcal{M}}$ can be canonically enriched over $\mathbb{D}_{\mathcal{N}}$. Recall from the proof of that result that the enrichment is actually given by $\mathbb{R}\mathbf{Hom}$. \square

Let us take up again our three classes of examples.

Corollary 3.14. *Let \mathcal{M} be a combinatorial model category.*

- i) *If \mathcal{M} is a simplicial model category, then the associated derivator $\mathbb{D}_{\mathcal{M}}$ is canonically tensored, cotensored, and enriched over the derivator $\mathbb{D}_{\mathbf{Set}_{\Delta}}$ of simplicial sets.*
- ii) *If \mathcal{M} is a spectral model category, then the associated derivator $\mathbb{D}_{\mathcal{M}}$ is canonically tensored, cotensored, and enriched over the derivator $\mathbb{D}_{\mathbf{Sp}}$ of spectra.*
- iii) *If \mathcal{M} is a dg model category over the ground ring k , then the associated derivator $\mathbb{D}_{\mathcal{M}}$ is canonically tensored, cotensored, and enriched over the derivator \mathbb{D}_k of chain complexes over k .*

We close by mentioning the more specific examples of the earlier sections. Again, these examples are completely parallel and could be given for any nice monoidal, combinatorial model category. We only stick to the cases of chain complexes and spectra.

Example 3.15. Let k be a commutative ground ring. The derivator \mathbb{D}_k of chain complexes over k is canonically tensored, cotensored, and enriched over itself. More generally, let C be a commutative differential graded algebra such that C is cofibrant as a chain complex, then the derivator \mathbb{D}_C of differential-graded C -modules is canonically tensored, cotensored, and enriched over itself. For a non-monoidal example, let us consider a non-commutative differential-graded algebra A . Then we can deduce that the associated derivator \mathbb{D}_A of differential-graded A -modules is canonically tensored, cotensored, and enriched over \mathbb{D}_k . Moreover, if A is cofibrant as a chain complex then \mathbb{D}_A is also canonically tensored, cotensored, and enriched over $\mathbb{D}_{A \otimes A^{\text{op}}}$.

Example 3.16. The derivator $\mathbb{D}_{\mathbf{Sp}}$ of spectra is canonically tensored, cotensored, and enriched over itself. More generally, let us consider a commutative symmetric ring spectrum E which has a cofibrant underlying symmetric spectrum. The derivator \mathbb{D}_E of E -module spectra is also canonically tensored, cotensored, and enriched over itself. If we are considering a symmetric ring spectrum R which is not necessarily commutative then we still obtain that the associated derivator \mathbb{D}_R of R -module spectra is canonically tensored, cotensored, and enriched over $\mathbb{D}_{\mathbf{Sp}}$. Finally, if R is cofibrant as a symmetric spectrum then \mathbb{D}_R is also canonically tensored, cotensored, and enriched over $\mathbb{D}_{R \wedge R^{\text{op}}}$.

APPENDIX A. THE 2-CATEGORICAL GROTHENDIECK CONSTRUCTION

In this appendix we will give a short description of the Grothendieck construction in the setting of 2-categories. As an input for that construction, one starts with a 2-functor $F: I \rightarrow 2\text{-CAT}$ where 2-CAT denotes the 2-category of 2-categories. Remark that we ignore the modifications [Bor94a] which would give us a 3-category of 2-categories. The basic idea behind the Grothendieck construction is that one wants to glue the different 2-categories $F(i)$ together to obtain a single new 2-category $\int F$. This is done in a way that an object ‘remembers in which category $F(i)$ it lived before’: $\int F$ will be canonically endowed with a ‘projection 2-functor’ $p: \int F \rightarrow I$. Before we give the actual construction in our 2-categorical situation, let us begin with a short recap of two ‘lower dimensional’ cases. For this purpose we let I be a category and replace 2-categories first by sets and then by categories.

Example A.1. (two dimensions less: the category of elements)

Let us consider a set-valued functor $F: I \rightarrow \mathbf{Set}$. Then one can construct the *category $el(F)$ of elements of F* . An object in $el(F)$ is a pair (i, X) consisting of an object $i \in I$ and an element $X \in F(i)$. Given two such objects, a morphism $(i, X) \rightarrow (j, Y)$ in $el(F)$ is a morphism $f: i \rightarrow j$ in I such that the induced map $F(f): F(i) \rightarrow F(j)$ maps X to Y . In the special case where the indexing category is the simplicial index category, i.e., if $I = \Delta^{\text{op}}$, we are starting with a simplicial set $F: \Delta^{\text{op}} \rightarrow \mathbf{Set}$. In that case the category $el(F)$ is just the category ΔF of simplices of F (cf. [GJ99] for the importance of this construction). Note that there is a canonical functor $p: el(F) \rightarrow I$ sending an object (i, X) to i and keeping the morphisms. This functor has the property that we have a canonical bijection $p^{-1}(i) \cong F(i)$ where we identified the discrete category $p^{-1}(i)$ with its set of objects.

Climbing up the dimension ladder by one, let us now consider categories instead of sets.

Example A.2. (one dimension less: the classical Grothendieck construction)

Let us consider a category-valued functor $F: I \rightarrow \mathbf{CAT}$. The Grothendieck construction $\int F$ of F is the following category. An object of $\int F$ is a pair (i, X) consisting of an object $i \in I$ and an object $X \in F(i)$. The fact that our functor F takes values in categories allows for a more general notion of morphisms than in the last example. So, let (i, X) and (j, Y) be two objects of $\int F$. A morphism $(i, X) \rightarrow (j, Y)$ is a pair (f, u) consisting of a morphism $f: i \rightarrow j$ in I and a morphism $u: F(f)X \rightarrow Y$ in $F(j)$. Given two composable morphisms $(f, u): (i, X) \rightarrow (j, Y)$ and $(g, v): (j, Y) \rightarrow (k, Z)$, their composition is defined to be $(g \circ f, v \circ F(g)(u))$. It is immediate to check that this is a category with the obvious identity morphisms. Again, we have a canonical projection functor $p: \int F \rightarrow I$. By definition, p sends an object (i, X) to i and a morphism (f, u) to f . Moreover, let us note that we have a canonical isomorphism of categories $p^{-1}(i) \cong F(i)$.

These projection functors p are not arbitrary functors but have particularly nice properties. In fact, they are examples of Grothendieck opfibrations [Vis05] and we will comment shortly on this after the next construction.

Having recalled these two classical cases the 2-categorical version will now go as expected. However, we also increase the dimension of the domain of F by one, so let us consider a 2-category-valued 2-functor $F: I \rightarrow 2\text{-CAT}$. The *2-categorical Grothendieck construction* $\int F$ is the following 2-category. The underlying category of $\int F$ will be as in the last example, so that we will only make explicit the 2-morphisms and the vertical and horizontal composition laws. Thus, let $(f, u), (g, v): (i, X) \rightarrow (j, Y)$ be two parallel morphisms in $\int F$. A 2-morphism $(\alpha, \phi): (f, u) \rightarrow (g, v)$ is a pair consisting of a 2-morphism $\alpha: f \rightarrow g$ in I and a 2-cell ϕ in $F(j)$

as indicated in:

$$\begin{array}{ccc}
 f_*X & \xrightarrow{u} & Y \\
 \searrow & \Downarrow & \uparrow v \\
 \alpha_*X & \xrightarrow{\quad} & g_*X
 \end{array}$$

Given three parallel morphisms (f, u) , (g, v) and (h, w) in $\int F$ and two vertically composable 2-morphisms $(\alpha, \phi): (f, u) \rightarrow (g, v)$ and $(\beta, \psi): (g, v) \rightarrow (h, w)$, their vertical composition is defined by

$$(\beta, \psi) \cdot (\alpha, \phi) = (\beta \cdot \alpha, \psi \alpha_* \cdot \phi).$$

Finally, let us consider two horizontally composable 2-morphisms $(\alpha, \phi): (f_1, u_1) \rightarrow (f_2, u_2)$ and $(\beta, \psi): (g_1, v_1) \rightarrow (g_2, v_2)$ as in:

$$\begin{array}{ccccc}
 & \xrightarrow{(f_1, u_1)} & & \xrightarrow{(g_1, v_1)} & \\
 (i, X) & \Downarrow & (j, Y) & \Downarrow & (k, Z) \\
 & \xrightarrow{(f_2, u_2)} & & \xrightarrow{(g_2, v_2)} &
 \end{array}$$

Then their horizontal composition is defined by the following formula

$$(\beta, \psi) * (\alpha, \phi) = (\beta * \alpha, \psi * g_{1*} \phi).$$

The corresponding diagram in $F(k)$ looks like:

$$\begin{array}{ccccc}
 g_{1*}f_{1*}X & \xrightarrow{u_1} & g_{1*}Y & \xrightarrow{v_1} & Z \\
 \alpha_* \downarrow & \searrow & \downarrow \beta_* & \searrow & \uparrow v_2 \\
 g_{1*}f_{2*}X & \xrightarrow{u_2} & g_{2*}Y & & \\
 \beta_* \downarrow & \searrow & \uparrow & & \\
 g_{2*}f_{2*}X & \xrightarrow{u_2} & & &
 \end{array}$$

It is now a straightforward calculation to verify that these two composition laws satisfy the interchange law, i.e., that $\int F$ is indeed a 2-category. Let us note that the projection on the first variable gives us a 2-functor $p: \int F \rightarrow I$ such that we have canonical isomorphisms $p^{-1}(i) \cong F(i)$ of 2-categories.

Already in the 1-dimensional case, the functor $p: \int F \rightarrow I$ is not an arbitrary functor but is a Grothendieck opfibration. A similar Grothendieck construction can be applied to a contravariant category-valued functor in which case the projection functor would be a Grothendieck fibration. Recall that a Grothendieck opfibration is by definition a functor $p: \mathcal{C} \rightarrow I$ which allows for a sufficient supply of p -coCartesian arrows (cf. [Bor94b, Vis05]). These p -coCartesian arrows are morphisms satisfying a universal property which is expressed by the bijectivity of a certain canonical map of sets. In the 2-categorical picture, there are now different ways of introducing a notion of coCartesian arrows: the role of the canonical map of sets is taken by a canonical functor and one could demand this functor to be an *isomorphism* or an *equivalence*. Since we will not need these p -coCartesian arrows we will not get into this. Nevertheless, depending on the variance with respect to 1-morphisms of the 2-category-valued 2-functor F we started with, we will call the associated projection functor p the *2-Grothendieck (op)fibration associated to F* .

Let us close this appendix by remarking that Grothendieck (op)fibrations and also the Grothendieck construction were generalized to the setting of ∞ -categories by Joyal and Lurie ([Lur09]). A short introduction to these notions is given in [Gro10b].

APPENDIX B. MONOIDAL AND CLOSED MONOIDAL 2-CATEGORIES

B.1. The 2-categories of monoidal objects and modules in a monoidal 2-category. In this subsection, let $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{S}, \alpha, \lambda, \rho)$ be a monoidal 2-category which is given by a 2-category \mathcal{C} , a 2-functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ called the monoidal pairing, a monoidal unit $\mathbb{S} \in \mathcal{C}$, and invertible 2-natural transformations α, λ and ρ called the associativity constraint and the unitality constraints respectively. The coherence conditions are the same as in the 1-categorical case. Again, we will not distinguish notationally between these 2-natural transformations and their respective inverses. We will quickly recall the notions of monoidal objects and modules in \mathcal{C} . This is done since we need some details about these notions in the construction of certain 2-categories of modules which are important in the next subsection.

So, let us begin with the monoidal objects. A *monoidal object* $X = (X, \mu_X, u_X, \alpha_X, \lambda_X, \rho_X)$ in \mathcal{C} is a sextuple consisting of an object $X \in \mathcal{C}$, morphisms $\mu_X: X \otimes X \rightarrow X$ and $u_X: \mathbb{S} \rightarrow X$ and invertible 2-cells λ_X, α_X and ρ_X as indicated in:

$$\begin{array}{ccccc}
 \mathbb{S} \otimes X & \xrightarrow{u_X} & X \otimes X & & (X \otimes X) \otimes X & \xrightarrow{\alpha} & X \otimes (X \otimes X) & & X \otimes X & \xleftarrow{u_X} & X \otimes \mathbb{S} \\
 \searrow \lambda & \nearrow \mu_X & \downarrow \mu_X & & \downarrow \mu_X & \Leftarrow & \downarrow \mu_X & & \downarrow \mu_X & \nearrow \rho & \downarrow \mu_X \\
 & & X & & X \otimes X & \xrightarrow{\mu_X} & X & \xleftarrow{\mu_X} & X \otimes X & & X
 \end{array}$$

These data are subject to certain coherence conditions which are the same as in [ML98, pp.162-163] suitably adapted to the context of a general monoidal 2-category. Let us express these coherence conditions as conditions on 2-cells in $\text{Hom}_{\mathcal{C}}(X^{\otimes n}, X)$ where we denote by $X^{\otimes n}$ the n -fold tensor power of X w.r.t. \otimes where we moved all brackets as far to the left as possible. For this purpose, let us denote the map μ_X by $(-) \cdot (-)$, the identity id_X by $(-)$, u_X by itself and similarly for combinations of these maps. In this notation the 2-cells $\lambda_X \lambda^{-1}$, α_X resp. $\rho_X \rho^{-1}$ are hence denoted by:

$$(-) \rightarrow u_X \cdot (-), \quad (-) \cdot ((-) \cdot (-)) \rightarrow ((-) \cdot (-)) \cdot (-) \quad \text{resp.} \quad (-) \rightarrow (-) \cdot u_X$$

The coherence conditions on the above 2-cells are given by the commutativity of the pentagon

$$\begin{array}{ccc}
 (((-) \cdot (-)) \cdot (-)) \cdot (-) & \xrightarrow{\quad} & ((-) \cdot (-)) \cdot ((-) \cdot (-)) \\
 \downarrow & & \downarrow \\
 ((-) \cdot ((-) \cdot (-))) \cdot (-) & \xrightarrow{\quad} & (-) \cdot (((-) \cdot (-)) \cdot (-)) \xrightarrow{\quad} (-) \cdot ((-) \cdot ((-) \cdot (-))),
 \end{array}$$

the equality of the two 2-cells $u_X \cdot u_X \xrightarrow{\quad} u_X$ and the commutativity of:

$$\begin{array}{ccc}
 (u_X \cdot (-)) \cdot (-) & \xrightarrow{\quad} & (-) \cdot (-) \\
 \downarrow & \nearrow & \\
 u_X \cdot ((-) \cdot (-)) & &
 \end{array}$$

Let now M and N be two monoidal objects in \mathcal{C} . A *monoidal morphism* $f: M \rightarrow N$ is a triple (f, m_f, u_f) consisting of a morphism $f: M \rightarrow N$ in \mathcal{C} and two invertible 2-cells m_f and u_f as

indicated in:

$$\begin{array}{ccc}
 M \otimes M & \xrightarrow{\mu_M} & M \\
 f \otimes f \downarrow & \nearrow & \downarrow f \\
 N \otimes N & \xrightarrow{\mu_N} & N
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{S} & \xrightarrow{u_M} & M \\
 \searrow u_N & \nearrow & \downarrow f \\
 & & N
 \end{array}$$

Using a similar notation as in the previous case the coherence conditions on such a triple are given by the commutativity of the following two diagrams:

$$\begin{array}{ccc}
 (f(-) \cdot f(-)) \cdot f(-) \longrightarrow f(-) \cdot (f(-) \cdot f(-)) & & u_N \cdot f(-) \longrightarrow f(-) \longleftarrow f(-) \cdot u_N \\
 \downarrow & & \downarrow & & \downarrow \\
 f((-) \cdot (-)) \cdot f(-) & & f(-) \cdot f((-) \cdot (-)) & & f(u_M) \cdot f(-) & = & f(-) \cdot f(u_M) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 f(((-) \cdot (-)) \cdot (-)) \longrightarrow f((-) \cdot ((-) \cdot (-))) & & f(u_M \cdot (-)) \longrightarrow f(-) \longleftarrow f((-) \cdot u_M)
 \end{array}$$

The composition of monoidal morphisms is defined by composition of the underlying morphisms and by splicing of 2-cells. Thus, for a pair (g, f) of composable monoidal morphisms we have

$$(g, m_g, u_g) \circ (f, m_f, u_f) = (gf, m_{gf}, u_{gf}) = (gf, gm_f \cdot m_g(f \otimes f), gu_g \cdot u_f).$$

Here, we use the central dot \cdot to denote the vertical composition of 2-cells in \mathcal{C} .

Finally, a *monoidal 2-morphism* $\phi: f \rightarrow g$ between two parallel monoidal morphisms (f, m_f, u_f) and (g, m_g, u_g) is a 2-morphism $\phi: f \rightarrow g$ in \mathcal{C} making the following two diagrams commute:

$$\begin{array}{ccc}
 \mu_N \circ (f \otimes f) & \xrightarrow{m_f} & f \circ \mu_M \\
 \phi \otimes \phi \downarrow & & \downarrow \phi \\
 \mu_N \circ (g \otimes g) & \xrightarrow{m_g} & g \circ \mu_M
 \end{array}
 \qquad
 \begin{array}{ccc}
 u_N & \xrightarrow{u_f} & f \circ u_M \\
 \searrow u_g & & \downarrow \phi \\
 & & g \circ u_M
 \end{array}$$

The vertical and the horizontal composition of monoidal 2-morphisms is the same as the corresponding one in \mathcal{C} . It is straightforward to check that we obtain a 2-category this way so let us make the following definition.

Definition B.1. Let \mathcal{C} be a monoidal 2-category. The *2-category* $\text{Mon}(\mathcal{C})$ of monoidal objects in \mathcal{C} consists of the monoidal objects together with the monoidal morphisms and the monoidal 2-morphisms.

Example B.2. i) For the Cartesian 2-category $\mathcal{C} = \text{CAT}$ we have $\text{Mon}(\text{CAT}) = \text{MonCAT}$, the 2-category of monoidal categories.

ii) For the Cartesian 2-category $\mathcal{C} = \text{PDer}$ resp. $\mathcal{C} = \text{Der}$ we have $\text{Mon}(\text{PDer}) = \text{MonPDer}$ resp. $\text{Mon}(\text{Der}) = \text{MonDer}$, the 2-category of monoidal prederivators resp. monoidal derivators.

Let us now turn to modules. A *(left) module* X over a monoidal object $M \in \text{Mon}(\mathcal{C})$ is a quadruple (X, a_X, m_X, u_X) consisting of an object $X \in \mathcal{C}$, an action map $a_X: M \otimes X \rightarrow X$ and

two invertible 2-cells u_X and m_X as indicated in:

$$\begin{array}{ccc}
 \mathbb{S} \otimes X & \xrightarrow{u_M} & M \otimes X \\
 \searrow \lambda & \nearrow \! \! \nearrow & \downarrow a_X \\
 & & X
 \end{array}
 \quad
 \begin{array}{ccc}
 (M \otimes M) \otimes X & \xrightarrow{\alpha} & M \otimes (M \otimes X) \\
 \mu_M \downarrow & \Leftarrow & \downarrow a_X \\
 M \otimes X & \xrightarrow{a_X} & X \xleftarrow{a_X} M \otimes X
 \end{array}$$

These are again subject to certain coherence conditions which we will depict using a similar notation as in the case of monoidal objects. Note, that in this context the outer right central dot corresponds to the action while the other ones correspond to multiplications on the monoid. The coherence conditions for a module consist again of a pentagon diagram as in the case of monoids and also the following two triangles:

$$\begin{array}{ccc}
 u_M \cdot ((-) \cdot (-)) & \longrightarrow & (-) \cdot (-) \\
 \uparrow & \nearrow & \uparrow \\
 (u_M \cdot (-)) \cdot (-) & & (-) \cdot (u_M \cdot (-)) \\
 & & \uparrow \\
 & & ((-) \cdot u_M) \cdot (-)
 \end{array}$$

Given two M -modules X and Y , a *lax morphism of modules* $f: X \rightarrow Y$ is a pair (f, m_f) consisting of an underlying morphism $f: X \rightarrow Y$ in \mathcal{C} and a (not necessarily invertible) 2-cell m_f as in:

$$\begin{array}{ccc}
 M \otimes X & \xrightarrow{a_X} & X \\
 f \downarrow & \nearrow \! \! \nearrow & \downarrow f \\
 M \otimes Y & \xrightarrow{a_Y} & Y
 \end{array}$$

This 2-cell is subject to the following two coherence conditions:

$$\begin{array}{ccc}
 ((-) \cdot (-)) \cdot f(-) & \longrightarrow & (-) \cdot ((-) \cdot f(-)) \longrightarrow (-) \cdot f((-) \cdot (-)) & u_M \cdot f(-) \longrightarrow f(-) \\
 \downarrow & & \downarrow & \downarrow \\
 f(((-) \cdot (-)) \cdot (-)) & \longrightarrow & f((-) \cdot ((-) \cdot (-))) & f(u_M \cdot (-))
 \end{array}$$

It is important that we allow m_f to be a non-invertible 2-cell here since this will be needed in the construction of the 2-category $\text{Mod}(\mathcal{C})^{\text{lax}}$ of modules via the 2-categorical Grothendieck construction. If the 2-cell m_f is invertible then let us call f a *strong morphism of modules* (or simply a *morphism of modules*). The composition of lax module morphisms is again defined by composition of the underlying morphisms and by splicing of 2-cells. Thus, for two composable lax morphisms g and f we set:

$$(g, m_g) \circ (f, m_f) = (gf, m_{gf}) = (gf, gm_f \cdot m_g(1 \otimes f))$$

Finally, given two parallel lax morphisms $f, g: X \rightarrow Y$ a *2-morphism of modules* $\phi: f \rightarrow g$ is just such a 2-cell ϕ in \mathcal{C} . This 2-cell has to satisfy the coherence condition:

$$\begin{array}{ccc}
 a_Y \circ (1 \otimes f) & \xrightarrow{\phi} & a_Y \circ (1 \otimes g) \\
 m_f \downarrow & & \downarrow m_g \\
 f \circ a_X & \xrightarrow{\phi} & g \circ a_X
 \end{array}$$

Thus, the 2-cell ϕ has to satisfy the following equation:

$$m_g \cdot a_Y(1 \otimes \phi) = \phi a_X \cdot m_f$$

It is again immediate that these definitions can be assembled to give us a 2-category.

Definition B.3. Let \mathcal{C} be a monoidal 2-category and let M be a monoidal object in \mathcal{C} . The 2-category $M - \text{Mod}^{\text{Lax}}$ of (left) M -modules is given by the M -modules, the lax M -module morphisms and the 2-morphisms of M -modules.

We now want to show that the association which sends a monoidal object $M \in \text{Mon}(\mathcal{C})$ to the 2-category $M - \text{Mod}^{\text{Lax}}$ is 2-functorial. This allows us then to apply the 2-categorical Grothendieck construction of Appendix A in order to obtain the 2-category $\text{Mod}(\mathcal{C})^{\text{Lax}}$ of modules in \mathcal{C} .

Let us begin by defining the behavior of the 2-functor on morphisms of monoids. So, let us consider a morphism $f: M \rightarrow N$ in $\text{Mon}(\mathcal{C})$ and let us construct the associated 2-functor $f^*: N - \text{Mod}^{\text{Lax}} \rightarrow M - \text{Mod}^{\text{Lax}}$ which basically is a restriction of scalar 2-functor. For this purpose, let $X = (X, a_X, m_X, u_X)$ be an N -module. The underlying object of f^*X is again just X while a_{f^*X} and u_{f^*X} are defined by the following diagrams:

$$\begin{array}{ccc} M \otimes X & \xrightarrow{f \otimes 1} & N \otimes X \\ & \searrow a_{f^*X} & \downarrow a_X \\ & & X \end{array} \quad \begin{array}{ccc} S \otimes X & \xrightarrow{u_M} & M \otimes X \\ & \searrow u_N & \downarrow f \\ & & N \otimes X \\ & \searrow \lambda & \downarrow a_X \\ & & X \end{array}$$

The upper unlabeled 2-cell is given by $u_f \otimes 1$ while the lower one is u_X . Thus, in formulas we are setting:

$$a_{f^*X} = a_X(f \otimes 1) \quad \text{and} \quad u_{f^*X} = a_X(u_f \otimes 1) \cdot u_X$$

Finally, in order to construct m_{f^*X} let us consider the following diagram in which the left 2-cell is induced by m_f while the other one is just m_X :

$$\begin{array}{ccccc} (M \otimes M) \otimes X & \xrightarrow{\alpha} & M \otimes (M \otimes X) & \xrightarrow{a_{f^*X}} & \\ \downarrow \mu_M & & \downarrow f & & \downarrow f \\ M \otimes X & \xrightarrow{f \otimes f} & M \otimes (N \otimes X) & \xrightarrow{a_X} & M \otimes X \\ & \searrow m_f & \downarrow f & & \downarrow f \\ & & (N \otimes N) \otimes X & \xrightarrow{\alpha} & N \otimes (N \otimes X) \\ & & \downarrow \mu_N & & \downarrow a_X \\ & & N \otimes X & \xrightarrow{a_X} & N \otimes X \\ & & & & \downarrow a_X \\ & & & & X \end{array}$$

The 2-cell obtained by splicing from this diagram is taken to be m_{f^*X} , i.e., we set

$$m_{f^*X} = a_X(m_f \otimes 1) \cdot m_X((f \otimes f) \otimes 1).$$

This concludes the definition of f^* on objects. Let us now define its behavior on morphisms. So, for a morphism $h = (h, m_h): X \rightarrow Y$ in $N - \text{Mod}^{\text{la}x}$ let us set:

$$f^*h = (h, m_{f^*h}) = (h, m_h(f \otimes 1))$$

Finally, given a 2-morphism $\phi: h_1 \rightarrow h_2$ of morphisms $h_1, h_2: X \rightarrow Y$ of modules, let f^* just map ϕ to itself. Then, in order to check that $f^*\phi: f^*h_1 \rightarrow f^*h_2$ has the necessary coherence property let us consider the following chain of equalities.

$$\begin{aligned} m_{f^*h_2} \cdot a_{f^*Y}(1 \otimes f^*\phi) &= m_{h_2}(f \otimes 1) \cdot a_Y(f \otimes 1)(1 \otimes \phi) \\ &= (m_{h_2} \cdot a_Y(1 \otimes \phi))(f \otimes 1) \\ &= (\phi a_X \cdot m_{h_1})(f \otimes 1) \\ &= \phi a_X(f \otimes 1) \cdot m_{h_1}(f \otimes 1) \\ &= f^*\phi a_{f^*X} \cdot m_{f^*h_1} \end{aligned}$$

Here, the third equation uses the fact that ϕ is a 2-cell in $N - \text{Mod}^{\text{la}x}$ while the composite equality precisely says that $f^*\phi = \phi$ is also a 2-cell $f^*h_1 \rightarrow f^*h_2$ in $M - \text{Mod}^{\text{la}x}$. This concludes the definition of f^* and it is easy to verify that it in fact defines a 2-functor. Now, given two composable morphisms $M \xrightarrow{f} N \xrightarrow{g} P$ in $\text{Mon}(\mathcal{C})$ we want to check that we have an equality $f^*g^* = (gf)^*$ of 2-functors $P - \text{Mod}^{\text{la}x} \rightarrow M - \text{Mod}^{\text{la}x}$. But this is obvious for their behavior on morphisms and 2-morphisms and hence also for their behavior on objects.

Finally, let us consider a 2-cell ψ in $\text{Mon}(\mathcal{C})$ as in:

$$M \begin{array}{c} \xrightarrow{f} \\ \Downarrow \psi \\ \xrightarrow{g} \end{array} N$$

We want to associate a 2-natural transformation $\psi^*: g^* \rightarrow f^*$ to ψ . So, let us consider an N -module X and the associated M -modules f^*X, g^*X . We claim that the pair $(\text{id}_X, a_X(\psi \otimes \text{id}_X))$ defines a morphism $g^*X \rightarrow f^*X$ in $M - \text{Mod}^{\text{la}x}$. Let us only check the unitality condition, i.e., let us consider the following diagram:

$$\begin{array}{ccc} \mathbb{S} \otimes X & \xrightarrow{\text{id}} & \mathbb{S} \otimes X \\ \downarrow u_M & & \downarrow u_M \\ \begin{array}{ccc} \lambda & \begin{array}{ccc} \xrightarrow{u_{g^*X}} M \otimes X & \xrightarrow{\text{id}} & M \otimes X \xleftarrow{u_{f^*X}} \\ \downarrow a_{g^*X} & \leftarrow & \downarrow a_{f^*X} \end{array} & \lambda \end{array} \\ \downarrow a_{g^*X} & & \downarrow a_{f^*X} \\ X & \xrightarrow{\text{id}} & X \end{array}$$

Here, the unlabeled 2-cell is given by $a_X(\psi \otimes \text{id}_X)$. But the unitality coherence condition satisfied by ψ as a 2-morphism in $\text{Mon}(\mathcal{C})$ implies that we have the equality:

$$u_{g^*X} = a_X(\psi \otimes \text{id}_X)(u_M \otimes 1) \cdot u_{f^*X}$$

Thus, $\psi_X^* = (\text{id}_X, a_X(\psi \otimes \text{id}_X))$ defines a morphism $g^*X \rightarrow f^*X$ in $M - \text{Mod}^{\text{la}x}$. It is now easy to verify that these ψ_X^* assemble to define a 2-natural transformation $\psi^*: g^* \rightarrow f^*$.

This concludes the construction of our 2-functor. Before we can summarize the construction by the following proposition let us quickly recall that given an arbitrary 2-category \mathcal{D} , the 2-category obtained from \mathcal{D} by inverting both the direction of the 1-cells and of the 2-cells is denoted by $\mathcal{D}^{\text{op,co}}$.

Proposition B.4. *Let \mathcal{C} be a monoidal 2-category and let us consider a 2-cell $\psi: f \rightarrow g: M \rightarrow N$ in $\text{Mon}(\mathcal{C})$. The following assignments define a 2-functor $(-) - \text{Mod}^{\text{ lax}}: \text{Mon}(\mathcal{C})^{\text{op,co}} \rightarrow 2\text{-CAT}$:*

$$M \mapsto M - \text{Mod}^{\text{ lax}}, \quad f \mapsto f^* \quad \text{and} \quad \psi \mapsto \psi^*$$

Having established this proposition we can now apply the 2-categorical Grothendieck construction of Appendix A to the 2-category-valued 2-functor $(-) - \text{Mod}^{\text{ lax}}$. This gives us the 2-category $\text{Mod}(\mathcal{C})^{\text{ lax}}$ of modules in \mathcal{C} . Let us be a bit more specific about this 2-category. An object is a pair (M, X) consisting of a monoidal object M and an M -module X . Similarly, a morphism $(f, u): (M, X) \rightarrow (N, Y)$ is a pair consisting of monoidal morphism $f: M \rightarrow N$ and a lax morphism of M -modules $u: X \rightarrow f^*Y$. Finally, given two parallel such morphisms (f, u) and (g, v) , a 2-cell $(\beta, \phi): (f, u) \rightarrow (g, v)$ is a monoidal 2-cell $\beta: f \rightarrow g$ together with a 2-cell ϕ of M -modules as in:

$$\begin{array}{ccc} X & \xrightarrow{u} & f^*Y \\ & \searrow \phi & \uparrow \beta^* \\ & & g^*Y \\ & \swarrow v & \end{array}$$

This 2-category is endowed with a projection functor $p: \text{Mod}(\mathcal{C})^{\text{ lax}} \rightarrow \text{Mon}(\mathcal{C})$ which we call the 2-Grothendieck fibration of modules in \mathcal{C} .

Of more importance in the next subsection is the 2-subcategory $\text{Mod}(\mathcal{C}) \subseteq \text{Mod}(\mathcal{C})^{\text{ lax}}$. By definition this consists of all objects (M, X) , the morphisms (f, u) such that u is a strong morphism of modules and all 2-cells between such morphisms. The inclusion endows $\text{Mod}(\mathcal{C})$ with a projection functor $p: \text{Mod}(\mathcal{C}) \rightarrow \text{Mon}(\mathcal{C})$ which we still call the 2-Grothendieck fibration of modules in \mathcal{C} .

Example B.5. i) For the Cartesian 2-category $\mathcal{C} = \text{CAT}$ we have $\text{Mod}(\text{CAT}) = \text{MonCAT}$, the 2-category of left-tensored categories.
ii) For the Cartesian 2-category $\mathcal{C} = \text{PDer}$ resp. $\mathcal{C} = \text{Der}$ we have $\text{Mod}(\text{PDer}) = \text{ModPDer}$ resp. $\text{Mod}(\text{Der}) = \text{ModDer}$, the 2-category of left-tensored prederivators resp. derivators.

In addition to this 2-functor p , we also have a canonical 2-functor $U: \text{Mod}(\mathcal{C}) \rightarrow \mathcal{C}$. This 2-functor sends an object (M, X) to the underlying object of X and a morphism $(f, u): (M, X) \rightarrow (N, Y)$ to the underlying morphism of $u: X \rightarrow f^*Y$. Similarly, U sends a 2-cell (β, ϕ) just to the underlying 2-cell $\phi: u \rightarrow \beta^*v$ in \mathcal{C} . It is immediate that this defines a 2-functor U . As an upshot, we thus obtain the following diagram of 2-categories:

$$\begin{array}{ccc} \text{Mod}(\mathcal{C}) & \xrightarrow{U} & \mathcal{C} \\ p \downarrow & & \\ \text{Mon}(\mathcal{C}) & & \end{array}$$

Now, given an object $X \in \mathcal{C}$ let $\text{Mod}(X) = U^{-1}(X)$ denote the fiber of U over X , i.e., it is the 2-category defined by the following pullback diagram:

$$\begin{array}{ccc} \text{Mod}(X) & \longrightarrow & \text{Mod}(\mathcal{C}) \\ \downarrow & & \downarrow U \\ e & \xrightarrow{X} & \mathcal{C} \end{array}$$

Here, e denotes the terminal 2-category and by abuse of notation X denotes at the same time the object X and the unique 2-functor $e \rightarrow \mathcal{C}$ classifying the object X . Let us call the 2-category $\mathbf{Mod}(X)$ the *2-category of module structures on X* . Since the underlying object of an arbitrary object in this 2-category is X let us agree that we denote such an object by (M, a_X) where a_X is the action belonging to the M -module structure on X . In Subsection B.2 we will show that for *closed* monoidal 2-categories \mathcal{C} this 2-category of module structures always has a terminal object in a suitable bicategorical sense.

B.2. Closed monoidal 2-categories. The main aim of this subsection is to give a 2-categorical analog of the following result about closed monoidal categories. Let \mathcal{C} be a closed monoidal category with symmetric monoidal pairing \otimes , monoidal unit \mathbb{S} and internal homomorphism functor \mathbf{HOM} . Plugging in twice the same object $X \in \mathcal{C}$ into \mathbf{HOM} we obtain internal endomorphism objects $\mathbf{END}(X) \in \mathcal{C}$. Since we assumed the monoidal structure to be closed we have natural isomorphisms

$$\mathbf{hom}_{\mathcal{C}}(X \otimes Y, Z) \cong \mathbf{hom}_{\mathcal{C}}(X, \mathbf{HOM}(Y, Z)).$$

The adjunction counit gives us in particular a map $\epsilon = \mathbf{ev}: \mathbf{END}(X) \otimes X \rightarrow X$ which we call an *evaluation map*. A combination of this map with the associativity constraint of the monoidal structure gives us the following map:

$$(\mathbf{END}(X) \otimes \mathbf{END}(X)) \otimes X \xrightarrow{\alpha} \mathbf{END}(X) \otimes (\mathbf{END}(X) \otimes X) \xrightarrow{\mathbf{ev}} \mathbf{END}(X) \otimes X \xrightarrow{\mathbf{ev}} X$$

Let us denote the associated adjoint map by $\circ_X: \mathbf{END}(X) \otimes \mathbf{END}(X) \rightarrow \mathbf{END}(X)$. Moreover, let us write $\iota_X: \mathbb{S} \rightarrow \mathbf{END}(X)$ for the map which is adjoint to the unitality constraint $\lambda: \mathbb{S} \otimes X \rightarrow X$. Finally, similar to the last subsection one can construct the category $\mathbf{Mod}(X)$ of module structures on X in the context of a monoidal category.

Proposition B.6. *Let \mathcal{C} be a closed monoidal category and let $X \in \mathcal{C}$ be an object. The triple $(\mathbf{END}(X), \circ_X, \iota_X)$ is a monoid in \mathcal{C} and the evaluation map $\mathbf{ev}: \mathbf{END}(X) \otimes X \rightarrow X$ turns X into a module over $\mathbf{END}(X)$. The pair $(\mathbf{END}(X), \mathbf{ev})$ is the terminal object in the category $\mathbf{Mod}(X)$ of module structures on X .*

We want to give a similar result in the setting of 2-categories which will be applied in Subsection 2.2 to the Cartesian closed 2-category \mathbf{Der} of derivators. Besides working with 2-categories, there is an additional technical difficulty resulting from the following fact. Before we can formulate this let us recall that there are two different notions of adjointness for 2-functors. The stricter one of these notions which we shall call a *2-adjunction* is just a special case of an enriched adjunction. In such a situation the adjointness is expressed by the fact that we have natural *isomorphisms* between the respective categories of morphisms. A more general and –morally speaking– more correct notion is the notion of a *biadjunction*. In this case we instead have natural *equivalences* of categories of morphisms. To mention only one difference between biadjunctions and 2-adjunctions, note that a biadjunction does in general not induce an adjunction on underlying 1-categories. Now, the additional technical difficulty results from the fact that the closedness of the Cartesian monoidal 2-category \mathbf{Der} given by Proposition 2.7 is expressed by a special instance of a biadjunction. With this example in mind, let us make the following definition.

Definition B.7. Let \mathcal{C} be a symmetric monoidal 2-category. The monoidal structure is *closed* if the functor $X \otimes -: \mathcal{C} \rightarrow \mathcal{C}$ has a right biadjoint $\mathbf{HOM}(X, -)$ for each object $X \in \mathcal{C}$, i.e., if there are natural equivalences of categories

$$\mathbf{Hom}_{\mathcal{C}}(X \otimes Y, Z) \simeq \mathbf{Hom}_{\mathcal{C}}(X, \mathbf{HOM}(Y, Z)).$$

For the rest of this subsection let \mathcal{C} be a closed monoidal 2-category and let us choose inverse equivalences of categories:

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \xrightleftharpoons[r]{l} \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{HOM}(Y, Z))$$

As for any biadjunction one can describe the equivalences l and r by the unit η and the counit ϵ . In fact, we have isomorphisms $r \cong \epsilon_* \circ (- \otimes Y)$ and $l \cong \eta^* \circ \mathrm{HOM}(-, Y)$.

By precisely the same formulas as in the 1-categorical case we can now use these equivalences and the constraints of the monoidal structure in order to obtain morphisms \circ_X , ι_X and ev . In the proof of the next proposition we will use EX as an abbreviation for $\mathrm{END}(X)$.

Proposition B.8. *Let \mathcal{C} be a closed monoidal 2-category and let $X \in \mathcal{C}$ be an object. The triple $(\mathrm{END}(X), \circ_X, \iota_X)$ can be extended into a monoidal object in \mathcal{C} and the map $\mathrm{ev}: \mathrm{END}(X) \otimes X \rightarrow X$ is part of an $\mathrm{END}(X)$ -module structure on X .*

Proof. We will only construct the 2-cells which will turn EX into a monoidal object and X into a module over EX . We will leave it to the reader to check the necessary coherence conditions.

So, let us begin by the unitality of the action, i.e., we want to construct an invertible 2-cell u_X as indicated in:

$$\begin{array}{ccc} \mathbb{S} \otimes X & \xrightarrow{\iota \otimes X} & \mathrm{EX} \otimes X \\ & \searrow \cong & \downarrow \mathrm{ev} \\ & & X \\ & \swarrow \lambda & \nearrow \\ & & X \end{array}$$

But this is given by $\mathrm{ev} \circ (\iota \otimes X) = \mathrm{ev} \circ (l(\lambda) \otimes X) \cong r(l(\lambda)) \cong \lambda$. The construction of the –say– left unitality of the multiplication \circ_X on EX is slightly more complicated. We have to show that there is an invertible 2-cell u_{EX} as in:

$$\begin{array}{ccc} \mathbb{S} \otimes \mathrm{EX} & \xrightarrow{\iota \otimes \mathrm{EX}} & \mathrm{EX} \otimes \mathrm{EX} \\ & \searrow \cong & \downarrow \circ_X \\ & & \mathrm{EX} \\ & \swarrow \lambda & \nearrow \\ & & \mathrm{EX} \end{array}$$

If we can show that the images of these two compositions under r are isomorphic we can use the fully-faithfulness of r to conclude that this isomorphism comes from a unique isomorphism of morphisms $\mathbb{S} \otimes \mathrm{EX} \rightarrow \mathrm{EX}$. To calculate the image of $\circ_X \circ (\iota \otimes \mathrm{EX})$ under r let us consider the left diagram:

$$\begin{array}{ccc} (\mathbb{S} \otimes \mathrm{EX}) \otimes X & \xrightarrow{\iota} & (\mathrm{EX} \otimes \mathrm{EX}) \otimes X \\ \alpha \downarrow & & \downarrow \alpha \\ \mathbb{S} \otimes (\mathrm{EX} \otimes X) & \xrightarrow{\iota} & \mathrm{EX} \otimes (\mathrm{EX} \otimes X) \\ \mathrm{ev} \downarrow & & \downarrow \mathrm{ev} \\ \mathbb{S} \otimes X & \xrightarrow{\iota} & \mathrm{EX} \otimes X \\ & \searrow \cong & \downarrow \mathrm{ev} \\ & & X \\ & \swarrow \lambda & \nearrow \\ & & X \end{array} \qquad \begin{array}{ccc} (\mathbb{S} \otimes \mathrm{EX}) \otimes X & & \\ \alpha \downarrow & \searrow \lambda & \\ \mathbb{S} \otimes (\mathrm{EX} \otimes X) & \xrightarrow{\lambda} & \mathrm{EX} \otimes X \\ \mathrm{ev} \downarrow & & \downarrow \mathrm{ev} \\ \mathbb{S} \otimes X & \xrightarrow{\lambda} & X \end{array}$$

The invertible 2-cell in the left diagram is u_X . Using the commutative diagram on the right we have thus obtained an isomorphism as intended:

$$r(\circ_X \circ (\iota \otimes \text{EX})) \cong \text{ev} \circ \text{ev} \circ \alpha \circ \iota \cong \lambda \circ (\mathbb{S} \otimes \text{ev}) \circ \alpha = \text{ev} \circ (\lambda \otimes X) \cong r(\lambda)$$

Let us now turn to the associativity of \circ_X and the multiplicativity of ev . Again we will begin with the action ev since that 2-cell will be used in the construction of the 2-cell expressing the associativity of the multiplication on EX . So, let us show that there is an invertible 2-cell m_X :

$$\begin{array}{ccc} (\text{EX} \otimes \text{EX}) \otimes X & \xrightarrow{\alpha} & \text{EX} \otimes (\text{EX} \otimes X) \\ \circ_X \downarrow & \Leftarrow & \downarrow \text{ev} \\ \text{EX} \otimes X & \xrightarrow{\text{ev}} & X \xleftarrow{\text{ev}} \text{EX} \otimes X \end{array}$$

But this is again just the observation that we have invertible 2-cells

$$\text{ev} \circ (\circ_X \otimes X) \cong r(\circ_X) = r(l(\text{ev} \circ \text{ev} \circ \alpha)) \cong \text{ev} \circ \text{ev} \circ \alpha.$$

Thus, let us now show that the multiplication \circ_X is associative, i.e., let us construct an invertible 2-cell α_{EX} as in:

$$\begin{array}{ccc} (\text{EX} \otimes \text{EX}) \otimes \text{EX} & \xrightarrow{\alpha} & \text{EX} \otimes (\text{EX} \otimes \text{EX}) \\ \circ_X \otimes 1 \downarrow & \Leftarrow & \downarrow 1 \otimes \circ_X \\ \text{EX} \otimes \text{EX} & \xrightarrow{\circ_X} & \text{EX} \xleftarrow{\circ_X} \text{EX} \otimes \text{EX} \end{array}$$

Similarly to the proof of the unitality, let us show that the two morphisms $(\text{EX} \otimes \text{EX}) \otimes \text{EX} \rightarrow \text{EX}$ have isomorphic images under r . For this purpose, let us consider the following diagram:

$$\begin{array}{ccccccc} & & (\text{EX} \otimes (\text{EX} \otimes \text{EX})) \otimes X & \xleftarrow{\cong} & ((\text{EX} \otimes \text{EX}) \otimes \text{EX}) \otimes X & & \\ & \circ_X \swarrow & \downarrow \cong \alpha & & \downarrow \cong \alpha & \circ_X \searrow & \\ (\text{EX} \otimes \text{EX}) \otimes X & & \text{EX} \otimes ((\text{EX} \otimes \text{EX}) \otimes X) & & (\text{EX} \otimes \text{EX}) \otimes (\text{EX} \otimes X) & & (\text{EX} \otimes \text{EX}) \otimes X \\ \downarrow \cong \alpha & \circ_X \swarrow & \downarrow \cong \alpha & \alpha \swarrow & \downarrow \text{ev} & \circ_X \searrow & \downarrow \cong \alpha \\ \text{EX} \otimes (\text{EX} \otimes X) & & \text{EX} \otimes (\text{EX} \otimes (\text{EX} \otimes X)) & & (\text{EX} \otimes \text{EX}) \otimes X & & \text{EX} \otimes (\text{EX} \otimes X) \\ \cong \downarrow & \cong \swarrow & \downarrow \text{ev} & \cong \swarrow & \downarrow \circ_X & \cong \swarrow & \downarrow \cong \alpha \\ & & \text{EX} \otimes (\text{EX} \otimes X) & & \text{EX} \otimes X & & \text{EX} \otimes (\text{EX} \otimes X) \\ & \cong \swarrow & \downarrow \text{ev} & \cong \swarrow & \downarrow \text{ev} & \cong \swarrow & \downarrow \cong \alpha \\ & & \text{EX} \otimes X & & X & & \text{EX} \otimes X \\ & \text{ev} \swarrow & \downarrow \text{ev} & \text{ev} \swarrow & \downarrow \text{ev} & \text{ev} \swarrow & \downarrow \text{ev} \\ & & X & & X & & X \end{array}$$

Up to an implicit use of the invertible 2-cell m_X the two possible paths through the boundary leading from $((\text{EX} \otimes \text{EX}) \otimes \text{EX}) \otimes X$ to X are the images under r of the maps which we want to compare. In this diagram there are three more instances of the invertible 2-cell m_X . The remaining part commutes on the nose by naturality in four cases and by the coherence property of the associativity constraint α in the last case. The resulting invertible 2-cell gives us the intended 2-cell expressing the associativity of \circ_X . This concludes the extension of \circ_X to a monoidal structure on EX and of the evaluation ev to a module structure on X and hence the proof of this proposition. \square

In the notation of the last subsection, this proposition shows that in the context of a closed monoidal 2-category the 2-category $\text{Mod}(X)$ of module structures on an object X contains the object $(\text{END}(X), \text{ev})$. The remaining aim of this subsection is to show that this object is a terminal object in the following sense.

Definition B.9. Let \mathcal{D} be a 2-category. An object $X \in \mathcal{D}$ is *terminal* if for all objects Y the category $\text{Hom}_{\mathcal{D}}(Y, X)$ of morphisms from Y to X is equivalent to the category e .

Proposition B.10. Let \mathcal{C} be a closed monoidal 2-category and let X be an object of \mathcal{C} . The canonical action $(\text{END}(X), \text{ev})$ is a terminal object of the 2-category $\text{Mod}(X)$ of module structures on X .

Proof. Let us begin by showing the following. For an arbitrary object (M, a_X) of $\text{Mod}(X)$ there is a morphism $(M, a_X) \rightarrow (\text{END}(X), \text{ev})$ in $\text{Mod}(X)$. Similar to the proof of Proposition B.8 we only give the construction of the 1-cells and the 2-cells of the morphism and do not check the necessary coherence conditions. So, our aim is to construct a pair $(f, u): (M, a_X) \rightarrow (\text{END}(X), \text{ev})$ where $f = (f, m_f, u_f): M \rightarrow \text{END}(X)$ is a morphism in $\text{Mon}(\mathcal{C})$ and $u = (u, m_u): (X, a_X) \rightarrow f^*(X, \text{ev})$ is a morphism in $M\text{-Mod}$. Since the morphism (f, u) has to lie in $\text{Mod}(X)$ we have $u = (\text{id}_X, m_u)$.

We construct the monoidal part first. By closedness, $a_X: M \otimes X \rightarrow X$ corresponds to a unique map $f = l(a_X): M \rightarrow \text{END}(X)$. Here, l again denotes a natural equivalence of categories $l: \text{Hom}(M \otimes X, X) \rightarrow \text{Hom}(M, \text{END}(X))$ given by the biadjunction. The construction of u_f is similar. As part of the M -module structure on X we have the following invertible 2-cell u_X :

$$\begin{array}{ccc} \mathbb{S} \otimes X & \xrightarrow{u_M \otimes 1} & M \otimes X \\ & \searrow \lambda & \downarrow a_X \\ & & X \end{array}$$

Recall from the proof of the last proposition that the unit $u_{\text{END}(X)}: \mathbb{S} \rightarrow \text{END}(X)$ is given by $l(\lambda)$. Thus, by adjointness, the 2-cell u_X gives us an invertible 2-cell u_f as in:

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{u_M} & M \\ & \searrow u_{\text{end}(X)} & \downarrow f \\ & & \text{END}(X) \end{array}$$

In the construction of m_f we will of course use the multiplicativity constraint of the module structure on X . So, let us consider the invertible 2-cell m_X :

$$\begin{array}{ccc} (M \otimes M) \otimes X & \xrightarrow{\alpha} & M \otimes (M \otimes X) \\ \mu_M \otimes 1 \downarrow & \leftarrow & \downarrow 1 \otimes a_X \\ X \otimes X & \xrightarrow{a_X} & X \xleftarrow{a_X} X \otimes X \end{array}$$

The 1-cell $l(a_X \circ (\mu_M \otimes 1)) = l(a_X) \circ \mu_M = f \circ \mu_M: M \otimes M \rightarrow \text{END}(X)$ gives us already the target of m_f . We now want to identify $l(a_X \circ (1 \otimes a_X) \circ \alpha)$ with the source $\circ_X \circ (f \otimes f)$ of m_f . For this let us recall from the proof of the last proposition that the multiplication \circ_X is given by

$l(\text{ev} \circ \text{ev} \circ \alpha)$. A calculation of $r(\circ_X \circ (f \otimes f))$ thus leads to the following diagram:

$$\begin{array}{ccccccc}
(M \otimes M) \otimes X & \xrightarrow{(f \otimes f) \otimes 1} & (\text{END}(X) \otimes \text{END}(X)) \otimes X & \xrightarrow{\circ_X} & \text{END}(X) \otimes X & \xrightarrow{\text{ev}} & X \\
\alpha \downarrow & & \alpha \downarrow & & \cong & & \nearrow \text{ev} \\
M \otimes (M \otimes X) & \xrightarrow{f \otimes (f \otimes 1)} & \text{END}(X) \otimes (\text{END}(X) \otimes X) & \xrightarrow{\text{ev}} & \text{END}(X) \otimes X & &
\end{array}$$

Here, the 2-cell is the invertible 2-cell m'_X expressing the multiplicativity of ev as constructed in the last proof. But the composition $\text{ev} \circ (1 \otimes \text{ev}) \circ (f \otimes (f \otimes 1))$ can be rewritten as:

$$\begin{array}{ccccccc}
M \otimes (M \otimes X) & \xrightarrow{f} & M \otimes (\text{END}(X) \otimes X) & \xrightarrow{\text{ev}} & M \otimes X & \xrightarrow{f} & \text{END}(X) \otimes X & \xrightarrow{\text{ev}} & X \\
& & \cong & & & & \cong & & \\
& \searrow & & \nearrow & & \searrow & & \nearrow & \\
& & 1 \otimes a_X & & & & a_X & &
\end{array}$$

The two invertible 2-cells are both instances of $\text{ev} \circ (f \otimes 1) = r(f) = r(l(a_X) \cong a_X)$. Thus, splicing these three invertible 2-cells together we obtain an isomorphism $r(\circ_X \circ (f \otimes f)) \cong a_X \circ (1 \otimes a_X) \circ \alpha$. This allows us to construct the invertible 2-cell m_f as the following composite

$$\circ_X \circ (f \otimes f) \cong l(r(\circ_X \circ (f \otimes f))) \cong l(a_X \circ (1 \otimes a_X) \circ \alpha) \cong l(a_X \circ (\mu_M \otimes 1)) = f \circ \mu_M.$$

The last invertible 2-cell in this composition is given by $l(m_X)$. The construction of the monoidal morphism $f: M \rightarrow \text{END}(X)$ is complete.

Let us now construct the morphism $u = (\text{id}, m_u): (X, a_X) \rightarrow f^*(X, \text{ev})$ in $M - \text{Mod}$. But this means that we only have to construct an invertible 2-cell m_u as in:

$$\begin{array}{ccc}
M \otimes X & \xrightarrow{a_X} & X \\
f \downarrow & \nearrow & \downarrow \text{id} \\
\text{END}(X) \otimes X & \xrightarrow{\text{ev}} & X
\end{array}$$

We take this to be $\text{ev} \circ (f \otimes 1) = r(f) = r(l(a_X)) \cong a_X$. One can now check that this pair (f, u) defines a morphism $(M, a_X) \rightarrow (\text{END}(X), \text{ev})$ in $\text{Mod}(X)$.

Now, given two parallel morphisms $(f, u), (g, v): (M, a_X) \rightarrow (\text{END}(X), \text{ev})$ in $\text{Mod}(X)$ it remains to show that there is a unique 2-cell $(f, u) \rightarrow (g, v)$. Thus, we have to construct a pair (β, ϕ) consisting of a monoidal 2-cell $\beta: f \rightarrow g$ and an invertible 2-cell ϕ in $M - \text{Mod}$ as in:

$$\begin{array}{ccc}
(X, a_X) & \xrightarrow{u} & f^*(X, \text{ev}) \\
& \searrow \phi & \uparrow \beta^* \\
& & g^*(X, \text{ev})
\end{array}$$

But from the module morphisms u and v we obtain invertible 2-cells $m_u: \text{ev} \circ (f \otimes 1) \rightarrow a_X$ and $m_v: \text{ev} \circ (g \otimes 1) \rightarrow a_X$. These can be combined to the invertible 2-cell

$$m_v^{-1} \circ m_u: r(f) = \text{ev} \circ (f \otimes 1) \rightarrow \text{ev} \circ (g \otimes 1) = r(g).$$

Thus, we obtain a unique invertible 2-cell $\beta: f \rightarrow g$ with $r(\beta) = m_v^{-1} \circ m_u$. One checks now that this β is a monoidal 2-cell and that the pair $(\beta, \phi = \text{id}_{\text{id}_X})$ gives us the intended unique 2-cell. This concludes the proof that $(\text{END}(X), \text{ev})$ is terminal in $\text{Mod}(X)$. \square

As a summary of this subsection we have thus established the following theorem.

Theorem B.11. *Let \mathcal{C} be a closed monoidal 2-category and let X be an object in \mathcal{C} . The internal endomorphism object $\text{END}(X)$ can be canonically made into a monoidal object and the adjunction counit $\text{ev}: \text{END}(X) \otimes X \rightarrow X$ is part of a canonical module structure on X . Moreover, the pair $(\text{END}(X), \text{ev})$ is a terminal object in the 2-category $\text{Mod}(X)$ of module structures on X .*

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