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The consistency strength of $\aleph_\omega$ and $\aleph_{\omega_1}$ being Rowbottom cardinals without the Axiom of Choice

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Abstract We show that for all natural numbers n, the theory "ZF + $\mathrm{DC}_{\aleph_n}$ + $\aleph_\omega$ is a Rowbottom cardinal carrying a Rowbottom filter" has the same consistency strength as the theory "ZFC + There exists a measurable cardinal". In addition, we show that the theory "ZF + $\aleph_{\omega_1}$ is an ω_2-Rowbottom cardinal carrying an ω_2-Rowbottom filter and ω_1 is regular" has the same consistency strength as the theory "ZFC + There exist ω_1 measurable cardinals". We also discuss some generalizations of these results.

Keywords Jonsson cardinal · Rowbottom cardinal · Rowbottom filter · Prikry forcing · Coherent sequence of Ramsey cardinals · Core model

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1 Introduction and Preliminaries

One of the longest standing open questions in large cardinals and forcing, dating back to Silver's 1965 thesis (see the published version [12]), is whether the theory "ZFC + $\aleph_\omega$ is a Rowbottom cardinal" is relatively consistent. In spite of numerous attempts to obtain a solution to this vexing and intriguing problem, by Shelah, Foreman, and others, no solution in either a positive or negative vein is in sight.

If we are willing to drop the Axiom of Choice from our assumptions, i.e., if we are willing to settle for the relative consistency of the theory "ZF + $\neg$AC + $\aleph_\omega$ is a Rowbottom cardinal", then the situation is quite different. Everett Bull (unpublished, see [4]) showed that, relative to "ZFC + There exists a measurable cardinal", the theory "ZF + $\neg\mathrm{AC}_\omega$ + GCH holds below $\aleph_\omega$ + $\aleph_\omega$ is a Rowbottom cardinal carrying a Rowbottom filter" is consistent.[1] The first author improved the amount of choice in Bull's model [2] to show that, relative to "ZFC+ There is an ω sequence of measurable cardinals", for an arbitrary $n < \omega$, the theory "ZF + $\mathrm{DC}_{\aleph_n}$ + $\aleph_\omega$ is a Rowbottom cardinal carrying a Rowbottom filter" is consistent.

The purpose of this paper is to obtain equiconsistency results concerning the theory "ZF + $\neg$AC + $\aleph_\omega$ is a Rowbottom cardinal carrying a Rowbottom filter" and some generalizations thereof. Specifically, we prove the following two theorems.

Theorem 1 *The theories "ZFC+ There exists a measurable cardinal" and "ZF+ $\neg AC + DC_{\aleph_n} + \aleph_\omega$ is a Rowbottom cardinal carrying a Rowbottom filter" are equiconsistent for every $n < \omega$.*

Theorem 2 *The theories "ZFC+ There exist ω_1 measurable cardinals" and "ZF + $\neg AC + \omega_1$ is regular + $\aleph_{\omega_1}$ is an ω_2-Rowbottom cardinal carrying an ω_2-Rowbottom filter" are equiconsistent.*

In showing the forward direction of Theorem 2, we will indicate how to construct different models in which various weak forms of the Axiom of Choice are true. Also, Theorems 1 and 2 above represent our main focus. We will in addition discuss at various junctures throughout the course of the paper generalizations of the theorems mentioned above, along with proving some other related results.

We work using forcing and core model theory. We will force to construct the relevant choiceless inner models in which $\aleph_\omega$, $\aleph_{\omega_1}$, etc. satisfy the desired properties. The construction of these choiceless inner models will be based in large part on the techniques set forth in [2]. As such, we will be assuming some familiarity with the methods of this paper, to which we will refer when appropriate.

[1] By GCH holding below $\aleph_\omega$, we literally mean, as in the situation when the Axiom of Choice is true, that for every $n < \omega$, there is a bijection between the power set of $\aleph_n$ and $\aleph_{n+1}$.

An overview of the proof of Theorem 1 is as follows. For the forward direction, if a measurable cardinal κ is made singular of cofinality ω by Prikry forcing, an end segment of the Prikry sequence $\langle \lambda_0, \lambda_1, \ldots \rangle$, which we denote by $\langle \kappa_0, \kappa_1, \ldots \rangle$, is a *coherent sequence of Ramsey cardinals* as defined in [9]. (Note that the definition of a coherent sequence of Ramsey cardinals can be found in the statement of Theorem 3.) The supremum κ of such a sequence is a Rowbottom cardinal carrying a Rowbottom filter. It is then turned into $\aleph_\omega$ by a product of Lévy collapses which collapses the Ramsey cardinals $\kappa_0, \kappa_1, \ldots$ to $\aleph_i, \aleph_{i+2}, \ldots$ for $i < \omega$, $i > 0$ a fixed but arbitrary natural number. We then as in [2] define a symmetric submodel of the generic extension in which ZF holds and in which $\kappa = \aleph_\omega$ is still a Rowbottom cardinal carrying a Rowbottom filter. For the converse, we use the Dodd–Jensen core model K as presented in the original articles [6] and [7] and in the monograph [5] to get an inner model with a measurable cardinal from $\aleph_\omega$ being Jonsson.

The proof of Theorem 2 is handled slightly differently, since when ω_1 is regular, $\aleph_{\omega_1}$ has uncountable cofinality. For the forward direction, if $\langle \kappa_i \mid i < \omega_1 \rangle$ is a sequence of ω_1 measurable cardinals with supremum κ, then κ is turned into $\aleph_{\omega_1}$ as before by a product of Lévy collapses which collapses κ_0, the first measurable cardinal in the sequence, to some fixed but arbitrary $\aleph_i$ for $i < \omega_1$, $i > 0$. The remaining measurable cardinals are collapsed in a manner to be described later, depending on how much of the Axiom of Choice we wish to be true in our final symmetric submodel of the generic extension in which ω_1 is regular and $\aleph_{\omega_1}$ is ω_2-Rowbottom and carries an ω_2-Rowbottom filter. The proofs necessary to establish the converse are based on short core models as presented in [8]. Short core models are constructed from short sequences $\bar{\mathcal{U}} = \langle \mathcal{U}_\kappa \mid \kappa \in \mathrm{dom}(\bar{\mathcal{U}}) \rangle$ of normal measures $\mathcal{U}_\kappa$ with measurable cardinal κ. Note that we say the sequence $\bar{\mathcal{U}}$ is *short* if its order type satisfies $\mathrm{otp}(\mathrm{dom}(\bar{\mathcal{U}})) < \min(\mathrm{dom}(\bar{\mathcal{U}}))$.

The structure of this paper is as follows. Section 1 contains our introductory comments and preliminary remarks concerning notation and terminology. Section 2 contains a discussion of coherent sequences of Ramsey cardinals that will be critical to the proof of the forward direction of Theorem 1. Section 3 contains our proof of Theorem 1. Section 4 contains our proof of Theorem 2, as well as a brief discussion of a generalization of this theorem. Section 5 contains a further generalization of our work, along with our final comments.

We conclude Sect. 1 with a few brief words concerning the conventions we will be following. Basically, our notation and terminology are standard. Exceptions to this will be duly noted. We do wish, however, to state explicitly that for $\alpha < \beta$ ordinals, $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$, and (α, β) are as in standard interval notation. For $\kappa < \lambda$ cardinals with κ regular, $\mathrm{Coll}(\kappa, {<}\lambda)$ is the standard Lévy partial ordering for collapsing every $\delta \in (\kappa, \lambda)$ to κ. For such a δ and any $S \subseteq \mathrm{Coll}(\kappa, {<}\lambda)$, we define $S \upharpoonright \delta = \{p \in S \mid \mathrm{dom}(p) \subseteq \kappa \times \delta\}$.

We also wish to recall for the benefit of readers the definitions of what it means for a cardinal to be Jonsson or Rowbottom. The cardinal κ is said to be *Jonsson* if it satisfies the partition relation $\kappa \to [\kappa]^{<\omega}_\kappa$, i.e., given a partition $f : [\kappa]^{<\omega} \to \kappa$, there is a homogeneous set $X \subseteq \kappa$ such that $f''[X]^{<\omega} \neq \kappa$. The filter $\mathcal{F}$ is called a *Jonsson filter* if some homogeneous set X for f may always be chosen so that $X \in \mathcal{F}$. The definition of Jonsson cardinal is equivalent in ZF to saying that any structure in a countable language whose domain has cardinality κ has a proper elementary substructure of cardinality κ.

The cardinal κ is said to be *Rowbottom* if for every cardinal $\lambda < \kappa$, it satisfies the partition relation $\kappa \to [\kappa]^{<\omega}_{\lambda,\omega}$, i.e., given a partition $f : [\kappa]^{<\omega} \to \lambda$, there is a homogeneous set $X \subseteq \kappa$ such that $|f''[X]^{<\omega}| \leq \omega$. The filter $\mathcal{F}$ is called a *Rowbottom filter* if some homogeneous set X for f may always be chosen so that $X \in \mathcal{F}$. The definition of Rowbottom cardinal is equivalent in ZF to saying that for any structure $\langle A, R, \ldots\rangle$ in a countable language such that $|A| = \kappa$ and R is a unary relation having cardinality $\lambda < \kappa$, there is a (proper) elementary substructure $\langle A', R', \ldots\rangle$ such that $|A'| = \kappa$ and $|R'| \leq \omega$.

The notion of what it means for a cardinal κ to be δ-*Rowbottom* for some uncountable cardinal $\delta < \kappa$ is a generalization of the definition of Rowbottom cardinal given in the preceding paragraph. This will hold if for every λ with $\delta \leq \lambda < \kappa$, κ satisfies the partition relation $\kappa \to [\kappa]^{<\omega}_{\lambda,<\delta}$, i.e., given a partition $f : [\kappa]^{<\omega} \to \lambda$, there is a homogeneous set $X \subseteq \kappa$ such that $|f''[X]^{<\omega}| < \delta$. The definition of δ-Rowbottom cardinal is equivalent in ZF to saying that for any structure $\langle A, R, \ldots\rangle$ in a countable language such that $|A| = \kappa$ and R is a unary relation having cardinality $\lambda < \kappa$, there is a (proper) elementary substructure $\langle A', R', \ldots\rangle$ such that $|A'| = \kappa$ and $|R'| < \delta$. By this definition, a Rowbottom cardinal is ω_1-Rowbottom. Also, the notion of δ-*Rowbottom filter* is defined as in the preceding two paragraphs. Further, note that for an uncountable cardinal $\delta < \kappa$, we have the chain of implications κ is Rowbottom $\Longrightarrow$ κ is δ-Rowbottom $\Longrightarrow$ κ is Jonsson.

To prove lower bounds on consistency strength, we employ the theory of short core models [8], which extends the theory of the Dodd–Jensen core model [5–7]. The theory of short core models is developed under the assumption that a certain object 0^{long}, which transcends short core models in the same way $0^\sharp$ transcends the constructible universe, does *not* exist. 0^{long} will be described further before the proof of Theorem 9. Under the assumption 0^{long} does not exist, which is denoted by $\neg 0^{\text{long}}$, the core model has the form $K = L[E]$, where E is some canonical sequence of total and partial measures. One also writes $K = K[\bar{\mathcal{U}}_{\text{can}}]$, where $\bar{\mathcal{U}}_{\text{can}}$ is the sequence consisting of the total measures in E. For each ordinal α, we define the α^{th} level of the core model by $K_\alpha = K_\alpha[\bar{\mathcal{U}}_{\text{can}}] = L_\alpha[E]$.

Core model theory is usually developed assuming the Axiom of Choice. For our study of choiceless combinatorics, we employ some workarounds which are based on building core models within the inner model HOD of *hereditarily ordinal definable sets* or some variants. Such methods were used, e.g., by Schindler in [11].

If $a \subseteq$ HOD is a set, let HOD$[a]$ be the smallest inner model such that HOD $\cup \{a\} \subseteq$ HOD$[a]$. We then have the following (see also Lemmas 3 and 4 of [11]).

Proposition 1.1 *(ZF) Let $a \subseteq$ HOD be a set. Then*

1. HOD$[a]$ *is a set-generic extension of* HOD, *so* HOD$[a] \models$ ZFC.

2. *If* $\neg 0^{\text{long}}$*, and if* K *is (the canonical term for) the core model, then* $K^{\text{HOD}[a]} = K^{\text{HOD}}$*. This equality holds for every level of the K-hierarchy, i.e.,* $K_\alpha^{\text{HOD}[a]} = K_\alpha^{\text{HOD}}$ *for every* $\alpha \in \text{Ord}$.

Proof Clause (1) follows from Vopěnka's genericity theorem (see page 142 of [11]). Clause (2) follows from the absoluteness of (small) core model constructions with respect to set-generic extensions. □

2 Coherent sequences of Ramsey cardinals

Assume κ is a measurable cardinal with normal measure $\mathcal{U}$. Let

$$\mathbb{P} = \{\langle a, X\rangle \mid a \in [\kappa]^{<\omega}, X \in \mathcal{U}, \max(a) < \min(X)\}$$

be the set of Prikry conditions for κ and $\mathcal{U}$ with the usual order. Let G be $\mathbb{P}$-generic over V, with $\langle \lambda_i \mid i < \omega\rangle$ the Prikry sequence induced by G. In [9], the following was proved as Theorem 3.2.

Theorem 3 *In* $V[G]$*, there is an ascending sequence* $\langle \kappa_i \mid i < \omega\rangle$ *of regular cardinals cofinal in* κ *which forms a coherent sequence of Ramsey cardinals, i.e., for every regressive* $f : [\kappa]^{<\omega} \to \kappa$*, there is* $\langle A_i \mid i < \omega\rangle$ *such that:*

1. $A_i \subseteq \kappa_i \setminus \kappa_{i-1}$ *is cofinal in* κ_i*, where for convenience, we set* $\kappa_{-1} = 0$.
2. *If* $x, y \in [\kappa]^{<\omega}$*,* $x, y \subseteq \bigcup\{A_i : i < \omega\}$*, and* $|x \cap A_i| = |y \cap A_i|$ *for* $i < \omega$*, then* $f(x) = f(y)$.

Note that $f : [\kappa]^{<\omega} \to \kappa$ is *regressive* if $f(x) = 0$ if 0 is the minimal member of x, or $f(x) < \min(x)$ for every $x \in [\kappa]^{<\omega}$ for which 0 is not the minimal member of x. Also, Lemma 3.1 of [9] tells us that $\langle \kappa_i \mid i < \omega\rangle$ is actually of the form $\langle \lambda_i \mid j \le i < \omega\rangle$, i.e., the coherent sequence of Ramsey cardinals is an end segment of the Prikry sequence induced by G. Lemma 3.1 of [9] further tells us that $\langle A_i \mid i < \omega\rangle$ can be taken so that the A_i's are mutually disjoint and $A = \bigcup_{i<\omega} A_i \cup \{\kappa_i \mid i < \omega\} \in \mathcal{U}$.

We shall use a slightly different technical characterization of coherent Ramseyness.

Proposition 2.1 $\langle \kappa_i \mid i < \omega\rangle$ *is a coherent sequence of Ramsey cardinals with supremum* κ *if for all regressive* $f : [\kappa]^{<\omega} \to \kappa$*, there is* $\langle A_i \mid i < \omega\rangle$ *such that:*

1. $A_i \subseteq \kappa_i \setminus \kappa_{i-1}$ *is cofinal in* κ_i*, where for convenience, we set* $\kappa_{-1} = 0$.
2. *For every* $m < \omega$*, if* $x, y \in [\kappa]^{<\omega}$*,* $x, y \subseteq \bigcup_{i<m} A_i$*, and* $\forall i < m[|x \cap A_i| = |y \cap A_i| = m]$*, then* $f(x) = f(y)$.

Proof Property (2) of Proposition 2.1 is a special case of property (2) in Theorem 3 in which only arguments with m elements in the first m sets A_i are considered.

Conversely it implies the original definition. To see this, let $f : [\kappa]^{<\omega} \to \kappa$ be a regressive function. Given $x \in [\kappa]^{<\omega}$, define x's *type* to be the countable sequence of integers whose i^{th} member is given by $|x \cap [\kappa_{i-1}, \kappa_i]|$. Since for any $x \in [\kappa]^{<\omega}$, there is some $m < \omega$ such that $\forall i \geq m[x \cap [\kappa_{i-1}, \kappa_i] = \emptyset]$, we may infer that there are only countably many types. For any $x \in [\kappa]^{<\omega}$ of some fixed type, let $m < \omega$ and $y \subseteq \bigcup_{i<m}[\kappa_{i-1}, \kappa_i]$ be such that $\forall i < m[|y \cap A_i| = m]$, $\forall i < m[x \cap [\kappa_{i-1}, \kappa_i]$ is an initial segment of $y \cap [\kappa_{i-1}, \kappa_i]]$, and $\forall i \geq m[x \cap [\kappa_{i-1}, \kappa_i] = \emptyset]$. Note that the finiteness of x ensures that m and y as just stipulated exist. In addition, the fact there are only a countable number of different types implies that a unique m may be chosen for each distinct type. One can find $g : [\kappa]^{<\omega} \to \kappa$ regressively such that for each $x \in [\kappa]^{<\omega}$ of some fixed type, there are m and y as just described with $g(y) = f(x)$. A homogeneous sequence for g in the sense of property (2) of Proposition 2.1 will also be fully homogeneous for f. □

Proposition 2.2 *If $\langle \kappa_i \mid i < \omega \rangle$ is a coherent sequence of Ramsey cardinals with supremum κ, then κ is a Rowbottom cardinal.*

Proof Suppose $\lambda < \kappa$ and $f : [\kappa]^{<\omega} \to \lambda$. Without loss of generality, replace κ with $B = \kappa \setminus \lambda$. $f : [B]^{<\omega} \to \lambda$ is regressive, so let $\langle A_i \mid i < \omega \rangle$ be homogeneous for f in the sense of Theorem 3. Define $A = \bigcup_{i<\omega} A_i$. Since by homogeneity, $f''[A]^{<\omega}$ depends only upon the number of distinct sequences $\langle |x \cap A_i| \mid i < \omega \rangle$ for $x \in [\kappa]^{<\omega}$, and since there are only countably many such sequences, $|f''[A]^{<\omega}| \leq \omega$. Thus, A is homogeneous for f in the sense of Rowbottomness. □

The following preservation result for coherent sequences of Ramsey cardinals will be essential for the construction to be given in Sect. 3.

Theorem 4 *Let $\langle \kappa_i \mid i < \omega \rangle$ be a coherent sequence of Ramsey cardinals with supremum κ. Let $\langle \delta_i \mid i < \omega \rangle$ be a sequence of inaccessible cardinals such that $\forall i < \omega[\delta_i \in (\kappa^+_{i-1}, \kappa_i)]$, where $\kappa_{-1} = \omega_\ell$ for some $\ell < \omega$. Let $\mathbb{P} = \{\langle p_i \mid i < \omega \rangle \mid p_i \in \mathrm{Coll}(\kappa^+_{i-1}, <\delta_i)$ for $i < \omega\}$, ordered componentwise. Let G be $\mathbb{P}$-generic over V. Then in $V[G]$, $\langle \kappa_i \mid i < \omega \rangle$ is a coherent sequence of Ramsey cardinals.*

Proof Let $p = \langle p_i \mid i < \omega \rangle \in \mathbb{P}$ and $p \Vdash$ "$\dot{g} : [\kappa]^{<\omega} \to \kappa$ is regressive". It suffices to show that some extension of p forces the existence of a homogeneous sequence for $\dot{g}$ in the sense of the characterization of coherent Ramseyness given in Proposition 2.1.

For $m < \omega$, let $R_m = \{r \in [\kappa]^{m \cdot m} \mid \forall i < m[|r \cap (\kappa_i \setminus \delta_i)| = m]\}$. $R = \bigcup_{m<\omega} R_m$ is then the set of all arguments relevant for the characterization of coherent Ramseyness given in Proposition 2.1. Well-order R by $r <' s$ iff either (a) $|r| < |s|$, or (b) $|r| = |s|$ and $\exists \beta[r \setminus \beta = s \setminus \beta$ and $\beta \notin r$ and $\beta \in s]$. Part (b) corresponds to the usual well-ordering of $[\mathrm{Ord}]^{<\omega}$ by largest difference. $\langle R, <' \rangle$ has order type κ. We construct by recursion on $<'$ a sequence $\langle p(r) \mid r \in R \rangle$, $p(r) = \langle p_i(r) \mid i < \omega \rangle \in \mathbb{P}$, and a sequence $\langle \omega(r) \mid r \in R \rangle$ such that the following "growth condition" holds:

(1) If $s <' r$ and $\forall j \geq i[s \cap [\delta_j, \kappa_j]$ is an initial segment of $r \cap [\delta_j, \kappa_j]]$, then $p_i \subseteq p_i(s) \subseteq p_i(r)$.

The growth condition will be essential for the compatibility requirements of the subsequent construction.

Assume that $r \in R$ and that for $s <' r$, the condition $p(s)$ is constructed so that (1) holds. Define $\bar{p}_i(r) = p_i \cup \bigcup\{p_i(s) \mid s <' r$ and for $j \geq i$, $s \cap [\delta_j, \kappa_j]$ is an initial segment of $r \cap [\delta_j, \kappa_j]\}$. $\bar{p}_i(r)$ is a condition, since $\mathrm{Coll}\left(\kappa_{i-1}^+, <\delta_i\right)$ is closed under unions of $\subseteq$-increasing sequences of length κ_{i-1}, and we are taking a union of a $\subseteq$-increasing chain of conditions having size at most κ_{i-1}. Choose $\omega(r) \in \kappa$ and $p(r) = \langle p_i(r) \mid i < \omega\rangle \leq \bar{p}(r)$ such that $p(r) \Vdash$ "$\dot{g}(r) = \omega(r)$". The definition of $p(r)$ is consistent with property (1), and so the recursion works.

For $i < \omega$ and $t \subseteq \bigcup_{i \leq j < \omega}[\delta_j, \kappa_j]$, we take the union of all the $p_i(r)$ where $t = r \setminus \delta_i$ and define $p'_i(t) = \bigcup\{p_i(r) \mid r \in R$ and $t = r \setminus \delta_i\}$. $p'_i(t) \in \mathrm{Coll}(\kappa_{i-1}^+, <\delta_i)$, since $\mathrm{Coll}(\kappa_{i-1}^+, <\delta_i)$ is κ_{i-1}-closed, and we are taking a union of at most κ_{i-1} many compatible forcing conditions. Note that p'_i can be viewed as a regressive function.

Since $\langle \kappa_i \mid i < \omega\rangle$ is a coherent sequence of Ramsey cardinals, by coding ω partitions into one in (3) below and then applying Theorem 3 twice, there is $\langle A_i \mid i < \omega\rangle$, each A_i cofinal in κ_i, such that for $A = \bigcup_{i<\omega} A_i$, the following homogeneity properties hold:

(2) If $r, s \in R$, $r, s \subseteq A$, $\forall j < \omega[|r \cap A_j| = |s \cap A_j|]$, then $\omega(r) = \omega(s)$.
(3) For $i < \omega$, if $r, s \in R$, $r, s \subseteq A$, $r, s \subseteq \bigcup_{i \leq j < \omega}[\delta_j, \kappa_j]$, $\forall j < \omega[|r \cap A_j| = |s \cap A_j|]$, then $p'_i(r) = p'_i(r \setminus \delta_i) = p'_i(s) = p'_i(s \setminus \delta_i)$.

Now, for $i < \omega$, define $\bar{p}_i = \bigcup\{p'_i(r \setminus \delta_i) \mid r \in R,\ r \subseteq A\}$. By (3), this is just a countable union. Consequently, $\bar{p}_i \in \mathrm{Coll}(\kappa_{i-1}^+, <\delta_i)$. Set $\bar{p} = \langle \bar{p}_i \mid i < \omega\rangle$. It is then the case that $\bar{p} \in \mathbb{P}$ and $\bar{p} \leq p$.

We show that $\bar{p}$ forces that $\langle A_i \mid i < \omega\rangle$ is homogeneous for $\dot{g}$ in the sense of the characterization of coherent Ramseyness given in Proposition 2.1. To do this, let $r \in R$, $r \subseteq \bigcup_{i<m} A_i$, where $\forall i < m[|r \cap A_i| = m]$. It is then the case that for every $i < \omega$, $p_i(r) \subseteq p'_i(r \setminus \delta_i) \subseteq \bar{p}_i$. Hence, $\bar{p} = \langle \bar{p}_i \mid i < \omega\rangle \leq \langle p_i(r) \mid i < \omega\rangle = p(r)$, and $\bar{p} \Vdash$ "$\dot{g}(r) = \omega(r)$". If $s \in \bigcup_{i<m} A_i$, where $\forall i < m[|s \cap A_i| = m]$, then the same calculation yields $\bar{p} \Vdash$ "$\dot{g}(s) = \omega(s)$". Thus, $\bar{p} \Vdash$ "$\dot{g}(r) = \dot{g}(s)$". □

Take $\lambda < \kappa$ and $g : [\kappa]^{<\omega} \to \lambda$. Consider the first-order structure $\mathfrak{B} = \langle \kappa, R, g, \kappa_0, \kappa_1, \ldots\rangle$, where we have distinguished as constants each member of a coherent sequence of Ramsey cardinals $\langle \kappa_i \mid i < \omega\rangle$ generated via a Prikry sequence with respect to the normal measure $\mathcal{U}$ over κ, and R is the unary relation composed of $g''[\kappa]^{<\omega}$. By Proposition 2.2, let $\mathfrak{A} \prec \mathfrak{B}$ be a Rowbottom elementary substructure. By the proof of Proposition 2.2 and the remarks made in the paragraph immediately following the statement of Theorem 3, we may take $A' = \mathrm{dom}(\mathfrak{A})$ to be such that $A = A' \cup \{\kappa_i \mid i < \omega\} \in \mathcal{U}$. Since each member of the set $\{\kappa_i \mid i < \omega\}$ was a distinguished constant in $\mathfrak{B}$, it is the case that $|g''[A]^{<\omega}| \leq \omega$, i.e., $A \in \mathcal{U}$ is Rowbottom homogeneous for g. The proof of Theorem 4 therefore yields that if we first do Prikry forcing and follow this by the forcing indicated by Theorem 4, then not only is κ a Rowbottom cardinal, but $\mathcal{U}$ generates a Rowbottom filter over κ. This will be critical in the proof of Theorem 5 to be given in the next section.

3 The proof of Theorem 1

In this section, we will prove Theorem 1. We break the proof up into both its forward and reverse directions, which we will establish separately. We begin with the forward direction, which we state as a separate theorem.

Theorem 5 *Let $V_0 \vDash$ "ZFC + κ is a measurable cardinal". Let $n < \omega$ be fixed but arbitrary. There is then a generic extension V of V_0, a notion of forcing $\mathbb{P}$, and a symmetric inner model $N \subseteq V^{\mathbb{P}}$ such that $N \vDash$ "ZF + $DC_{\aleph_n}$ + $\aleph_\omega$ is a Rowbottom cardinal carrying a Rowbottom filter".*

We note that Theorem 5 represents a significant reduction in consistency strength of the hypotheses used for the main result (Theorem 1) of [2]. As was mentioned in Sect. 1, in that paper, a model witnessing the same conclusions as in Theorem 5 was constructed, but assuming the consistency of the theory "ZFC + There is an ω sequence of measurable cardinals".

We turn now to the proof of Theorem 5.

Proof Let $V_0 \vDash$ "ZFC + κ is a measurable cardinal". We assume that V_0 has been extended generically via Prikry forcing using a normal measure $\mathcal{U}$ over κ to a model, which we denote by V, containing a Prikry sequence $\langle \kappa_i \mid i < \omega \rangle$ through κ.

Take V as our ground model. Let $n < \omega$ be fixed but arbitrary. Let $\mathbb{P}_0 = \text{Coll}(\aleph_{n+1}, {<}\kappa_0)$, and for $1 \le i < \omega$, let $\mathbb{P}_i = \text{Coll}(\kappa_{i-1}^+, {<}\kappa_i)$. We then define $\mathbb{P} = \prod_{i<\omega} \mathbb{P}_i$ with full support.

Let G be $\mathbb{P}$-generic over V. $V[G]$, being a model of AC, is not our desired model N. In order to define N, we first note that by the Product Lemma, G_i, the projection of G onto $\mathbb{P}_i$, is V-generic over $\mathbb{P}_i$. Next, working in V, let $\mathcal{F} = (\aleph_{n+1}, \kappa_0) \times (\kappa_0^+, \kappa_1) \times (\kappa_1^+, \kappa_2) \times \cdots$. For each $f \in \mathcal{F}, f = \langle f(0), f(1), \ldots \rangle$, define $G \restriction f = G_0 \restriction f(0) \times G_1 \restriction f(1) \times \cdots$. By the Product Lemma and the properties of the Lévy collapse, $G \restriction f$ is $\prod_{i<\omega}(\mathbb{P}_i \restriction f(i))$-generic over V. N can now intuitively be described as the least model of ZF extending V which contains, for each $f \in \mathcal{F}$, the set $G \restriction f$.

In order to define N more formally, we let $\mathcal{L}_1$ be the ramified sublanguage of the forcing language $\mathcal{L}$ with respect to $\mathbb{P}$ which contains symbols $\check{v}$ for each $v \in V$, a unary predicate symbol $\check{V}$ (to be interpreted $\check{V}(\check{v}) \leftrightarrow v \in V$), and symbols $\dot{G} \restriction f$ for each $f \in \mathcal{F}$. N is then defined as follows.

$$N_0 = \emptyset.$$

$$N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha \text{ if } \alpha \text{ is a limit ordinal.}$$

$$N_{\alpha+1} = \left\{ x \subseteq N_\alpha \;\middle|\; \begin{array}{l} x \text{ is definable over the model } \langle N_\alpha, \in, c\rangle_{c\in N_\alpha} \\ \text{via a term } \tau \in \mathcal{L}_1 \text{ of rank } \le \alpha \end{array} \right\}.$$

$$N = \bigcup_{\alpha \in \text{Ord}^V} N_\alpha.$$

The standard arguments show $N \models$ ZF. Further, by Lemmas 1.1, 1.2, and 1.4 of [2], which remain valid in the context of this paper, $N \models$ "$\mathrm{DC}_{\aleph_n} + \kappa = \aleph_\omega$". The proof of Theorem 5 will thus be complete once we have shown that $N \models$ "$\aleph_\omega$ is a Rowbottom cardinal carrying a Rowbottom filter".

To do this, fix $\lambda < \kappa$, and suppose $g \in N$ is such that $g : [\kappa]^{<\omega} \to \lambda$. By Lemma 1.1 of [2], there is $f \in \mathcal{F}$ such that $g \in V[G \upharpoonright f]$. Consequently, by Theorem 4 and the remarks immediately following its proof, there is a set $A \in \mathcal{U} \in V \subseteq N$ which is Rowbottom homogeneous for g. Thus, $N \models$ "κ is a Rowbottom cardinal and $\mathcal{U}$ generates the Rowbottom filter $\mathcal{U}^* = \{X \subseteq \kappa \mid \exists Y \in \mathcal{U}[X \supseteq Y]\}$ for κ". □

We continue now with the reverse direction of Theorem 1. We state this as a separate, stronger theorem which implies our desired result.

Theorem 6 *Let κ be a singular Jonsson cardinal in a model V of ZF. Then κ is measurable in some inner model of ZFC.*

Proof Since HOD is a model of the Axiom of Choice, take $H \in$ HOD to be a sufficiently elementary submodel of HOD such that $\kappa + 1 \subseteq H$. Let X be a Jonsson substructure of H, i.e., let $X \prec H$ be such that $|X \cap \kappa| = \kappa$ and $X \cap \kappa \neq \kappa$.

The remainder of the argument will be carried out in the inner model $M =$ HOD$[X, a]$, where $a \subseteq \kappa$ is cofinal in κ with order type less than κ. Note that Proposition 1.1 applies to M.

Let $\pi : \langle \bar{H}, \in \rangle \cong \langle X, \in \rangle \prec \langle H, \in \rangle$ be the Mostowski collapse of X, where $\bar{H}$ is transitive. We then have that $\pi \upharpoonright \kappa \neq \mathrm{id} \upharpoonright \kappa$ and $\pi(\kappa) = \kappa$.

Let $\bar{K} = K^{\bar{H}}$, where K is the Dodd–Jensen core model. Consider the following two cases.

Case 1 $K_\kappa \subseteq \bar{K}$. Then $K_\kappa = K_\kappa^{\bar{H}}$, and the elementary map $\pi \upharpoonright K_\kappa : \langle K_\kappa, \in \rangle \to \langle K_\kappa, \in \rangle$ can be extended to an elementary map $\tilde{\pi} : \langle K, \in \rangle \to \langle K, \in \rangle$. The rigidity theorem for the Dodd–Jensen core model implies that there is an inner model with a measurable cardinal less than κ. Iterating that model, one obtains an inner model with measurable cardinal κ.

Case 2 $K_\kappa \nsubseteq \bar{K}$. Then we can take a mouse N with $|N| < \kappa$ and $N \notin \bar{K}$. N has exactly one (active) measure, by which it may be iterated. We therefore let $\langle \kappa_i \mid i \in \mathrm{Ord} \rangle$ be the sequence of iteration points in the iteration of N. $\{\kappa_i \mid i \in \mathrm{Ord}\} \cap \bar{K}$ is a set of order-indiscernibles for $\bar{K}$ which is closed, unbounded in every sufficiently large cardinal less than or equal to κ. Each element κ_i of that set is strongly inaccessible in $\bar{K}$. In particular, κ is a limit of the iteration points, and hence is regular in $\bar{K}$.

We claim now that κ is regular in K. To see this, first note that because κ is regular in $\bar{K} = K^{\bar{H}}$ and π is elementary, $\kappa = \pi(\kappa)$ is regular in K^H. Since H is a sufficiently elementary substructure of HOD formed in the original universe,

κ is consequently regular in the original K^{HOD}. By clause (2) of Proposition 1.1, K^{HOD} is the Dodd–Jensen core model K of M, the present universe of discourse.

By the facts that κ is singular in M but is regular in K, the covering property fails for K at κ. By the Dodd–Jensen Covering Theorem, there exists an inner model with a measurable cardinal less than or equal to κ. By eventually iterating that inner model, one obtains an inner model with a measurable cardinal exactly equal to κ. □

Theorem 6 clearly implies the reverse direction of Theorem 1. Thus, the proof of Theorem 1 is now complete. □

We remark that Bull's result of [4], together with Theorem 6, yield the following theorem.

Theorem 7 *The theories "ZFC+ There exists a measurable cardinal" and "ZF+ $\neg AC_\omega + \aleph_\omega$ is a Rowbottom cardinal carrying a Rowbottom filter" are equiconsistent.*

4 The proof of Theorem 2

In this section, we will prove Theorem 2. As in Sect. 3, we break the proof up into both its forward and reverse directions, which we will establish separately. We begin with the forward direction, which we once again state as a separate theorem.

Theorem 8 *Let $V \models$ "ZFC + $\langle \kappa_i \mid i < \omega_1 \rangle$ is a sequence of ω_1 measurable cardinals with supremum κ". There is then a notion of forcing $\mathbb{P}$ and a symmetric inner model $N \subseteq V^{\mathbb{P}}$ such that $N \models$ "ZF + $\neg AC$ + ω_1 is regular + $\aleph_{\omega_1}$ is an ω_2-Rowbottom cardinal carrying an ω_2-Rowbottom filter".*

Proof Suppose $V \models$ "ZFC + $\langle \kappa_i \mid i < \omega_1 \rangle$ is a sequence of ω_1 measurable cardinals". Without loss of generality, we assume in addition that $V \models$ GCH.

We will give two proofs of Theorem 8, one in which the desired model satisfies $\text{DC}_{\aleph_\ell}$ for a fixed but arbitrary $\ell < \omega_1$, and one in which the desired model satisfies only DC but also witnesses that GCH holds below $\aleph_{\omega_1}$. Our arguments are slight generalizations of those given in [2] and [4].

For the first of these models, we proceed in analogy to the proof of Theorem 5. Specifically, let $\ell < \omega_1$ be fixed but arbitrary. Take $\langle \lambda_i \mid i < \omega_1 \rangle$ as the sequence $\langle \kappa_i \mid i < \omega_1 \rangle$, together with its limit points. Let $I = \{i < \omega_1 \mid i$ is either a successor ordinal or $0\}$. For $i \in I$, let $\mathbb{P}_i = \text{Coll}(\lambda^+_{i-1}, {<}\lambda_i)$, where we take $\lambda^+_{-1} = \aleph_{\ell+1}$. We then define $\mathbb{P} = \prod_{i \in I} \mathbb{P}_i$ with full support, and take G as being $\mathbb{P}$-generic over V. By the definition of $\mathbb{P}$ and the properties of the Lévy collapse, $V[G] \models$ "$\omega_1 = \omega_1^V$".

$V[G]$, being a model of AC, is once again not our desired model N. In order to define N, we first note that by the Product Lemma, for $i \in I$, G_i, the projection of G onto $\mathbb{P}_i$, is V-generic over $\mathbb{P}_i$. Next, let $\mathcal{F} = \prod_{i\in I}(\lambda^+_{i-1}, \lambda_i)$. For each $f \in \mathcal{F}$, define $G \restriction f = \prod_{i\in I}(G_i \restriction f(i))$. Once again, by the Product Lemma and the properties of the Lévy collapse, $G \restriction f$ is $\prod_{i\in I}(\mathbb{P}_i \restriction f(i))$-generic over V. As before, N can now intuitively be described as the least model of ZF extending V which contains, for each $f \in \mathcal{F}$, the set $G \restriction f$.

In order to define N more formally, we let $\mathcal{L}_1$ be the ramified sublanguage of the forcing language $\mathcal{L}$ with respect to $\mathbb{P}$ which contains symbols $\check{v}$ for each $v \in V$, a unary predicate symbol $\check{V}$ (to be interpreted $\check{V}(\check{v}) \leftrightarrow v \in V$), and symbols $\dot{G} \restriction f$ for each $f \in \mathcal{F}$. N is then defined in the same way as in the proof of Theorem 5, i.e., as follows.

$$N_0 = \emptyset.$$
$$N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha \text{ if } \alpha \text{ is a limit ordinal.}$$
$$N_{\alpha+1} = \left\{ x \subseteq N_\alpha \;\middle|\; \begin{array}{l} x \text{ is definable over the model } \langle N_\alpha, \in, c\rangle_{c\in N_\alpha} \\ \text{via a term } \tau \in \mathcal{L}_1 \text{ of rank } \le \alpha \end{array} \right\}.$$
$$N = \bigcup_{\alpha\in \mathrm{Ord}^V} N_\alpha.$$

As earlier, $N \vDash$ ZF. Since $V \subseteq N \subseteq V[G]$ and $V[G] \vDash$ "$\omega_1 = \omega_1^V$", $N \vDash$ "$\omega_1 = \omega_1^V$". Further, by Lemmas 1.1, 1.2, and 1.4 of [2], which remain valid even when our sequence $\langle \lambda_i \mid i < \omega_1\rangle$ is uncountable, $N \vDash$ "$\mathrm{DC}_{\aleph_\ell} + \kappa = \aleph_{\omega_1}$". In addition, since $N \vDash$ DC, $N \vDash$ "ω_1 is regular". The first proof of Theorem 8 will thus be complete once we have shown that $N \vDash$ "$\aleph_{\omega_1}$ is an ω_2-Rowbottom cardinal carrying an ω_2-Rowbottom filter".

To do this, let $\langle \mu_i \mid i < \omega_1\rangle \in V$ be such that μ_i is a normal measure over κ_i. In N, define $\mathcal{F} = \{A \subseteq \kappa \mid \exists i < \omega_1 \forall j \in [i, \omega_1)[A \cap \kappa_j \in \mu_j]$. By Lemma 1.3 of [2], which again remains valid working with an uncountable sequence of cardinals $\langle \lambda_i \mid i < \omega_1\rangle$, $\mathcal{F}$ is in N an ω_2-Rowbottom filter over $\kappa = \aleph_{\omega_1}$. This completes our first proof of Theorem 8. □

For our second proof of Theorem 8, let $\langle \kappa_i \mid i < \omega_1\rangle$, $\langle \lambda_i \mid i < \omega_1\rangle$, λ_{-1}, I, $\mathbb{P}_i$, and $\mathbb{P}$ be as in the first proof of Theorem 8. (The exact value of $\ell < \omega_1$ will be irrelevant.) Let G be $\mathbb{P}$-generic over V, and for $i \in I$, let G_i be the projection of G onto $\mathbb{P}_i$. For $j \in I$, let $\mathbb{Q}_j = \prod_{i\le j, i\in I} \mathbb{P}_i$ and $H_j = \prod_{i\le j, i\in I} G_i$. It is again the case, by the properties of the Lévy collapse and the Product Lemma, that H_j is $\mathbb{Q}_j$-generic over V. The N for our second proof of Theorem 8 can now be intuitively described as the least model of ZF extending V which contains, for every $j \in I$, the set H_j.

In order to define N more formally, we let $\mathcal{L}_1$ be the ramified sublanguage of the forcing language $\mathcal{L}$ with respect to $\mathbb{P}$ which contains symbols $\check{v}$ for each $v \in V$, a unary predicate symbol $\check{V}$ (to be interpreted $\check{V}(\check{v}) \leftrightarrow v \in V$), and symbols $\dot{H}_j$ for every $j \in I$. N is then defined as follows.

$$N_0 = \emptyset.$$
$$N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha \text{ if } \alpha \text{ is a limit ordinal.}$$
$$N_{\alpha+1} = \left\{ x \subseteq N_\alpha \;\middle|\; \begin{array}{l} x \text{ is definable over the model } \langle N_\alpha, \in, c\rangle_{c\in N_\alpha} \\ \text{via a term } \tau \in \mathcal{L}_1 \text{ of rank } \le \alpha \end{array} \right\}.$$
$$N = \bigcup_{\alpha\in \mathrm{Ord}^V} N_\alpha.$$

As earlier, $N \models$ ZF. Also, in analogy to our first proof of Theorem 8, for $j \in I$, $\omega_1^V = \omega_1^{V[H_j]} = \omega_1^N = \omega_1^{V[G]}$. In addition, Lemma 1 of [3] (using the homogeneity properties of the Lévy collapse instead of the homogeneity properties of Cohen forcing) or Lemma 1.4 of [4] (for an uncountable sequence of cardinals) gives us the fundamental homogeneity property that if $x \in N$ is a set of ordinals, then $x \in V[H_j]$ for some $j \in I$. (The proofs of) Lemmas 1.2 and 1.4 of [2], which again remain valid even when our sequence of cardinals $\langle \lambda_i \mid i < \omega_1\rangle$ is uncountable, then once more tell us that $N \models$ "DC + $\kappa = \aleph_{\omega_1}$". Further, as before, since $N \models$ DC, $N \models$ "ω_1 is regular". Our second proof of Theorem 8 will thus be complete once we have shown that $N \models$ "GCH holds below $\aleph_{\omega_1}$ + $\aleph_{\omega_1}$ is an ω_2-Rowbottom cardinal carrying an ω_2-Rowbottom filter".

To see that the first of these facts is true, by the properties of the Lévy collapse, since $\mathbb{P}$ is a full support product, if $j \in I$ and $x \subseteq \lambda_j$, $x \in V[G]$, then $x \in V[H_j]$. Since $V \models$ GCH, we may once again use the properties of the Lévy collapse and the fact $j < \omega_1$ to infer that $V[H_j] \models$ "There are no cardinals in any of the open intervals $(\lambda_{i-1}^+, \lambda_i)$ for $0 \le i \le j$ + λ_{i-1}^+ and λ_i remain cardinals for $0 \le i \le j$ + GCH holds for any cardinal less than or equal to λ_j + $\lambda_j < \aleph_{\omega_1}$". Since $V[H_j] \subseteq N \subseteq V[G]$, these facts remain true in N as well. Consequently, since $N \models$ "$\aleph_{\omega_1} = \kappa = \sup(\langle \lambda_i \mid i < \omega_1\rangle)$", $N \models$ "GCH holds below $\aleph_{\omega_1}$".

To see that the second of these facts is true, as before, let $\langle \mu_i \mid i < \omega_1\rangle \in V$ be such that μ_i is a normal measure over κ_i. In N, define $\mathcal{F} = \{A \subseteq \kappa \mid \exists i < \omega_1 \forall j \in [i, \omega_1)[A \cap \kappa_j \in \mu_j]$. By the fundamental homogeneity property mentioned above, since any $f : [\kappa]^{<\omega} \to \lambda$ for any $\lambda < \kappa$ can be coded as a set of ordinals, for some $j \in I, f \in V[H_j]$. Since by the definition of $\mathbb{Q}_j$, there is some $j' > j, j' \in I$ such that $|\mathbb{Q}_j| < \kappa_{j'}$, by the Lévy-Solovay results [10], the sequence $\langle \kappa_i \mid j' \le i < \omega_1\rangle$ is composed of cardinals which are measurable in $V[H_j]$, and for any i with $j' \le i < \omega_1$, $\mu_i^* = \{X \subseteq \kappa_i \mid \exists Y \in \mu_i[X \supseteq Y]\}$ is a normal measure over κ_i in $V[H_j]$. As in the last paragraph of the proof of Lemma 1.3 of [2], which still remains valid working with the uncountable sequence of cardinals $\langle \kappa_i \mid j' \le i < \omega_1\rangle$, we may infer that there is $A \subseteq \kappa$, $A \in V[H_j] \subseteq N$ which is ω_2-Rowbottom homogeneous for f such that $\forall i \in [j', \omega_1)[A \cap \kappa_i \in \mu_i^*]$. Without loss of generality, as in [2], we may further assume that $\forall i \in [j', \omega_1)[A \cap \kappa_i \in \mu_i]$. It then immediately follows that $A \in \mathcal{F}$. Thus, $\mathcal{F}$ is in N an ω_2-Rowbottom filter over $\kappa = \aleph_{\omega_1}$. Our second proof of Theorem 8 is now complete. □

We continue now with the reverse direction of Theorem 2. As before, we state this as a separate, stronger theorem which implies our desired result.

Theorem 9 *Let κ be a singular Jonsson cardinal of uncountable cofinality in a model V of ZF. Then there is an inner model whose class of measurable cardinals is cofinal in κ.*

The proof of Theorem 9 will use the theory of short core models, but in the context of models *not* satisfying the Axiom of Choice. We therefore briefly mention now some terminology and notation attendant to this theory. The smallest mouse which is not an element of a short core model is uniquely determined and is the set 0^{long} mentioned in Sect. 1. It is a countable iterable structure of the form $0^{\text{long}} = L_\delta[\bar{\mathcal{U}}^{\text{long}}]$, in which the measure sequence $\bar{\mathcal{U}}^{\text{long}}$ is not short. Indeed, it is the case that $\text{otp}(\text{dom}(\bar{\mathcal{U}}^{\text{long}})) = \min(\text{dom}(\bar{\mathcal{U}}^{\text{long}}))$. The existence of 0^{long} is a large cardinal axiom.

We begin our discussion of the proof of Theorem 9 with a useful preliminary result.

Proposition 4.1 *Assume that 0^{long} exists. Then for any singular cardinal κ, there is an inner model whose class of measurable cardinals is cofinal in κ.*

Proof Let $\lambda = \text{cof}(\kappa) < \kappa$. Choose a strictly monotone sequence $\langle \kappa_i \mid i < \lambda \rangle$ which is cofinal in κ such that $\kappa_0 = \lambda$. Construct a minimal iterate $L_\eta[\bar{\mathcal{U}}]$ of 0^{long} such that for every $i < \lambda$, it is the case that the ith measurable cardinal of $\bar{\mathcal{U}}$ is greater than κ_i. Since $\bar{\mathcal{U}}$ has a λth measurable cardinal greater than or equal to κ which can be iterated out of the ordinals, one gets that $L[\bar{\mathcal{U}} \restriction \kappa] \models$ "$\bar{\mathcal{U}} \restriction \kappa$ is a sequence of measures". The model $L[\bar{\mathcal{U}} \restriction \kappa]$ is as desired. □

We turn now to the proof of Theorem 9.

Proof If 0^{long} exists, Theorem 9 holds by the previous proposition. Consequently, we assume $\neg 0^{\text{long}}$. Then the theory of short core models of [8] is adequate for models of the form HOD[a] with $a \subseteq$ HOD. Let $K[\bar{\mathcal{U}}_{\text{can}}]$ be the canonical short core model formed in HOD, with measure sequence $\bar{\mathcal{U}}_{\text{can}}$. $K[\bar{\mathcal{U}}_{\text{can}}] \models$ "$\bar{\mathcal{U}}_{\text{can}}$ is a sequence of measures", so it suffices to show that dom $(\bar{\mathcal{U}}_{\text{can}} \restriction \kappa)$ is cofinal in κ. We therefore assume towards a contradiction that $\text{dom}(\bar{\mathcal{U}}_{\text{can}} \restriction \kappa)$ is bounded below κ. Set $\lambda = \text{cof}(\kappa)$ and $\theta = \sup(\text{dom}(\bar{\mathcal{U}}_{\text{can}} \restriction \kappa))$.

We begin by showing there is a closed, unbounded $D \subseteq \kappa$ such that every element of D is singular in $K[\bar{\mathcal{U}}_{\text{can}}]$. For this, we consider two cases.

Case 1 κ is a singular successor cardinal. Then let $\bar{\kappa}$ be the cardinal predecessor of κ, so $\kappa = \bar{\kappa}^+$. Choose a strictly monotone sequence $\langle \kappa_i \mid i < \lambda \rangle$ which is closed, unbounded in κ. Let $\kappa' = (\bar{\kappa}^+)^{\text{HOD}[\langle \kappa_i | i < \lambda \rangle]} \leq \bar{\kappa}^+ = \kappa$. Since κ' is regular in HOD[$\langle \kappa_i \mid i < \lambda \rangle$] and κ is singular in HOD[$\langle \kappa_i \mid i < \lambda \rangle$], we have $\kappa' < \kappa$. Since $\kappa' > \bar{\kappa} \geq \text{cof}^V(\kappa) = \lambda \geq \omega_1^V$, we have HOD[$\langle \kappa_i \mid i < \lambda \rangle$] $\models$ "$\kappa' \geq \omega_2$".

Work in HOD[$\langle \kappa_i \mid i < \lambda \rangle$]. Let $D = \{\kappa_{\omega \cdot i} \mid i < \lambda$ and $\kappa_{\omega \cdot i} > \kappa'$ and $\kappa_{\omega \cdot i} > \theta\}$. Consider $\kappa_{\omega \cdot i} \in D$. It is then the case that $\text{cof}(\kappa_{\omega \cdot i}) \leq \omega \cdot i < \lambda \leq \bar{\kappa} \leq |\kappa_{\omega \cdot i}|$, $\kappa_{\omega \cdot i} > \omega_2$, and $\kappa_{\omega \cdot i} > \sup(\text{dom}(\bar{\mathcal{U}}_{\text{can}} \restriction (\kappa_{\omega \cdot i} + 1)))$. By the Covering Theorem 3.20(i) of [8], $\kappa_{\omega \cdot i}$ is singular in $K[\bar{\mathcal{U}}_{\text{can}}]$. The claim follows, since $K[\bar{\mathcal{U}}_{\text{can}}]^{\text{HOD}[\langle \kappa_i | i < \lambda \rangle]} = K[\bar{\mathcal{U}}_{\text{can}}]^{\text{HOD}}$.

Case 2 κ is a singular limit cardinal. Choose a closed, unbounded set $D \subseteq \kappa$ of order type λ. Since the limit cardinals are closed, cofinal in κ, we may assume that, in HOD[D], every $\tau \in D$ is a limit cardinal of cofinality less than or equal to $\lambda < \tau$ which is greater than $\max(\omega_2, \theta, \lambda)$. Then, by the Covering Theorem 3.20(ii) of [8], τ is singular in $K[\bar{\mathcal{U}}_{\mathrm{can}}]^{\mathrm{HOD}[D]} = K[\bar{\mathcal{U}}_{\mathrm{can}}]^{\mathrm{HOD}}$. D is hence as desired.

Continuing with the proof of Theorem 9, take $H \in \mathrm{HOD}$ to be a sufficiently elementary submodel of HOD such that $\kappa + 1 \subseteq H$. Let X be a Jonsson substructure of the first-order structure $\mathfrak{H} = \langle H, \in, D\rangle$, where D is taken as above. By the choice of X, $X \prec \langle H, \in, D\rangle$, $|X \cap \kappa| = \kappa$, and $X \cap \kappa \neq \kappa$.

The remainder of the argument will be carried out inside the structure HOD[D, X]. Note that by Proposition 1.1, notions of short core model theory are absolute between HOD and HOD[D, X].

Let $\pi : \langle \bar{H}, \in, \bar{D}\rangle \cong \langle X, \in, X \cap D\rangle \prec \langle H, \in, D\rangle$ be the Mostowski collapse of X, where $\bar{H}$ is transitive. As in the proof of Theorem 6, it is then the case that $\pi \restriction \kappa \neq \mathrm{id} \restriction \kappa$ and $\pi(\kappa) = \kappa$.

Let $\bar{\mathcal{U}} = \pi^{-1}(\bar{\mathcal{U}}_{\mathrm{can}} \restriction \kappa)$, and let $\bar{K} = K[\bar{\mathcal{U}}]^{\bar{H}}$ be the short core model over $\bar{\mathcal{U}}$ as defined in $\bar{H}$. Since being closed, unbounded can be defined absolutely in the $\in$-language, $\bar{D}$ is closed, unbounded in κ.

We note now that $\forall \gamma \in \bar{D}[\bar{K} \models$ "γ is singular"]. To see this, recall that $\forall \gamma \in D[K[\bar{\mathcal{U}}_{\mathrm{can}}] \models$ "γ is singular"], from which it follows by the definition of π and elementarity that $\forall \gamma \in D[K[\bar{\mathcal{U}}_{\mathrm{can}} \restriction \kappa] \models$ "γ is singular"]. Here, the core model can be considered as being defined in the model HOD. Since H is a sufficiently elementary submodel of HOD, we have $\forall \gamma \in D[(K[\bar{\mathcal{U}}_{\mathrm{can}} \restriction \kappa])^H \models$ "γ is singular"]. This is downwards absolute to $\bar{H}$, so $\forall \gamma \in \bar{D}[(K[\bar{\mathcal{U}}])^{\bar{H}} \models$ "γ is singular"].

It is also easily seen that $K_\kappa[\bar{\mathcal{U}}] \subseteq \bar{K}$. This is verified by checking the Condensation Criterion 3.24(ii) of [8]. Consider a closed, unbounded set $C \subseteq \kappa$. Since $\mathrm{cof}(\kappa) = \lambda > \omega$, there is $\gamma \in C \cap \bar{D}$. By the preceding paragraph, $\bar{K} \models$ "γ is singular", as required.

In addition, it is easily shown that $\bar{\mathcal{U}}$ is a strong measure sequence, i.e., $K[\bar{\mathcal{U}}] \models$ "$\bar{\mathcal{U}}$ is a sequence of measures". To see this, by elementarity, $\bar{\mathcal{U}}$ is a sequence of measures in $\bar{K}$. If $\xi \in \mathrm{dom}(\bar{\mathcal{U}})$, then $\bar{\mathcal{U}}_\xi$ is a measure in $\bar{K}$. By the fact that $K_\kappa[\bar{\mathcal{U}}] \subseteq \bar{K}$, $\bar{\mathcal{U}}_\xi$ is a measure in $K_\kappa[\bar{\mathcal{U}}]$. Since $K_\kappa[\bar{\mathcal{U}}] = H_\kappa^{K[\bar{\mathcal{U}}]}$, $\bar{\mathcal{U}}_\xi$ is a measure in $K[\bar{\mathcal{U}}]$. Further, since $K_\kappa[\bar{\mathcal{U}}] = H_\kappa^{\bar{K}}$ and $\pi \restriction H_\kappa^{\bar{K}} : H_\kappa^{\bar{K}} \to H_\kappa^{K[\bar{\mathcal{U}}_{\mathrm{can}}]^H} = K_\kappa[\bar{\mathcal{U}}_{\mathrm{can}}]$, we have that $\pi \restriction K_\kappa[\bar{\mathcal{U}}] : K_\kappa[\bar{\mathcal{U}}] \to K_\kappa[\bar{\mathcal{U}}_{\mathrm{can}}]$ is elementary.

By Theorem 3.16 of [8], there is an iterated ultrapower $\sigma : K[\bar{\mathcal{U}}_{\mathrm{can}}] \to K[\mathcal{U}']$ such that $\bar{\mathcal{U}}$ is an initial segment of the measure sequence $\mathcal{U}'$. We may assume that $\bar{\mathcal{U}} = \mathcal{U}' \restriction \kappa$, by possibly further iteration of measures in $\mathcal{U}'$ above $\bar{\mathcal{U}}$. Then $K_\kappa[\mathcal{U}'] = K_\kappa[\bar{\mathcal{U}}]$, and $\pi \restriction K_\kappa[\mathcal{U}'] : K_\kappa[\mathcal{U}'] \to K_\kappa[\bar{\mathcal{U}}_{\mathrm{can}}]$ is elementary.

Since $\mathrm{cof}(\kappa)$ is uncountable, the upward extension embeddings techniques known from the standard proof of the Covering Theorem (see Theorem 3.25 of [8]) may be applied to lift the map $\pi \restriction K_\kappa[\mathcal{U}']$ up to $K[\mathcal{U}']$. In particular, there is a map $\tilde{\pi}$ and a transitive inner model W such that $\tilde{\pi} : K[\mathcal{U}'] \to W$ is elementary, $\tilde{\pi} \supseteq \pi \restriction K_\kappa[\bar{\mathcal{U}}]$, and $\tilde{\pi}(\kappa) = \kappa$.

By Theorem 3.13 of [8], $W = K[\tilde{\mathcal{U}}]$, where $\tilde{\mathcal{U}} = \tilde{\pi}(\mathcal{U}')$. Hence, $\tilde{\pi} : K[\mathcal{U}'] \to K[\tilde{\mathcal{U}}]$ is elementary. In addition, $\tilde{\mathcal{U}} \restriction \kappa = \bar{\mathcal{U}}_{\text{can}} \restriction \kappa$. This is since by the choice of $\mathcal{U}'$, we have $\bar{\mathcal{U}} = \mathcal{U}' \restriction \kappa$. Thus, $\tilde{\mathcal{U}} \restriction \kappa = \tilde{\pi}(\mathcal{U}') \restriction \kappa = \tilde{\pi}(\mathcal{U}') \restriction \tilde{\pi}(\kappa) = \tilde{\pi}(\mathcal{U}' \restriction \kappa) = \tilde{\pi}(\bar{\mathcal{U}}) = \bar{\mathcal{U}}_{\text{can}} \restriction \kappa$. It then immediately follows that $\tilde{\pi} \circ \sigma : K[\bar{\mathcal{U}}_{\text{can}}] \to K[\tilde{\mathcal{U}}]$ is elementary.

By Theorem 3.17 of [8], $\tilde{\pi} \circ \sigma$ is a normal iterated ultrapower of $K[\bar{\mathcal{U}}_{\text{can}}]$ which, since $\pi \restriction \kappa \neq \text{id} \restriction \kappa$, is not the identity on κ. Let $\alpha < \kappa$ be the critical point of $\tilde{\pi} \circ \sigma$. Then α is measurable in $K[\bar{\mathcal{U}}_{\text{can}}]$, and $\alpha \in \text{dom}(\bar{\mathcal{U}}_{\text{can}})$. Since $\bar{\mathcal{U}}_{\text{can}}(\alpha)$ is the first measure used in the normal iteration, $\alpha \notin \text{dom}(\tilde{\mathcal{U}})$. This, however, contradicts that $\tilde{\mathcal{U}} \restriction \kappa = \bar{\mathcal{U}}_{\text{can}} \restriction \kappa$, thereby completing the proof of Theorem 9. □

Theorem 9 clearly implies the reverse direction of Theorem 2. Thus, the proof of Theorem 2 is now complete. □

We conclude this section by noting that our methods of proof for Theorem 2 routinely generalize to the situation where α is an ordinal such that $\omega_\alpha > \alpha$ and ω_α is regular. More specifically, the methods of this section allow us to prove the following theorem.

Theorem 10 *Suppose α is a definable ordinal whose definition is absolute between transitive models of ZF. Suppose further that for any transitive model V of ZFC, $V \vDash$ "$\omega_\alpha > \alpha$ and ω_α is regular". The theories "ZFC + There exist ω_α measurable cardinals" and "ZF + $\neg AC$ + ω_α is regular + $\aleph_{\omega_\alpha}$ is an $\omega_{\alpha+1}$-Rowbottom cardinal carrying an $\omega_{\alpha+1}$-Rowbottom filter" are then equiconsistent.*

5 Some generalizations and additional remarks

We begin this section by noting that in Theorem 2, we require for our equiconsistency that ω_1 be regular. That this is not a superfluous requirement is shown by the following theorem.

Theorem 11 *Let $V \vDash$ "ZFC + κ is a measurable cardinal". There is then a notion of forcing $\mathbb{P}$ and a symmetric inner model $N \subseteq V^{\mathbb{P}}$ such that $N \vDash$ "ZF + $\neg AC_\omega$ + ω_1 is singular + $\aleph_{\omega_1}$ is a Rowbottom cardinal carrying a Rowbottom filter".*

Proof As in the proof of Theorem 5, we assume that the ground model V for the hypotheses of Theorem 11 has been extended generically via Prikry forcing using a normal measure $\mathcal{U}$ over κ to a model, which we also denote by V, containing a Prikry sequence $\langle \kappa_i \mid i < \omega \rangle$ through κ.

Let $\mathbb{P}_{-1} = \text{Coll}(\omega, <\aleph_\omega)$, $\mathbb{P}_0 = \text{Coll}(\aleph_{\omega+1}, <\kappa_0)$, and for $1 \le i < \omega$, let $\mathbb{P}_i = \text{Coll}(\kappa_{i-1}^{+\aleph_{i-1}+1}, <\kappa_i)$. We then define $\mathbb{P} = \mathbb{P}_{-1} \times \prod_{i<\omega} \mathbb{P}_i$ with full support.

Let G be $\mathbb{P}$-generic over V. To define our desired model N witnessing the conclusions of Theorem 11, we first note that as before, by the Product Lemma, for $-1 \le i < \omega$, G_i, the projection of G onto $\mathbb{P}_i$, is V-generic over $\mathbb{P}_i$. Next, let $\mathcal{F} = (\omega, \aleph_\omega) \times (\aleph_{\omega+1}, \kappa_0) \times (\kappa_0^{+\aleph_0+1}, \kappa_1) \times (\kappa_1^{+\aleph_1+1}, \kappa_2) \times \cdots$. For each $f \in \mathcal{F}$,

$f = \langle f(-1), f(0), f(1), \ldots \rangle$, define $G \restriction f = G_{-1} \restriction f(-1) \times G_0 \restriction f(0) \times G_1 \restriction f(1) \times \cdots$. By the Product Lemma and the properties of the Lévy collapse, $G \restriction f$ is $(\mathbb{P}_{-1} \restriction f(-1)) \times \prod_{i<\omega}(\mathbb{P}_i \restriction f(i))$-generic over V. N can now intuitively be described as the least model of ZF extending V which contains, for each $f \in \mathcal{F}$, the set $G \restriction f$.

In order to define N more formally, we let $\mathcal{L}_1$ be the ramified sublanguage of the forcing language $\mathcal{L}$ with respect to $\mathbb{P}$ which contains symbols $\check{v}$ for each $v \in V$, a unary predicate symbol $\check{V}$ (to be interpreted $\check{V}(\check{v}) \leftrightarrow v \in V$), and symbols $\dot{G} \restriction f$ for each $f \in \mathcal{F}$. N is then defined as follows.

$$N_0 = \emptyset.$$
$$N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha \text{ if } \alpha \text{ is a limit ordinal.}$$
$$N_{\alpha+1} = \left\{ x \subseteq N_\alpha \;\middle|\; \begin{array}{l} x \text{ is definable over the model } \langle N_\alpha, \in, c\rangle_{c\in N_\alpha} \\ \text{via a term } \tau \in \mathcal{L}_1 \text{ of rank } \leq \alpha \end{array} \right\}.$$
$$N = \bigcup_{\alpha\in\mathrm{Ord}^V} N_\alpha.$$

The standard arguments show $N \models$ ZF. Further, by Lemmas 1.1 and 1.2 of [2], which remain valid even with the current definition of $\mathbb{P}$, $N \models$ "$\kappa = \aleph_{(\aleph_\omega)^V}$". Since by Lemmas 1.1 and 1.2 of [2], $N \models$ "$(\aleph_\omega)^V = \aleph_1$", it is actually the case that $N \models$ "$\kappa = \aleph_{\omega_1}$ and $\mathrm{cof}(\kappa) = \omega$". As a consequence of this, $N \models$ "$\neg\mathrm{AC}_\omega$". Further, the proof of Theorem 4, as well as the remarks in the succeeding paragraph, remain valid and show that for any $f \in \mathcal{F}$ and any $g \in V[G \restriction f]$ with $g : [\kappa]^{<\omega} \to \lambda$ for $\lambda < \kappa$, there is $A \in \mathcal{U}$ which is Rowbottom homogeneous for g. The proof that $N \models$ "$\aleph_{\omega_1}$ is a Rowbottom cardinal carrying a Rowbottom filter" is therefore the same as the one given in Theorem 5. Hence, the proof of Theorem 11 is now complete. □

From Theorems 11 and 6, we may consequently now immediately infer the following theorem.

Theorem 12 *The theories "ZFC+ There exists a measurable cardinal" and "ZF + ¬AC + $\aleph_{\omega_1}$ is a Rowbottom cardinal carrying a Rowbottom filter" are equiconsistent.*

One might also wonder if it is possible to have additional instances of the Axiom of Choice holding in our aforementioned models in which $\aleph_\omega$ is Rowbottom or $\aleph_{\omega_1}$ is ω_2-Rowbottom and GCH holds below either $\aleph_\omega$ or $\aleph_{\omega_1}$. In particular, in Bull's model of [4], AC_ω fails, and in the model constructed for the second proof of Theorem 8, we can only show that DC is true. In fact, the degree of the Axiom of Choice in these models appears to be optimal for the large cardinal strength available in the relevant ground models. If one had models, e.g., in which $\aleph_\alpha$ were Jonsson for $\alpha = \omega$ or ω_1, AC_α were true, and for each $i < \alpha$, it were possible to well-order $\wp(\aleph_i)$, then it would be possible to well-order $\bigcup_{i<\alpha} \wp(\aleph_i)$ as well. This would then allow us to run the argument of Theorem 5.2 of [8] and infer the existence of 0^{long}.

In conclusion, we ask what the consistency strength is for the theory "$ZF + \neg AC+$ The least regular cardinal is both a limit and Jonsson cardinal". By the results of [1], by forcing over a model of AD, we may establish an upper bound of ω Woodin cardinals. Is it possible to lower this upper bound, and even establish an equiconsistency result, in analogy to what is done in this paper?

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