AG-SEMINAR WS 2011/12:
∞-CATEGORIES AND THEIR APPLICATIONS

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The aim of higher category theory is to develop a theory of 'categories with morphisms of arbitrarily high dimension' that could be useful in the areas of science where categorical structures of higher order arise naturally. By now, there are many different approaches to such a theory. The goal of this seminar is to discuss two approaches to higher category theory in the $(\infty, 1)$-case (i.e., all morphisms of dimension > 1 are invertible in a certain weak sense), review their connections with homotopical algebra and present some of their main applications so far.

Among these two approaches, the more rigid one is given by the theory of simplical categories, i.e., simplicially enriched categories, while the more flexible ('homotopy coherent') one is given by the theory of ∞-categories (aka. quasi-categories or weak Kan complexes). The basics of these two approaches will be presented in the first two sessions of the seminar. It will be shown that each of them defines a homotopy theory in the sense of Quillen's model categories. The equivalence between these two approaches is stated in terms of a Quillen equivalence between the associated model categories (this is a deep rigidification result!).

In the remainder of the seminar we focus on ∞-categories. In the third session, it will be shown how many key notions of classical category theory can be extended to the context of ∞-categories. For example, there are good notions of limits and colimits for ∞-categories, which should now be regarded as homotopy (co)limits. This connection with homotopical algebra here further emphasizes a central theme of the theory of ∞-categories, namely, that 'higher coherence laws are efficiently encoded in the underlying combinatorics'. Some aspects of the general theory of presentable ∞-categories will also be presented. As an example of a concrete result, we will discuss that the ∞-category of spaces is the free cocomplete ∞-category generated by a single generator, corresponding to the zero simplex.

From then on, we will restrict our attention to the stable theory, i.e., the theory of stable ∞-categories as an enhancement of triangulated categories (there are others as well: stable model categories, stable/triangulated derivators,...). An intrinsic description of the ∞-category Sp of spectra, as a special case of a general stabilization process, will be presented in the fourth session. This is again characterized by a beautiful universal property: it is the free stable presentable ∞-category on one generator.

In the fifth and sixth sessions, we discuss aspects of homological algebra in the ∞-categorical setting. The rudiments of the theory of t-structures and a generalization of the classical Dold-Kan correspondence will be presented. We shall see that there is a derived ∞-category enhancing the classical notion of the derived category. Moreover, it will be shown that the derived ∞-category together with its 'canonical' t-structure is characterized by a universal property.

In the last session, we will discuss an application of ∞-categories to higher algebraic K-theory. More specifically, it will be shown, using the language of ∞-categories, that higher algebraic K-theory is also characterized by a universal property.

1. First Session (20.10.2011): Two models for $(\infty, 1)$-categories

Talk 1: Introduction to ∞-categories. (Moritz Groth) This talk should begin by first introducing the abstract concept of an (∞, n)-category before it specializes to the $(\infty, 1)$-case. Then ∞-categories (aka. quasi-categories, weak Kan complexes) will be discussed as a model for $(\infty, 1)$-categories. Spaces and categories are important motivating examples of ∞-categories. Topics to be discussed: (a) the homotopy relation(s) on morphisms and a sketch of the construction of the homotopy category of an ∞-category, (b) mapping spaces between objects in an ∞-category, (c) equivalences (aka. isomorphisms, quasi-isomorphisms) in an ∞-category and the filling of special outer horns [Joy02, Theorem 1.3],[Joy08, Theorem 4.13], (d) Kan-complexes are ∞-groupoids, (e) a characterization of ∞-categories as those simplicial sets which 'allow for a composition law up to contractible space of choices' [Joy08, Proposition 2.24].
    References: [Lur09, Chapter 1], [Joy08, Chapters 1 and 2], [Joy02], [Gro10, Chapter 1].

Talk 2: Simplicial categories and the Bergner model category. (Georgios Raptis) This talk will introduce the category of simplicial categories together with the class of Dwyer-Kan equivalences ([DK80a], [DK80b]) as a model for the homotopy theory of homotopy theories. To explain this viewpoint, the construction of the simplicial (or hammock) localization of a category with weak equivalences should be presented and its main properties should be discussed [DK80a, Section 4]. Then the Dwyer-Kan model category of simplicial categories with a fixed set of objects ([DK80b, Section 7]) and the Bergner model category of all simplicial categories ([Ber07]) will be discussed.
    References: [DK80a], [DK80b], [Ber07].

2. Second Session (03.11.2011): The Joyal model category and the comparison with simplicial categories

Talk 3: The Joyal model category. (Viktoriya Ozornova) This talk will introduce the homotopy theory of ∞-categories which is a completely new way of looking at a simplicial set. First, the class of Joyal equivalences (aka. weak categorical equivalences in [Joy08]) will be defined following [Joy08, Section 1.2], [DS11a, Section 2.8] and some of their properties should be mentioned [Joy08, Sections 1.2 and 2.4]. The alternative definition, which resembles the definition of DK-equivalences between simplicial categories, should also be discussed [DS11a, Definition 7.1], [DS11a, Proposition 8.1, (i) and (iii) only]. The rest of the talk should give a sketch of the proof of the Joyal model structure (following [Joy08, Section 6], [DS11a, Appendix C]) focusing mostly on the various properties of pseudo-fibrations between ∞-categories (aka. special inner fibrations in [DS11a]), e.g. [Joy08, Theorem 5.22], [DS11a, Lemmas C.2 and C.3].
    References: [Joy08, Chapters 5 and 6], [DS11a].

Talk 4: The Quillen equivalence between the Joyal and the Bergner model categories. (Irakli Patchkoria) In this talk, we will discuss the Quillen equivalence between these two models for a theory of $(\infty, 1)$-categories following [DS11a]. The comparison adjunction should be carefully defined [Lur09, Section 1.1.5]. Then a sketch of the key fact that the derived counit map of the adjunction is a weak equivalence should be presented, following the approach of [DS11a, Proposition

5.9], [DS11b]. (Note that the same theorem in [Lur09, Theorem 2.2.0.1] is proved using completely different methods.) From this, the Quillen equivalence will be deduced [DS11a, Corollary 8.2].

References:[DS11a], [DS11b], [Lur09, Sections 1.1.5, 2.2.4 and 2.2.5].

3. Third Session (17.11.2011): (Co)limits in ∞-categories and presentable ∞-categories

Talk 5: Categorical constructions with ∞-categories. (Wolfgang Steimle) This talk will focus on the extension of some key notions from classical category theory to the setting of ∞-categories. More specifically, the goal is to talk about limits and colimits of diagrams in an ∞-category, discuss some of their properties and review their connection with homotopy limits and colimits. For this, the join and slice constructions will be first introduced [Lur09, Chapter 1], [Joy02], [Joy08, Chapter 3]. Using these constructions, limits and colimits of general diagrams are defined as special cases of final resp. initial objects, and they are unique up to a contractible space of choices if they exist. The specific examples of pushouts (and pullbacks) and retracts should be mentioned, as they will be important later on [Lur09, Sections 4.4.2 and 4.4.5]. If time permits, a sketch of the relation between this theory of (co)limits and the theory of homotopy (co)limits in simplicial categories should also be included [Lur09, Section 4.2.4].

References: [Lur09, Chapters 1 and 4], [Joy02], [Joy08, Chapter 3], [Gro10, Chapter 2].

Talk 6: Presentable ∞-categories. (Marcus Zibrowius) This talk should begin by recalling (very briefly) the notion of accessible and presentable categories from classical category theory and state the Representation Theorem of [AR94, Theorem 1.46,(i) and (iv) only]. A further possibility is to recall also the simplified form of the Special Adjoint Functor Theorem in this context. We then move to accessible and presentable ∞-categories. The ∞-categories of presheaves should be introduced as the main example [Lur09, Section 5.1] and the ∞-categorical Yoneda lemma should be mentioned [Lur09, Section 5.1.3]. In analogy with classical category theory, ∞-categories of presheaves are characterized as free cocompletions [Lur09, Theorem 5.1.5.6]. As an example, the ∞-category of spaces can be characterized as the free cocomplete ∞-category on one generator, namely, the zero-simplex Δ^0. A sketch of the strategy to establish the ∞-categorical version of the Representation Theorem should be attempted [Lur09, Theorem 5.5.1.1, (1) and (5) only]. If time permits, the relationship between presentable ∞-categories and combinatorial simplicial model categories could also be mentioned [Lur09, Proposition A.3.7.6].

References: [Lur09, Chapter 5], [Gro10, Chapter 2]. For the classical theory: [AR94, GU71, Bor94].

4. Fourth Session (01.12.2011): Stable ∞-categories and Stabilization

Talk 7: Stable ∞-categories. (Karol Szumilo) This talk introduces the important concept of a stable ∞-category (which is also an enhancement of the notion of a triangulated category). An important fact is that the homotopy category of a stable ∞-category is additive and can be canonically endowed with the structure of a triangulated category [Lur11, Theorem 1.1.2.13]. The 'key technical ingredient' [Lur09, Proposition 4.3.2.15] is so important in many places of this AG that it should be mentioned (the absolute case suffices!). Exact functors between stable ∞-categories should be defined and stable (simplicial) model cateogories should be presented as (extrinsic) examples [Lur11, pp.14-28].

As a preparation for the next talk we further discuss (pre)spectrum objects in a pointed ∞-category $\mathcal{C}$. In the finitely complete case, the ∞-category $Sp(\mathcal{C})$ of spectrum objects 'is given by ∞-loop objects' [Lur11, Proposition 1.4.2.6, pp.98-100].

References: [Lur11, Chapter 1], [Gro10, Chapter 5].

Talk 8: Stabilization and universal property of ∞-category of spectra. (Lennart Meier) We show that the ∞-category $Sp(\mathcal{C})$ of spectrum objects in a pointed ∞-category $\mathcal{C}$ is stable ([Lur11, pp.101-104]) and deduce the alternative characterization of stable ∞-categories given by [Lur11, Corollary 1.4.2.20]. The important special case where $\mathcal{C}$ is the ∞-category of pointed spaces gives the stable ∞-category Sp of spectra [Lur11, pp.105-109]. If we start with a finitely complete ∞-category $\mathcal{C}$, we can define its stabilization $Sp(\mathcal{C}_*)$ to be the the stable ∞-category of 'spectra in pointed objects of $\mathcal{C}$' [Lur11, p.109]. In the context of presentable stable ∞-categories, the stabilization enjoys a beautiful universal property [Lur11, Corollary 1.4.5.5]. This specializes to show that the ∞-category Sp is the 'free stable presentable ∞-category on one generator', namely, the sphere spectrum [Lur11, Corollary 1.4.5.6, pp.112-114].

References: [Lur11, Chapter 1], [Gro10, Chapter 5].

5. Fifth Session (15.12.2011): Higher Homological Algebra

Talk 9: t-structures. This talk should begin by recalling the notion of a t-structure on a triangulated category (in the homological notion) and give the 'smart truncation functors' as an example. A t-structure on a stable ∞-category is just a t-structure on the underlying homotopy category and gives rise to a family of localization and colocalization functors. The heart of a t-structure is the nerve of an abelian category (and hence makes the methods of homological algebra available). We discuss the notion of (left/right) bounded and left complete t-structures and mention the relation between these concepts [Lur11, pp.30-38]. As an example, the 'canonical' t-structure on the ∞-category Sp of spectra should be mentioned [Lur11, pp.105-106].

References: [Lur11, Chapter 1]. For the classical theory of t-structures: [Mil, Chapter 4].

Talk 10: The Dold-Kan correspondence. (Uwe Kranz) This talk should begin with the classical Dold-Kan correspondence which asserts an equivalence $(DK, N)\colon Ch_{\geq 0}\mathcal{A} \rightharpoonup s\mathcal{A}$ between the category of non-negatively graded chain complexes and simplicial objects in an abelian category $\mathcal{A}$. A proof of this equivalence will be sketched in the more general case of an idempotently complete additive category [Lur11, Thm.1.2.3.7]. Along the way, there are a few interesting observations to mention (for example Remarks 1.2.3.15 and 1.2.3.16, [Lur11, pp.44-50]).

The normalized chain complex functor is 'suitably lax comonoidal with respect to multi-additive functors' (generalization of Alexander-Whitney maps) which implies (by abstract nonsense) that the functor DK is 'suitably monoidal'. For our purposes, we can restrict attention to the additive bifunctor given by the tensor product [Lur11, pp.50-53].

Then the ∞-categorical version of the Dold-Kan correspondence should be discussed [Lur11, Thm.1.2.4.1].

References: [Lur11, Chapter 1].

6. Sixth Session (12.01.2012): Derived ∞-categories

Talk 11: Examples of stable ∞-categories induced by additive categories. This talk gives a new family of examples of stable ∞-categories. Using the (co)monoidal structure of the Dold-Kan correspondence, we introduce the stable ∞-category of unbounded chain complexes in an

arbitrary *additive* category [Lur11, Corollary 1.3.1.12]. Examples of nice subcategories of additive categories (i.e., satisfying certain closure properties) also give examples of stable ∞-categories. As a special case, we obtain the derived ∞-category $\mathcal{D}^-(\mathcal{A})$ of an abelian category $\mathcal{A}$ with enough projectives (it is given by the right-bounded levelwise projective complexes). This stable ∞-category has a 'canonical' t-structure ([Lur11, Proposition 1.3.1.23]) which has particularly nice properties [Lur11, Proposition 1.3.2.15] ,[Lur11, pp.62-70 and p.77]. If time permits, we also mention the examples given by unbounded chain complexes in Grothendieck abelian categories. In this context, there is always a model structure, which, however, is not simplicial in general [Lur11, pp.85-91].

References: [Lur11, Chapter 1].

Talk 12: Universal property of derived ∞-category. In this talk, it will be shown that the derived ∞-category $\mathcal{D}^-(\mathcal{A})$ of an abelian category $\mathcal{A}$ with enough projective objects, together with its 'canonical' t-structure, enjoys a universal property [Lur11, Thm.1.3.2.2]. This implies immediately that we have a theory of (left) derived functors and also generalized Eilenberg-MacLane spectrum functors. As a further consequence of this universal property, we have a construction of 'realization functors': under certain assumptions on a t-structure of a stable ∞-category $\mathcal{C}$ and its heart $N\mathcal{A}$, there is an essentially unique t-exact functor $\mathcal{D}^-(\mathcal{A}) \to \mathcal{C}$ extending the inclusion of the heart. In the case of $\mathcal{C} = Sp$, for example, the existence of integral cohomology operations of positive degree implies that this functor is *not* fully-faithful, i.e., that 'topology sees more than algebra'. The talk should include a proof of the universal property modulo [Lur11, Theorem 1.3.2.7, pp.70-76]. If time permits, a (sketch?) proof of [Lur11, Theorem 1.3.2.7] could also be attempted ([Lur11, pp.76-77] and [Lur09, Section 5.5.9]).

References: [Lur11, Chapter 1], [Lur09, Section 5.5.9].

7. SEVENTH SESSION (26.01.2012): A UNIVERSAL CHARACTERIZATION OF (CONNECTIVE) ALGEBRAIC K-THEORY

Talk 13: The universal additive invariant. (Justin Noël) This talk should begin with the definition of an additive invariant on the ∞-category Cat_∞^{ex} of small stable ∞-categories. The proof of the theorem about the existence of a universal additive invariant and the construction of the ∞-category of 'non-commutative motives' ([BGT10, Theorem 5.9, pp. 30-32]) should be sketched. (Drawing an analogy with the universal property of the Grothendieck group could be helpful here.) Then the K-theory of ∞-categories and the comparison with Waldhausen K-theory will be discussed [BGT10, pp.33-36]. The fact that K-theory is an example of an additive invariant [BGT10, Proposition 6.7] should be mentioned and, if time permits, it could also be shown, up to the technical facts about replacing ∞-categories by spectral categories ([BGT10, Sections 3 and 4]) that allow a reduction to a more familiar setting.

References: [BGT10, Sections 1,2 and 5].

Talk 14: Universal property of (connective) algebraic K-theory. (Jeremiah Heller) In this talk, the co-representability theorem ([BGT10, Theorem 6.9]) will be stated and the proof should be sketched [BGT10, pp. 37-39]. Then the universal property [BGT10, Theorem 10.2] will be obtained as an immediate application of the co-representability of K-theory and the ∞-categorical Yoneda lemma.

References: [BGT10, Sections 1, 6 and 10].

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