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ON THE CONSISTENCY STRENGTH OF 'ACCESSIBLE' JONSSON CARDINALS AND OF THE WEAK CHANG CONJECTURE

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Using the core model K we determine better lower bounds for the consistency strength of some combinatorial principles:

I. Assume that λ is a Jonsson cardinal which is 'accessible' in the sense that at least one of (1)–(4) holds: (1) λ is a successor cardinal; (2) $\lambda = \omega_\xi$ and $\xi < \lambda$; (3) λ is singular of uncountable cofinality; (4) λ is a regular but not weakly hyper-Mahlo.Then $0^\dagger$ exists.

II. For $\lambda = \rho^+$ a successor cardinal we consider the *weak Chang Conjecture*, wCC(λ), which is a consequence of the Chang transfer property $(\lambda^+, \lambda) \Rightarrow (\lambda, \rho)$.

III. If $\lambda = \rho^+ \geqslant \omega_2$, then wCC($\lambda$) implies the existence of $0^\dagger$.

IV. We can determine the consistency strength of wCC(ω_1).

We include a relatively simple definition of the core model which together with the results of Dodd and Jensen suffices for our proofs.

0. Introduction

The inner model L of constructible sets has been frequently used to investigate the consistency strength of combinatorial principles. In our paper some of these methods are adapted to the core model K to obtain stronger results.

The way in which we will apply the model K may be motivated by Kunen's proof that the existence of a Jonsson cardinal implies the existence of $0^\#$. We sketch the argument (a detailed account is in Jech [11, p. 396]):

A cardinal κ is called *Jonsson* if every first-order structure of cardinality κ whose language is countable possesses a *Jonsson substructure,* i.e. a proper elementary substructure of the same cardinality. So let X be a Jonsson substructure of $\langle L_\kappa, \in\rangle$. By the condensation lemma for the L_α-hierarchy, $X \cong L_\kappa$. The inverse of the isomorphism is a nontrivial elementary embedding $\pi: L_\kappa \to L_\kappa$. $U = \{x \subseteq \alpha \mid x \in L \text{ and } \alpha \in \pi(x)\}$ is an ultrafilter on $\mathfrak{P}(\alpha) \cap L$ where α is the first ordinal moved by π. A condensation argument shows that the ultrapower

$({}^{\alpha}L \cap L)/U$ is well-founded. Hence there is a nontrivial elementary embedding $\tilde{\pi}: L \to L$, and by a theorem of Kunen, $0^{\#}$ exists.

The core model K was invented by Dodd and Jensen [6]. K is an 'L-like' inner model of set theory which satisfies many of the combinatorial properties of L. But compared with L, the core model admits 'larger' cardinals; for example there may be Ramsey cardinals in K. Dodd and Jensen obtain the following *Covering Theorem for K* which strengthens the Jensen Covering Theorem for L (see [6]):

If there is no inner model with a measurable cardinal, then K *covers* V, i.e. for every set $X \subseteq \mathrm{On}$ there is some $Y \in K$ with $X \subseteq Y$ and $\mathrm{card}(Y) \leqslant \mathrm{card}(X) + \omega_1$. The core model is also L-like in that it satisfies an analogue of Kunen's theorem about elementary embeddings of L into L:

If there is a nontrivial elementary map from K into K, then there is an inner model with a measurable cardinal.

Hence, when Kunen's argument about Jonsson cardinals can be carried over to K, it yields an inner model with a measurable cardinal. We are able to do this transfer for certain 'accessible' Jonsson cardinals and for a weak form of a generalized Chang Conjecture. Actually we can strengthen this. Having obtained an inner model with a measurable cardinal we can repeat Kunen's argument with some inner model $L[U]$, U is a normal ultrafilter on a measurable cardinal α. We obtain a nontrivial elementary embedding $\tilde{\pi}: L[U] \to L[U]$ with its critical point above α. This is equivalent to the existence of $0^{\dagger}$, a set of Gödel numbers defined by Solovay (see [15, p. 132]).

Our main theorems now are:

Theorem A. *Let* λ *be a Jonsson cardinal such that at least one of* (1)–(4) *holds*:

(1) $\lambda = \rho^{+}$,

(2) $\lambda = \omega_{\xi}$ *and* $\xi < \lambda$,

(3) $\omega < \mathrm{cof}(\lambda) < \lambda$,

(4) λ *is regular but not weakly hyper-Mahlo.*

Then $0^{\dagger}$ *exists.*

See Drake [10] for a definition of weak hyper-Mahlo cardinals.

Definition. Let $\lambda = \rho^{+}$ be a successor cardinal. The *weak Chang Conjecture* for λ, $\mathrm{wCC}(\lambda)$, is the assertion: Whenever $\mathfrak{A}$ is a first-order structure with a countable language and $\lambda^{+} \subseteq \mathfrak{A}$, then there is $\alpha < \lambda$ such that for all $\beta < \lambda$ there is $X \prec \mathfrak{A}$ with $X \cap \lambda \subseteq \alpha$ and $\mathrm{otp}(X \cap \lambda^{+}) > \beta$.

The weak Chang Conjecture for ω_1 was, in an equivalent combinatorial form, considered in Shelah [18, section 35]. $\mathrm{wCC}(\lambda)$ is a trivial consequence of the Chang two-cardinal property $(\lambda^{+}, \lambda) \Rightarrow (\lambda, \rho)$ (see Chang–Keisler [3, p. 450]).

Theorem B. *Let* $\lambda = \rho^{+} \geqslant \omega_2$ *and assume* $\mathrm{wCC}(\lambda)$. *Then* $0^{\dagger}$ *exists.*

Corollary. *If* $(\omega_3, \omega_2) \Rightarrow (\omega_2, \omega_1)$, *then* $0^\dagger$ *exists.*

Many more corollaries can be drawn from Theorem B, also taking into account the various interdependences between the Chang properties $(\kappa, \lambda) \Rightarrow (\kappa', \lambda')$, (see 3]).

We determine the consistency strength of $\mathrm{wCC}(\omega_1)$ in terms of partition cardinals:

Definition. (1) Let $f:[S]^{<\omega} \to V$, $S \subseteq \mathrm{On}$. Assume that $X \subseteq S$ is an infinite homogeneous set for f. Then set:

$$\mathrm{tp}_f(X) = \langle y_n \mid \exists \gamma_1, \ldots, \gamma_n \in X(\gamma_1 < \cdots < \gamma_n \wedge f(\gamma_1, \ldots, \gamma_n) = y_n), n < \omega \rangle.$$

$\mathrm{tp}_f(X)$ is called the *type* of X (with respect to f).

A sequence $\langle X_\alpha \mid \alpha < \tau \rangle$ is called *homogeneous* for f (of order τ) iff for $\alpha < \beta < \tau : X_\alpha \subseteq S$; $\mathrm{otp}(X_\alpha) = \omega(1+\alpha)$; X_α is homogeneous for f; and $\mathrm{tp}_f(X_\alpha) = \mathrm{tp}_f(X_\beta)$.

(2) Let $\omega\tau = \tau$, $\tau \neq 0$.

(a) Set $\kappa \to (<\tau)^{<\omega}_\lambda$ iff for all $f:[\kappa]^{<\omega} \to \lambda$ there is a homogeneous sequence $\langle X_\alpha \mid \alpha < \tau \rangle$ for f.

(b) κ is called *almost* $<\tau$-*Erdös* iff κ is regular and $\kappa \to (<\tau)^{<\omega}_\lambda$ for all $\lambda < \kappa$.

This type of properties has been studied in Baumgartner–Galvin [2]. The property $\kappa \to (<\omega_1)^{<\omega}_2$ implies the existence of $0^\#$ but is strictly weaker than $\kappa \to (\omega_1)^{<\omega}_2$ (see Section 8).

Theorem C. *Let M be a countable transitive model of* ZFC *and let κ be almost* $<\omega_1$-*Erdös in M. Then there is a generic extension N of M such that* $N \models \mathrm{wCC}(\omega_1)$.

Theorem D. *Assume* $\mathrm{wCC}(\omega_1)$. *Let* $\kappa = \omega_2$ *and* $\tau = \omega_1^K$. *Then κ is almost* $<\tau$-*Erdös in K.*

'Accessible' Jonsson cardinals have been considered before. It is easy to see that no ω_n is Jonsson, $(n < \omega)$. Under GCH no successor cardinal is Jonsson. Shelah [19] gives a generalisation of this. Theorem A(3) strengthens results of Mitchell and Silver. Mitchell [16] shows that a Jonsson cardinal is Ramsey in K. Thus a singular Jonsson cardinal is regular in K and by the Covering Theorem for K there is an inner model with a measurable cardinal. Even before the introduction of K, Silver had constructed an inner model with a measurable from the assumption that ω_ω is Jonsson and $2^\omega < \omega_\omega$ (see Kanamori–Magidor [12]). On the other hand one can obtain singular Jonsson cardinals: A singular limit of measurable cardinals is Jonsson; Prikry forcing produces a Jonsson cardinal of confinality ω (Prikry [17]).

Theorem B contrasts with results of Silver and the first author about the consistency strength of the Chang Conjecture $(\omega_2, \omega_1) \Rightarrow (\omega_1, \omega_0)$. Silver constructed a model for the Chang Conjecture by forcing starting from a model for $\kappa \to (\omega_1)^{<\omega}$ (Kanamori–Magidor [12] exhibit an easier version of this, starting from a Ramsey cardinal). Donder has shown that if $(\omega_2, \omega_1) \Rightarrow (\omega_1, \omega_0)$ is true in the universe then, in K, $\kappa \to (\lambda)^{<\omega}$ holds, where $\kappa = \omega_2$ and $\lambda = \omega_1$ [9]. An upper bound for the consistency strength of $(\omega_3, \omega_2) \Rightarrow (\omega_2, \omega_1)$ is given by a huge cardinal: If the forcing in Kunen [14] is modified to yield an ω_3-saturated ideal on ω_2, then, in the extension, $(\omega_2, \omega_2) \Rightarrow (\omega_2, \omega_1)$ holds (see the remark at the end of [14]).

As one might expect, Theorems C and D are descendants of the aforementioned results of Silver and Donder.

We strongly suppose that with the introduction of generalised core models appropriate for inner models with several measurable cardinals the conclusion of Theorems A and B can be considerably strengthened.

Kunen's result on Jonsson cardinals rests heavily on the condensation properties of the constructible hierarchy, and the main point in the proofs of Theorems A and B is to define a structure such that certain elementary substructures of it 'condense' nicely. We want the condensation map to determine an ultrafilter on $\mathfrak{P}(\alpha) \cap K$ for some α. Hence the condensate has to contain $\mathfrak{P}(\alpha) \cap K$. Lemma 2.6. is the tool to show that the condensate contains enough sets.

This paper is organized as follows:

Section 1 gives a brief introduction into the core model. The main properties of K are stated without proof. We consider 'iterable premice', which allow us to define K in a rather elementary way.

Section 2 develops the machinery for our condensation arguments with K..

In Section 3 we derive from the assumptions of Theorem A the existence of an inner model with a measurable cardinal less than the Jonsson cardinal considered. This is strengthened in Section 4 where we show:

Theorem A2. *If κ is a Jonsson cardinal and some ordinal $<\kappa$ is measurable in an inner model, then $0^{\dagger}$ exists.*

Section 5 gives an equivalence of wCC(λ) which is better suited to the proofs of Theorem B and C. The proof of Theorem B is, as the proof of Theorem A, split into two steps.

Section 6 gets from wCC(λ), $\lambda = \rho^+ \geqslant \omega_2$, that there is an inner model with a measurable $<\lambda^+$. In Section 7 we prove:

Theorem B2. *Assume* wCC(λ), $\lambda = \rho^+$, *and that there is an inner model with a measurable cardinal* $<\lambda^+$. *Then* $0^{\dagger}$ *exists.*

Sections 8 and 9 contain the proofs of Theorems C and Theorem D.

We presuppose an acquaintance with (relative) constructibility, basic knowledge of iterated ultrapowers and of course, in Section 8, of the forcing method. Constructibility is done with the J_α-hierarchy.

We use standard set-theoretical notation throughout.

1. The core model

Dodd and Jensen [6] introduce the core model K in order to generalise the Jensen Covering Theorem for L.

1.1–1.6 state fundamental properties of K.

1.1. *K is transitive,* $\mathrm{On} \subseteq K$, *and* $K \models \mathrm{ZFC} + V = K + \mathrm{GCH}$.

K also satisfies various combinatorial principles which hold in L, like $\Diamond, \Box, \ldots$.

Definition. *A covers B* iff $\forall X \subseteq \mathrm{On}$,

$$X \in B\ \exists Y \in A\ (X \subseteq Y \text{ and } \mathrm{card}(Y) = \mathrm{card}(X) + \omega_1).$$

1.2 (The Covering Theorem for K). *If there is no inner model with a measurable cardinal, then K covers V.*

1.3 (The Covering Theorem for K, extended). *Assume* $0^\dagger$ *does not exist. Then one of* (1)–(3) *holds*:

(1) *K covers V.*

(2) *$L[U]$ covers V, for some U, such that*

$$L[U] \models \text{"}U \textit{ is a normal ultrafilter on some ordinal}\text{"}.$$

(3) *$L[U, C]$ covers V, for some U, C, such that*

$$L[U] \models \text{"}U \textit{ is a normal ultrafilter on some ordinal}\text{"}$$

and C is a Prikry-sequence for U over $L[U]$.

(A *normal* ultrafilter is always understood to be non-trivial.)

1.4. *Let $\pi : K \to M$ be elementary and let M be transitive. Then $M = K$.*

1.5. *Let $\pi : K \to K$ be nontrivial and elementary. Let α be the first ordinal moved by π. Then there is an inner model with a measurable cardinal β, such that*

$$\beta \leq \omega_1 \quad \textit{if } \alpha < \omega_1, \quad \textit{and} \quad \beta < \alpha^+ \quad \textit{if } \alpha \geq \omega_1.$$

Remark. Since this result is not explicitly proved in the published papers we sketch a proof of Claim 1.5 referring mainly to the proof of Lemma 16.21 in [5]. We may assume that π is an ultrapower by U. We need the following fact (see [7, Lemma 2.3])

(1) If $\mathrm{cf}((\alpha^+)^K) > \omega$, then α is measurable in an inner model.

So we may assume that $(\alpha^+)^K < \alpha^+$. But then, if π is ω_1-iterable, we get the conclusion as in [5]. So we may assume that π is not ρ-iterable ($\rho < \omega_1$) as in the main case of [5]. Let N_i, $i < \rho$, be defined as in that proof. A condensation argument shows that $|N_i| < \alpha^+$. Let C_i, C be defined as in [5] replacing τ by On. The proof shows

(2) (a) C is closed, $\sup(C \cap \alpha^+) = \alpha^+$.

(b) Let γ be a limit point of C and $\mathrm{cf}(\gamma) > \omega$. Then γ is measurable in an inner model.

So the conclusion of Claim 1.5 follows immediately.

1.6. *Assume* $L[U] \models$ "*U is a normal ultrafilter on κ*". *Then* $\mathfrak{P}(\kappa) \cap K = \mathfrak{P}(\kappa) \cap L[U]$. *This implies* $V_\kappa \cap K = V_\kappa \cap L[U]$, *and further that* $K = \bigcap_{i<\infty} (L[U])_i$, *where* $(L[U])_i$ *is the i-th iterated ultrapower of* $L[U]$.

1.6 indicates that the size of the core model depends on the large cardinal situation of the universe. By 1.6, K does not allow measurable cardinals. But the 'low part' of K agrees with the 'low part' of $L[U]$: $V_\kappa \cap K = V_\kappa \cap L[U]$. Thus one may think of the core model being an approximation to measurability from below. This is reflected in the definition of K that we will use. K will be the union of L together with the 'low parts' of certain '$L[U]$-like' structures which are called 'iterable premice'. A *premouse* is a structure $M = J^U_\alpha$ constructed from a filter U over a cardinal κ such that, in M, U is a normal ultrafilter on κ. M is called *iterable* if the iterated ultrapowers of M by U are all well-founded.

The core model may be obtained in several different ways. Dodd and Jensen define K as the inner model constructible from all 'mice'. Even the definition of mouse involves finestructure notions. A mouse possesses a particular, finestructure-preserving 'mouse-iteration', which is adequate for the finestructure investigations of K, leading up to the covering theorem.

Dodd and Jensen show that in ZF^- the original definition of K is equivalent to the one given here. Our definition is not at all suited to prove 1.1–1.6, but it suffices for our proofs.

Definition. A structure $M = J^U_\alpha$ is a *premouse at* κ, iff

$$M \models \text{“}U \text{ is a normal ultrafilter on } \kappa > \omega\text{”}.$$

Note that the 'measurable' κ of M is regular in M.

Let $M = J^U_\alpha$ be a fixed premouse at κ.

Definition. The *ultrapower* $\tilde{M}$ of M is defined by

$$f \sim g \quad \text{iff} \quad \{\nu < \kappa \mid f(\nu) = g(\nu)\} \in U \qquad (f, g \in {}^\kappa M \cap M),$$

$$\tilde{f} := \{g \mid g \sim f\} \qquad (f \in {}^\kappa M \cap M),$$

$$|\tilde{M}| := \{\tilde{f} \mid f \in {}^k M \cap M\},$$

$$\tilde{f} \,\tilde{\in}\, \tilde{g} \quad :\text{iff} \quad \{\nu < \kappa \mid f(\nu) \in g(\nu)\} \in U,$$

$$\tilde{f} \,\tilde{\in}\, \tilde{U} \quad :\text{iff} \quad \{\nu < \kappa \mid f(\beta) \in U\} \in U,$$

$$\tilde{M} = \langle |\tilde{M}|, \tilde{\in}, \tilde{U} \rangle.$$

Since M satisfies Σ_0-separation and the Axiom of Choice, we can prove a Łos-theorem for Σ_0-formulae:

1.7. Lemma. *Let ϕ be Σ_0 in the language for M, and $f_1, \dots, f_n \in {}^\kappa M \cap M$. Then*

$$\tilde{M} \models \phi[\tilde{f}_1, \dots, \tilde{f}_n] \quad \textit{iff} \quad \{\nu < \kappa \mid M \models \phi[f_1(\nu), \dots, f_n(\nu)]\} \in U.$$

Definition. For $x \in M$ set $c_x := \langle x \mid \nu < \kappa \rangle$. Define $\pi_M : M \to \tilde{M}$ by $\pi_M(x) = c_x$.

1.8. Lemma. $\pi_M : M \prec_{\Sigma_0} \tilde{M}$.

1.9. Lemma. *$\pi_M'' M$ is $\tilde{\in}$-cofinal in $\tilde{M}$, i.e. $\forall x \in \tilde{M}\, \exists y \in M\, x \tilde{\in} \pi_M(y)$.*

Proof. Let $x = \tilde{f} \in \tilde{M}$. Set $y = \text{range}(f) \in M$. $\{\nu < \kappa \mid f(\nu) \in y\} = \kappa$, hence $\tilde{f} \tilde{\in} \pi_M(y)$. $\square$

1.10 Lemma. $\pi_M : M \prec_{\Sigma_1} \tilde{M}$.

Proof. Let ϕ be Σ_0 and $x_1, \dots, x_n \in M$. Assume $\tilde{M} \models \exists x\; \phi[\pi_M(x_1), \dots, \pi_M(x_n)]$. By 1.9, there is $x_0 \in M$ such that $\tilde{M} \models \exists x \in \pi_M(x_0)\; \phi[\pi_M(x_1), \dots, \pi_M(x_n)]$. By 1.8, $M \models \exists x \in x_0\; \phi[x_1, \dots, x_n]$. $\square$

If $\tilde{M}$ is well-founded, identify $\tilde{M}$ with its transitive collapse.

1.11. Lemma. *Assume $\tilde{M}$ is well-founded, hence transitive. Then*

(1) *$\tilde{M} = J_{\tilde{\alpha}}^{\tilde{U}}$ for some $\tilde{\alpha}$, and $\tilde{M}$ is a premouse at $\tilde{\kappa} = \pi_M(\kappa)$.*
(2) $\pi_M : M \prec_{\Sigma_1} \tilde{M}$, $\pi_M \restriction \kappa = \text{id}$, $\tilde{\kappa} = \pi_M(\kappa) > \kappa$.
(3) *$V_\kappa \cap M = V_\kappa \cap \tilde{M}$ and $\pi_M \restriction (V_\kappa \cap M) = \text{id}$.*
(4) *$\tilde{f} = \pi_M(f)(\kappa)$, for $f \in {}^\kappa M \cap M$.*
(5) $\mathfrak{P}(\kappa) \cap M = \mathfrak{P}(\kappa) \cap \tilde{M}$.
(6) $x \in \mathfrak{P}(\kappa) \cap M \to (x \in U \leftrightarrow \kappa \in \pi_M(x))$.

Proof. (1) follows from 1.9, 1.10, and the absoluteness of building functions for the $S_\nu^{\tilde{U}}$-hierarchy. (2), (4), (5), (6) are standard for ultrapowers with normal ultrafilters.

(3) We show by induction on $\eta < \kappa$:

$$(*) \qquad \text{rn}(x) = \eta \to ((x \in M \leftrightarrow x \in \tilde{M}) \text{ and } (x \in M \to \pi_M(x) = x)).$$

Let $\eta < \kappa$ and assume $(*)$ holds for $\xi < \eta$. Let $\text{rn}(x) = \eta$ and $x \in M$. rn is uniformly Σ_1-definable over structures of the form J_β^B and over the universe (see [6], Lemma 2.2]). So $M \models \text{rn}(x) = \eta$, $\tilde{M} \models \text{rn}(\pi_M(x)) = \eta$, by (2), and $\text{rn}(\pi_M(x)) = \eta$.

$$\begin{aligned} \pi_M(x) &= \{y \in V_\eta \mid y \in \pi_M(x)\} = \{y \in V_\eta \cap \tilde{M} \mid y \in \pi_M(x)\} \\ &= \{y \in V_\eta \cap M \mid y \in \pi_M(x)\} = \{y \in V_\eta \cap M \mid \pi_M(y) \in \pi_M(x)\} \\ &= \{y \in V_\eta \cap M \mid y \in x\} = x. \end{aligned}$$

Also $x = \pi_M(x) \in \tilde{M}$.

Conversely let $x = \tilde{f} \in \tilde{M}$, $\mathrm{rn}(x) = \eta$. Then

$$\begin{aligned} x &= \{y \mid y \in x\} = \{y \in M \mid \pi_M(y) \in x\} \\ &= \{y \in M \mid \{\nu < \kappa \mid y \in f(\nu)\} \in u\} \\ &= \Big\{y \in \bigcup_{\nu<\kappa} f(\nu) \mid \{\nu < \kappa \mid y \in f(\nu)\} \in U\Big\} \in M. \qquad \square \end{aligned}$$

Definition. A premouse M is η-iterable ($\nu \leq \infty$) iff there is a system $\langle M_i, \pi_{ij}, \kappa_i, U_i \rangle_{i \leq j < \eta}$ such that for $i \leq j < \eta$:

(1) $M_0 = M$.

(2) M_i is a premouse at κ_i with measure U_i.

(3) $\pi_{ij}: M_i \prec_{\Sigma_1} M_j$, $\pi_{ii} = \mathrm{id} \restriction M_i$.

(4) The system π_{ij} commutes.

(5) $i+1 < \eta \rightarrow M_{i+1} = \tilde{M}_i$, $\pi_{i,i+1} = \pi_{M_i}$.

(6) $k < \eta$, $\mathrm{Lim}(k) \rightarrow \langle M_k, \pi_{ik} \rangle$ is the transitive direct limit of $\langle M_i, \pi_{ij} \rangle_{i \leq j < k}$.

If this system exists it is uniquely determined and is called the η-*iteration* of M. If M is ∞-iterable, we just call M *iterable* and we call the ∞-iteration of M the *iteration* of M.

Iterating Lemma 1.11 we obtain:

1.12. *Let M be η-iterable and let $\langle M_i, \pi_{ij}, \kappa_i, U_i \rangle_{i \leq j < \eta}$ be the η-iteration of M. Then for $i < j < \eta$:*

(1) $\pi_{ij}: M_i \prec_{\Sigma_1} M_j$, $\pi_{ij} \restriction \kappa_i = \mathrm{id}$, $\pi_{ij}(\kappa_i) = \kappa_j > \kappa_i$.

(2) $V_{\kappa_i} \cap M_i = V_{\kappa_i} \cap M_j$ *and* $\pi_{ij} \restriction (V_{\kappa_i} \cap M_i) = \mathrm{id}$.

(3) $\mathfrak{P}(\kappa_i) \cap M_i = \mathfrak{P}(\kappa_i) \cap M_j$.

(4) $x \in \mathfrak{P}(\kappa_i) \cap M_i \rightarrow (x \in U_i \leftrightarrow \kappa_i \in \pi_{ij}(x))$.

(5) $\{\kappa_i \mid i < \eta\}$ *is closed in* $\sup_{i<\eta} \kappa_i$ *as a set of ordinals, and* $\{\kappa_i \mid i < \eta\}$ *is cofinal in every cardinal θ such that* $\mathrm{card}(M) < \theta \leq \eta$.

(6) $k < \eta$, $\mathrm{Lim}(k) \rightarrow (x \in U_k \leftrightarrow \exists i < k\, \{\kappa_j \mid i \leq j < k\} \subseteq x)$.

(7) *Let θ be a regular cardinal such that* $\mathrm{card}(M) < \theta < \eta$. *Then $M = J_\beta^F$ for some β, where F is the closed unbounded filter on θ.*

1.12(7) allows us to 'compare' iterable premice:

1.13. Lemma. *Let M, N be iterable premice and θ a regular cardinal* $> \mathrm{card}(M)$, $\mathrm{card}(N)$. *Then either $M_\theta \in N_\theta$ or $M_\theta = N_\theta$ or $N_\theta \in M_\theta$.*

The structure of the iterates is present in the original premouse to a certain extent:

1.14. Lemma. *Let $\langle M_i, \pi_{ij}, \kappa_i, U_i \rangle_{i \leq j < \eta}$ be the η-iteration of M. Then for $j < \eta$:*

(1) $M_j = \{\pi_{0j}(f)(\kappa_{i_1}, \ldots, \kappa_{i_n}) \mid n < \omega, f: \kappa_0^n \rightarrow M_0, i_1 < \cdots < i_n < j\}$.

(2) *For ϕ a Σ_0-formula in the language for M_0, $x \in M_0$, and $i_1 < \cdots < i_n < j$:*

$M_j \models \phi[\pi_{0j}(x), \kappa_{i_1}, \ldots, \kappa_{i_n}]$ *iff*

$$\exists X \in U_0 \cap M_0 \ \forall x_1, \ldots, x_n \in X \ (x_1 < \cdots < x_n \to M_0 \models \phi[x, x_1, \ldots, x_n]).$$

(3) $\{\kappa_i \mid i < j\}$ *is a set of* Σ_0*-indiscernibles for* $\langle M_j, \langle \pi_{0j}(x) \mid x \in M_0 \rangle \rangle$.

(1) and (2) are proved by simultaneous induction on $j < \eta$. (3) is an immediate consequence of (2).

1.14(1) and (3) yield a criterion for iterability:

1.15. Lemma. *If M is ω_1-iterable, then it is iterable.*

This implies the following absoluteness property:

1.16. Lemma. *Let* ZF^{--} *be the system* ZF *without the power-set and the replacement axiom. Let A be a transitive model of* ZF^{--} *and $\omega_1 \subseteq A$. Then an iterable premouse in A is an iterable premouse in the universe.*

Proof. Let $\eta = A \cap \mathrm{On}$, $\eta \geq \omega_1$. Let $M \in A$ be an iterable premouse in A. ZF^{--} is strong enough to show that the iteration of M in A is the η-iteration of M in V. Thus, by 1.15, M is ∞-iterable. □

Remark. Note that the argument above depends on our specific definition of iterability. Since ZF^{--} is a very weak set theory many definitions which are equivalent in ZF are not equivalent in ZF^{--}.

1.17. Lemma. *Let $\sigma : \bar{M} \prec_{\Sigma_1} M$, where M is an η-iterable premouse and $\bar{M}$ is transitive. Then $\bar{M}$ is a premouse and η-iterable.*

This is [6, Lemma 3.24]. One obtains the iteration maps for $\bar{M}$ canonically from σ and the iteration maps for M.

Definition. For M a premouse at κ set $\mathrm{lp}(M) := M \cap V_\kappa$. $\mathrm{lp}(M)$ is called the *low part* of M.

Note that $\mathrm{lp}(M)$ is a class in M which is uniformly definable for all such M. By 1.12(2), $\mathrm{lp}(M)$ is preserved under iterations of M.

Definition. The *core model K* is the class

$$K := L \cup \bigcup \{\mathrm{lp}(M) \mid M \text{ is an iterable premouse}\}.$$

In ZF^-, this definition yields the core model defined by Dodd and Jensen. So $K \models$ "$V = K$", where "$V = K$" refers to our definition of K.

2. Condensation

Definition. For λ a cardinal in K set $K_\lambda := H_\lambda^K =$ the set of sets hereditarily of cardinality $<\lambda$ in K.

Recall that ZF^{--} was ZF without the power-set and replacement axiom.

2.1. Lemma. *Let λ be an uncountable cardinal in K. Then $K_\lambda \vDash$ "$V = K$" and $K_\lambda \vDash \mathrm{ZF}^{--}$.*

Proof. Work in K. Let $x \in K_\lambda$. $x \in \mathrm{lp}(M)$ for some iterable premouse M. Let X be the smallest substructure of M such that $\mathrm{TC}(\{x\}) \subseteq X$. Let $\sigma : \bar{M} \cong X \prec M$, where $\bar{M}$ is transitive. $\bar{M}$ is an iterable premouse by 1.17, and $x \in \mathrm{lp}(\bar{M})$. $\mathrm{card}(\bar{M}) < \lambda$ and for all $i < \lambda$ also $\mathrm{card}(\bar{M}_i) < \lambda$. Thus $\bar{M} \in K_\lambda$, $\bar{M}_i \in K_\lambda$ $(i < \lambda)$, and the λ-iteration of $\bar{M}$ is the iteration of $\bar{M}$ in the sense of K_λ. So $K_\lambda \vDash$ "$x \in K$". $\square$

2.2. Lemma. $K \vDash \mathrm{card}(K_\lambda) = \lambda$, *since* $K \vDash \mathrm{GCH}$.

2.3. Lemma. *Let $\lambda \geqslant \omega_1$ be a cardinal and assume there is no inner model with a measurable cardinal $<\lambda$. Then K_λ covers V_λ.*

Proof. If $0^\dagger$ exists, then some countable ordinal is measurable in an inner model. So $0^\dagger$ does not exist, and by the extended Covering Theorem, V is covered either by K, or some $L[U]$, or some $L[U, C]$, as in 1.3. The measurable of that $L[U]$ is $\geqslant\lambda$. For $\alpha < \lambda$: $\mathfrak{P}(\alpha) \cap K = \mathfrak{P}(\alpha) \cap L[U] = \mathfrak{P}(\alpha) \cap L[U, C]$, and hence K_λ covers V_λ. $\square$

2.4. Lemma. *Let λ be an uncountable cardinal and let K_λ cover V_λ. Then*

(1) *If $\rho < \lambda$ is a singular cardinal, then ρ is singular in K_λ and $(\rho^+)^K = \rho^+$.*

(2) *Let ρ be regular, $\omega_2 \leq \rho < \lambda$, $\alpha \in (\rho, \rho^+)$, and $\mathrm{cof}(\alpha) < \rho$. Then α is singular in K_λ.*

Proof. (1) Let $\rho < \lambda$ be a singular cardinal. Let X be a cofinal subset of ρ, $\mathrm{card}(X) < \rho$. Because K_λ covers V_λ, there is $Y \in K_\lambda$, $X \subseteq Y \subseteq \rho$ and $\mathrm{card}(Y) = \mathrm{card}(X) + \omega_1 < \rho$. Y is cofinal in ρ and $\mathrm{otp}(Y) < \rho$. Hence $K_\lambda \vDash$ "ρ is singular".

Assume that $\xi = (\rho^+)^K < \rho^+$. Then $\mathrm{cof}(\xi) < \rho$. Let X be a cofinal subset of ξ, $\mathrm{card}(X) < \rho$. There is $Y \in K_\lambda$ such that $X \subseteq Y \subseteq \xi$ and $\mathrm{card}(Y) = \mathrm{card}(X) + \omega_1 < \rho$. Hence $K_\lambda \vDash$ "$\xi = \rho^+$ is singular". Contradiction!

(2) Let $\rho < \lambda$ be regular, $\omega_2 \leq \rho < \alpha < \rho^+$, and $\mathrm{cof}(\alpha) < \rho$. Let X be a cofinal subset of α, $\mathrm{card}(X) < \rho$. There is $Y \in K_\lambda$, $X \subseteq Y \subseteq \alpha$ and $\mathrm{card}(Y) = \mathrm{card}(X) + \omega_1 < \rho$. Hence α is singular in K_λ. $\square$

2.5. Lemma. *Let A be a transitive model of $\mathrm{ZF}^{--} + V = K$, and let $\omega_1 \subseteq A$. Let λ be a cardinal $\geqslant A \cap \mathrm{On}$. Then $A \subseteq K_\lambda$.*

Proof. Let $x \in A$. If $x \in L$, then $x \in L_\lambda \subseteq K_\lambda$. So assume $x \notin L$. There is $M \in A$ such that, in A, M is an iterable premouse and $x \in \mathrm{lp}(M)$. Let $\eta = M \cap \mathrm{On} < \lambda$.

$$A \models \mathrm{card}(\mathrm{TC}(x)) \leq \mathrm{card}(M) \leq \eta.$$

Take $f \in A$ such that $f: \eta \to \mathrm{TC}(x)$ is surjective. Again using that $A \models V = K$, there is $N \in A$ such that, in A, N is an iterable premouse and x, $\mathrm{TC}(x)$, $f \in \mathrm{lp}(N)$. By 1.16, N is an iterable premouse. Hence $\mathrm{TC}(x), x, f \in K$, and $\mathrm{card}^K(\mathrm{TC}(x)) \leq \eta < \lambda$. So $x \in K_\lambda$ □

The following lemma is the tool which allows us to imitate condensation properties of L.

2.6. Lemma. *Let A be a transitive model of $\mathrm{ZF}^{--} + V = K$, and let $\omega_1 \subseteq A$. Let $M = J_\alpha^U$ be an iterable premouse at κ, and assume that κ is singular in A. Then* $\mathrm{lp}(M) \subseteq A$.

Proof. There is $f \in A$ such that $f: \gamma \to \kappa$ is cofinal and $\gamma < \kappa$. Let $\langle M_i, \pi_{ij} \rangle$ be the iteration of M.

Claim. $f \notin L$.

Proof. Assume $f \in L_\nu$. Then $f \in L_\nu \subseteq M_\nu$, and, by 1.12(3), $f \in M$. But, in M, κ is regular. Contradiction! □(Claim)

Since $A \models V = K$, there is $N \in A$, such that

$$A \models \text{"}N \text{ is an iterable premouse and } f \in \mathrm{lp}(N)\text{"}.$$

By Lemma 1.16, N is an iterable premouse and $f \in \mathrm{lp}(N)$. Let $\langle N_i, \rho_{ij} \rangle$ be the iteration of N, and let θ be a sufficiently large regular cardinal. $f \in N_\theta$ by 1.12(2). $f \notin M_\theta$ as in the proof of the Claim. Then 1.13 implies $M_\theta \in N_\theta$, and so

$$\mathrm{lp}(M) = V_\kappa \cap M = V_\kappa \cap M_\theta \subseteq V_\kappa \cap N_\theta = V_\kappa \cap N \subseteq A. \qquad \square$$

The following lemma brings this method into a form which we will use in the investigation of 'accessible' Jonsson cardinals.

2.7. Lemma. *Let A be a transitive model of $\mathrm{ZF}^{--} + V = K$, and let $\lambda = A \cap \mathrm{On}$ be a cardinal. Assume that for every C and every $\gamma < \lambda$ with the property, that C is closed unbounded in every cardinal $\mu \in (\gamma, \lambda]$, there exists $\kappa \in C$ which is singular in A. Then $A = K_\lambda$.*

Proof. By 2.5, $A \subseteq K_\lambda$, and we must show $K_\lambda \subseteq A$. Of course $J_\lambda \subseteq A$. So let $x \in K_\lambda$, $x \notin L$. By 2.1, there is an iterable premouse $M \in K_\lambda$ such that $x \in \mathrm{lp}(M)$, $\mathrm{card}(M) < \lambda$. Let $\langle M_i, \pi_{ij}, \kappa_i, U_i \rangle$ be the iteration of M. $\{\kappa_i \mid i < \lambda\}$ is closed cofinal in every cardinal $\mu \in (\mathrm{card}(M), \lambda]$. By our hypothesis there is $i < \lambda$ such that κ_i is singular in A. By 2.6, $\mathrm{lp}(M_i) \subseteq A$, and so $x \in \mathrm{lp}(M_i) \subseteq A$. □

2.8. Lemma. *Let U be an ultrafilter on $\mathfrak{P}(\alpha) \cap K$ and let λ be a cardinal such that $\lambda > \alpha$ and $\lambda \geq \omega_1$. Assume that the ultrapower $({}^{\alpha}K \cap K)/U$ is not well-founded. Then there are $f_0, f_1, \ldots \in K_\lambda$ such that, for $i < \omega$:*

$$\{\nu < \alpha \mid f_{i+1}(\nu) \in f_i(\nu)\} \in U.$$

Proof. There are $g_0, g_1, \ldots \in {}^{\alpha}K \cap K$ such that, for $i < \omega$, $\{\nu < \alpha \mid g_{i+1}(\nu) \in g_i(\nu)\} \in U$. Take a cardinal μ with $\{g_i \mid i < \omega\} \subseteq K_\mu$. By 2.2, there is a function $h \in K$ which maps μ onto K_μ. Let M be an iterable premouse with $h \in \mathrm{lp}(M)$. Take $X \prec M$ such that $\alpha \cup \{g_i \mid i < \omega\} \cup \{h\} \subseteq X$ and $\mathrm{card}(X) = \mathrm{card}(\alpha) < \lambda$. Let $\sigma : \bar{M} \cong X \prec M$, where $\bar{M}$ is transitive. Set $f_i = \sigma^{-1}(g_i)$, $(i < \omega)$, $\bar{h} = \sigma^{-1}(h)$, $\bar{\mu} = \sigma^{-1}(\mu) < \lambda$. $\bar{M}$ is an iterable premouse by 1.17, and so $f_i, \bar{h} \in K$ $(i < \omega)$. $\mathrm{TC}(f_i) \subseteq \mathrm{range}(\bar{h})$, thus $f_i \in K_\lambda$ for $i < \omega$. Note that $\sigma \restriction \alpha = \mathrm{id}$. For $i < \omega$:

$$\{\nu < \alpha \mid f_{i+1}(\nu) \in f_i(\nu)\} = \{\nu < \alpha \mid g_{i+1}(\nu) \in g_i(\nu)\} \in U. \qquad \square$$

2.9. Lemma. *Let λ be a cardinal $\geq \omega_2$ and let $\pi : K_\lambda \to K_\lambda$ be elementary with critical point α. Then there is an elementary map $\tilde{\pi} : K \to K$ with critical point α.*

Proof. Set $U := \{x \subseteq \alpha \mid x \in K \text{ and } \alpha \in \pi(x)\}$. U is a normal ultrafilter on $\mathfrak{P}(\alpha) \cap K$.

(1) $\quad ({}^{\alpha}K \cap K)/U$ is well-founded.

Proof. Assume not. According to 2.8, there are $f_0, f_1, \ldots \in K_\lambda$ such that, for $i < \omega$, $\{\nu < \alpha \mid f_{i+1}(\nu) \in f_i(\nu)\} \in U$. Then

$$\alpha \in \pi(\{\nu < \alpha \mid f_{i+1}(\nu) \in f_i(\nu)\} = \{\nu < \pi(\alpha) \mid \pi(f_{i+1}(\nu) \in \pi(f_i)(\nu)\}.$$

So, for $i < \omega$, $\pi(f_{i+1})(\alpha) \in \pi(f_i)(\alpha)$, an infinite descending $\in$-chain. Contradiction! $\square$(1)

Identify $({}^{\alpha}K \cap K)/U$ with its transitive collapse M. The canonical embedding from K into $({}^{\alpha}K \cap K)/U$ is a map $\tilde{\pi} : K \prec M$ with critical point α. By 1.4, $M = K$. $\square$

2.10. Lemma. *Let λ be a cardinal $\geq \omega_2$, and let $\pi : K_\lambda \to K_\lambda$ be a nontrivial elementary embedding. Then there is an inner model with a measurable cardinal $< \lambda$.*

Proof. π must move some ordinal since it moves the rank of some $x \in K_\lambda$. By 2.9, there is an elementary map $\tilde{\pi} : K \to K$ with critical point $< \lambda$. By 1.5, there is an inner model with a measurable cardinal $< \lambda$. $\square$

3. An inner model with a measurable cardinal

Theorem A1. *Assume λ is a Jonsson cardinal, and at least one of the following holds:*

(1) $\lambda = \rho^+$,

(2) $\lambda = \omega_\xi$ *and* $\xi < \lambda$,

(3) $\omega < \operatorname{cof} \lambda < \lambda$,

(4) λ *is regular but not weakly hyper-Mahlo.*

Then there is an inner model with a measurable cardinal $<\lambda$.

The following proof is analogous to Kunen's argument presented in the outset. In order to build the structure to which the Jonsson property will be applied, we have to assume that K_λ covers V_λ. By 2.3 this holds if there is no inner model with a measurable cardinal $<\lambda$. Thus we proceed by contradiction.

Proof of Theorem A1. Let λ be as above but assume that there is no inner model with a measurable cardinal $<\lambda$. Then K_λ covers V_λ (2.3). By 2.10 we get a contradiction if we prove:

Claim. There is a nontrivial elementary embedding $\pi: K_\lambda \to K_\lambda$.

The rest of this paragraph is devoted to the proof of this Claim. Note that $\lambda \geq \omega_\omega$. To demonstrate the main idea we consider the case (1), $\lambda = \rho^+$, separately although it is a subcase of (2).

Case 1: $\lambda = \rho^+$.

Case 1.1.: ρ *is singular.* By 2.4(1), $(\rho^+)^K = \rho^+ = \lambda$, and

$$(1) \qquad \forall \alpha \in (\rho, \lambda) \quad K_\lambda \models \text{“}\alpha \text{ is singular”}.$$

Let X be a Jonsson substructure of K_λ, i.e. $X \prec K_\lambda$, $\operatorname{card}(X) = \lambda$, $X \neq K_\lambda$. Let $\pi: A \cong X \prec K_\lambda$, A transitive. Of course, π is not the identity. It suffices to show $A = K_\lambda$. We use criterion 2.7. A is a transitive model of $\mathrm{ZF}^{--} + V = K$, and $A \cap \mathrm{On} = \lambda$. Let C be closed unbounded in λ. Take $\kappa \in C \cap (\rho, \lambda)$. $\pi(\kappa) \in (\rho, \lambda)$, and by (1), $K_\lambda \models$ “$\pi(\kappa)$ is singular”. Hence $A \models$ “κ is singular”. □ (Case 1.1)

Case 1.2: ρ *is regular.* By 2.4(2),

$$(2) \qquad \forall \alpha \in (\rho, \lambda) \quad (\operatorname{cof}(\alpha) \neq \rho \to K_\lambda \models \text{“}\alpha \text{ is singular”}).$$

Choose $g: \rho \times \lambda \to \lambda$ such that for all $\alpha < \lambda$ with $\operatorname{cof}(\alpha) = \rho$ the function $\beta \mapsto g(\beta, \alpha)$ maps ρ monotone cofinally into α. Let X be a Jonsson substructure of $\langle K_\lambda, g, \rho\rangle$, where ρ is considered to be a constant. Let $\pi: \langle A, \bar{g}, \bar{\rho}\rangle \cong X \prec \langle K_\lambda, g, \rho\rangle$, A transitive. It suffices to show $A = K_\lambda$, and we use 2.7:

$$A \models \mathrm{ZF}^{--} + V = K \quad \text{and} \quad A \cap \mathrm{On} = \lambda.$$

Let C be closed unbounded in λ. Take $\kappa \in C \cap (\bar{\rho}, \lambda)$ such that $\operatorname{cof}(\kappa) \neq \operatorname{cof}(\bar{\rho})$. We can do this: since $\lambda \geqslant \omega_2$ there exists at least two different cofinalities $<\lambda$.

$$\langle A, \bar{g}, \bar{\rho}\rangle \models \text{“the function } \beta \mapsto \bar{g}(\beta, \kappa) \text{ does } \textit{not} \text{ map } \bar{\rho} \text{ monotone cofinally into } \kappa\text{”.}$$

By the elementarity of π, the function $\beta \mapsto g(\beta, \pi(\kappa))$ does not map ρ monotone cofinally into $\pi(\kappa)$. By construction, $\operatorname{cof}(\pi(\kappa)) \neq \rho$, and $\pi(\kappa) \in (\rho, \lambda)$. By (2), $K_\lambda \models$ “$\pi(\kappa)$ is singular”, and so $A \models$ “κ is singular”. □ (Case 1.2)

Case 2: $\lambda = \omega_\xi$ *and* $\xi < \lambda$.

Define $c: \lambda \to \lambda$ by $c(\lambda) = \operatorname{card}(\lambda)$. Take $g: \lambda \times \lambda \to \lambda$ such that if $\alpha < \lambda$ and $\operatorname{cof}(\alpha) = \operatorname{card}(\alpha)$, then the function $\beta \mapsto g(\beta, \alpha)$ maps $\operatorname{card}(\alpha)$ monotone cofinally into α. Let X be a Jonsson substructure of $\langle K_\lambda, g, c\rangle$. Let $\pi: \langle A, \bar{g}, \bar{c}\rangle \cong X \prec \langle K_\lambda, g, c\rangle$, and A transitive. It suffices to show $A = K_\lambda$. We use 2.7: $A \models \mathrm{ZF}^{--} + V = K$ and $A \cap \mathrm{On} = \lambda$. Let $C \subseteq \lambda$, $\gamma < \lambda$ such that C is closed cofinal in every cardinal $\mu \in (\gamma, \lambda]$. The set $Z := \{\alpha < \lambda \mid \bar{c}(\alpha) = \alpha\}$ is a closed subset of λ of ordertype $\leqslant \xi$. Take a regular cardinal μ, such that $\omega_1 < \mu$, $\gamma < \mu$, $\xi < \mu$ and $\mu \leqslant \lambda$: If λ is a successor cardinal we can take $\mu = \lambda$; if λ is a limit cardinal, take $\mu =$ the first regular cardinal $> \omega_1, \gamma, \xi$. Z is bounded below μ. Let $\theta = \max(Z \cap \mu)$. Then $\pi(\theta)$ is a cardinal.

Case 2.1: $\pi(\theta)$ *is singular.* Take $\kappa \in (\theta, \mu) \cap C$. $\bar{c}(\kappa) = \theta$, hence $\operatorname{card}(\pi(\kappa)) = \pi(\theta)$, and $\pi(\theta) < \pi(\kappa) < (\pi(\theta))^+ \leqslant \lambda$. By 2.4(1), $(\pi(\theta))^+ = (\pi(\theta))^{+K}$. Then $K_\lambda \models$ “$\pi(\kappa)$ is singular”, and $A \models$ “κ is singular”. □ (Case 2.1)

Case 2.2: $\pi(\theta)$ *is regular.* Take $\kappa \in (\theta, \mu) \cap C$ such that $\operatorname{cof}(\kappa) \neq \operatorname{cof}(\theta)$. Since $\mu \geqslant \omega_2$, this is possible. $\bar{c}(\kappa) = \theta$, so $\pi(\kappa) \in (\pi(\theta), (\pi(\theta))^+)$.

$$\langle A, \bar{g}, \bar{c}\rangle \models \text{“the function } \beta \mapsto \bar{g}(\beta, \kappa) \text{ does not map } \theta = \bar{c}(\kappa) \text{ monotone cofinally into } \kappa\text{”.}$$

So $\beta \mapsto g(\beta, \pi(\kappa))$ does not map $\pi(\theta)$ montone cofinally into $\pi(\kappa)$. By the choice of $g: \operatorname{cof}(\pi(\kappa)) \neq \pi(\theta)$. By 2.4(2): $K_\lambda \models$ “$\pi(\kappa)$ is singular”. Hence $A \models$ “κ is singular”. □ (Case 2.2)

Case 3: $\omega < \operatorname{cof}(\lambda) < \lambda$.

There is $D \subseteq \lambda$ which is closed cofinal in λ, and every $\kappa \in D$ is a singular cardinal.

(3) $\langle K_\lambda, D\rangle \models$ “D is closed unbounded in the ordinals”.

Let X be a Jonsson substructure of $\langle K_\lambda, D\rangle$. Let $\pi: \langle A, \bar{D}\rangle \cong X \prec \langle K_\lambda, D\rangle$, A transitive. It suffices to show $A = K_\lambda$, and we use 2.7: $A \models \mathrm{ZF}^{--} + V = K$, and $A \cap \mathrm{On} = \lambda$. Let C be closed unbounded in λ. By (3), $\bar{D}$ is closed unbounded in λ. Take $\kappa \in C \cap \bar{D}$. $\pi(\kappa) \in D$. By 2.4(1), $K_\lambda \models$ “$\pi(\kappa)$ is singular”, and so $A \models$ “κ is singular”. □ (Case 3)

Case 4: λ *is regular but not weakly hyper-Mahlo.*

By Case 2 we may assume that $\lambda = \omega_\lambda$, hence λ is weakly inaccessible. For the moment, adjoin some distinguished set -1 as new least element to the ordinals. For $\alpha \in \mathrm{On}$ define its (weak) *Mahlo degree* $M(\alpha) \in [-1, \alpha]$ by

$$\begin{array}{lll} M(\alpha) \geqslant 0 & \text{iff} & \alpha \text{ is weakly inaccessible,} \\ M(\alpha) \geqslant \beta & \text{iff} & \text{for all } \gamma < \beta \text{ the set } \{\delta < \alpha \mid M(\delta) \geqslant \gamma\} \\ & & \text{is stationary in } \alpha, (\beta > 0). \end{array}$$

Drake [10, p. 116], calls an ordinal α with $M(\alpha) \geqslant \beta \geqslant 0$ *weakly Mahlo of kind* β. α is weakly inaccesssible iff $M(\alpha) \geqslant 0$. α is *weakly Mahlo* $\mathrm{iff}_{\mathrm{Def}} M(\alpha) \geqslant 1$. α is *weakly hyper-Mahlo* $\mathrm{iff}_{\mathrm{Def}} M(\alpha) = \alpha$. Hence $0 \leqslant M(\lambda) < \lambda$.

For every $\alpha \in \mathrm{On}$ with $0 \leqslant M(\alpha) < \alpha$ pick a closed unbounded set $D_\alpha \subseteq \alpha$ such that: $\gamma \in D_\alpha \rightarrow \gamma$ is a limit cardinal and $M(\gamma) < M(\alpha)$. Define $D \subset \lambda \times \lambda$ by $\langle \gamma, \alpha \rangle \in D \leftrightarrow \gamma \in D_\alpha$.

Let X be a Jonsson substructure of $\langle K_\lambda, D, D_\lambda, M \upharpoonright \lambda \rangle$. (We can of course assume that -1 is some element of K_λ like $\{\{0\}\}$.)

Let $\pi : \langle A, \bar{D}, \bar{D}_\lambda, \bar{M} \rangle \cong X \prec \langle K_\lambda, D, D_\lambda, M \upharpoonright \lambda \rangle$, A transitive. It suffices to show that $A = K_\lambda$ and we use 2.7 $A \models \mathrm{ZF}^{--} + V = K$ and $A \cap \mathrm{On} = \lambda$. By the elementarity of π we get:

(4) $\bar{D}_\lambda$ is closed unbounded in λ.

For $\alpha < \lambda$ set $\bar{D}_\alpha := \{\gamma < \alpha \mid \langle \gamma, \alpha \rangle \in \bar{D}\}$.

(5) Assume $0 \leqslant \bar{M}(\alpha) < \alpha < \lambda$. Then $\bar{D}_\alpha$ is closed unbounded in α, and if $\gamma \in \bar{D}_\alpha$, then $\bar{M}(\gamma) < \bar{M}(\alpha)$ and $\pi(\gamma)$ is a limit cardinal.

Let C be closed unbounded in λ. Do the following construction until it breaks down: $\alpha_0 := \lambda$, $\beta_0 := M(\lambda) < \lambda$. If α_n, β_n are constructed, put $\alpha_{n+1} :=$ the ω_{β_n+1}-st element of $C \cap \bar{D}_{\alpha_n}$, and $\beta_{n+1} := \bar{M}(\alpha_{n+1})$.

Obviously, α_1 and β_1 exist. Because $\pi(\alpha_1) \in D_\lambda$, $\beta_1 = \bar{M}(\alpha_1) \leqslant M(\pi(\alpha_1)) < M(\lambda) = \beta_0$. Hence

(6) $\mathrm{cof}(\alpha_1) = \omega_{\beta_0+1} > \omega_{\beta_1+1}$.

(7) Assume that α_n, β_n are constructed ($n \geqslant 1$), $\mathrm{cof}(\alpha_n) > \omega_{\beta_n+1}$, and $\bar{M}(\alpha_n) \geqslant 0$. Then $\alpha_{n+1}, \beta_{n+1}$ exist, and $\mathrm{cof}(\alpha_{n+1}) > \omega_{\beta_{n+1}+1}$. Also $\beta_{n+1} < \beta_n$.

Proof. Because $\beta_n = \bar{M}(\alpha_n) \geqslant 0$, $\bar{D}_{\alpha_n}$ is closed unbounded in α_n. $\mathrm{cof}(\alpha_n) > \omega_{\beta_n+1} \geqslant \omega_1$. Then $C \cap \bar{D}_{\alpha_n}$ is closed unbounded in α_n, and we find $\alpha_{n+1} :=$ the ω_{β_n+1}-st element of $C \cap \bar{D}_{\alpha_n}$. By (5), $\beta_{n+1} = \bar{M}(\alpha_{n+1}) < \bar{M}(\alpha_n) = \beta_n$, and so $\mathrm{cof}(\alpha_{n+1}) = \omega_{\beta_n+1} > \omega_{\beta_{n+1}+1}$. □ (7)

Because the β_n form a decreasing sequence of ordinals, the construction must stop. By (7), it only breaks down if $\beta_n = -1$. So there are α_n, β_n with $\beta_n = \bar{M}(\alpha_n) = -1$, $n \geqslant 1$. Set $\kappa = \alpha_n$. $\kappa \in C \cap \bar{D}_{\alpha_{n-1}}$. $\bar{M}(\kappa) = -1$ implies $M(\pi(\kappa)) = -1$. So $\pi(\kappa)$ is a singular limit cardinal $< \lambda$, and by 2.4 (1), $K_\lambda \models$ "$\pi(\kappa)$ is singular". Then $A \models$ "κ is singular". □

4. Proof of Theorem A2

To obtain Theorem A from Theorem A1 it is enough to prove:

Theorem A2. *Let λ be a Jonsson cardinal and assume there is an inner model with a measurable cardinal $<\lambda$. Then $0^\dagger$ exists.*

Proof. Let $\mu<\lambda$ be the smallest ordinal measurable in an inner model. Let U be a filter on μ such that $U\in L[U]$ and $L[U]\models$"U is a normal ultrafilter on μ". Set $\gamma:=\operatorname{card}(\mu)$. Let $f:\gamma\to\mu$ be surjective. Take $g:\omega_1^2\to\omega_1$ such that for $\zeta\in(0,\omega_1)$, $g''(\omega\times\{\zeta\})=\zeta$. (This function demonstrates that ω_1 is not a Jonsson cardinal.)

Let X be a Jonsson substructure of $\langle J_\lambda^U, f, g\rangle$. Let $\pi:\langle J_\lambda^{\bar U},\bar f,\bar g\rangle\cong X\prec\langle J_\lambda^U, f, g\rangle$. Set $\bar\mu=\pi^{-1}(\mu)$.

(1) $\quad J_\lambda^{\bar U}\models$"$\bar U$ is a normal ultrafilter on $\bar\mu$".

A condensation argument shows that $\mathfrak{P}(\bar\mu)\cap L[\bar U]\subseteq J_\lambda^{U}$. So

(2) $\quad L[\bar U]\models$"$\bar U$ is a normal ultrafilter on $\bar\mu$".

By the minimality of μ:

(3) $\quad \bar\mu=\mu$.

(4) $\quad \mu\cap X=\mu$.

Proof. Case 1: $\mu<\omega_1$. Then $\mu=f''\omega\subseteq X$.

Case 2: $\mu=\omega_1$.

$$\bar\mu=\pi^{-1}(\omega_1)=\{\pi^{-1}(\zeta)\mid\zeta\in X\cap\omega_1\}=\omega_1.$$

Hence $X\cap\omega_1$ is cofinal in ω_1, and

$$\omega_1=\bigcup\{\zeta\mid\zeta\in X\cup\omega_1\}=\bigcup\{g''(\omega\times\{\zeta\})\mid\zeta\in X\cap\omega_1\}\subseteq X.$$

Case 3: $\mu>\omega_1$. Assume that $\mu\cap X\neq\mu$. Then, since $\mu=f''\gamma$, $\gamma\cap X\neq\gamma$. Let α be the critical point of π. $\alpha<\gamma$. Let K_* be the term

$$\{x\mid x\in L\vee\exists\gamma,f,M\,(M\text{ is an iterable premouse}\wedge f:\xrightarrow{\text{onto}}\mathrm{TC}(\{x\}),$$
$$\wedge\,\gamma,f,\mathrm{TC}(\{x\})\in\mathrm{lp}(M))\}.$$

(a) $\quad K_\lambda=(K_*)^{J_\lambda^U}$.

Proof. By 1.6, $K\subseteq L[U]$. Hence $K_\lambda\subseteq H_\lambda^{L[U]}=J_\lambda^U$.

($\subseteq$) Let $x\in K_\lambda$. If $x\in L$, then obviously $x\in(K_*)^{J_\lambda}$. Otherwise there are $\gamma,f\in K_\lambda$ such that $f:\gamma\xrightarrow{\text{onto}}\mathrm{TC}(\{x\})$, and $\mathrm{TC}(\{x\})\in K_\lambda$. Because $K_\lambda\models V=K$, there is $M\in K_\lambda$ such that M is an iterable premouse and $\gamma,f,\mathrm{TC}(\{x\})\in\mathrm{lp}(M)$. Since $K_\lambda\subseteq J_\lambda^U$, $x\in(K_*)^{J_\lambda^U}$.

($\supseteq$) Let $x\in(K_*)^{J_\lambda^U}$. If $x\in L$, then obviously $x\in K_\lambda$. Otherwise there are $\gamma,f,M\in J_\lambda^U$ such that $J_\lambda^U\models$"M is an iterable premouse and $f:\gamma\xrightarrow{\text{onto}}\mathrm{TC}(\{x\})$ and $\gamma,f,\mathrm{TC}(\{x\})\in\mathrm{lp}(M)$". By absoluteness, in particular by 1.16, "$\cdots$" holds in the universe. Hence $x,\mathrm{TC}(\{x\}),f\in K$, and $x\in H_\lambda^K=K_\lambda$. $\square$ (a)

Analogously:

(b) $\quad K_\lambda=(K_*)^{J_\lambda^{\bar U}}$.

Since K_λ is definable in $J_\lambda^{\bar{U}}$ and J_λ^U, by the same term,

(c) $\pi \upharpoonright K_\lambda : K_\lambda \prec K_\lambda$ with critical point α.

By 2.9, there is an elementary embedding $\tilde{\pi}: K \to K$ with critical point α.

By 1.5, there is an inner model with a measurable cardinal $<\mu$: if $\gamma = \omega_1$, 1.5 yields an inner model with a measurable $\leqslant \gamma < \mu$, and if $\gamma \geqslant \omega_2$, 1.5 yields an inner model with a measurable $<\gamma \leqslant \mu$. This contradicts the minimality of μ. □ (4)

(5) $\pi \upharpoonright (\mu + 1) = \mathrm{id}$.

(6) $\bar{U} = U$.

Proof.

$$\bar{U} = \{x \in J_\lambda^{\bar{U}} \mid x \in \bar{U}\} = \{x \in J_\lambda^{\bar{U}} \mid \pi(x) = x \in U\} = U \cap J_\lambda^{\bar{U}}.$$

Then $J_\lambda^U = J_\lambda^{\bar{U}}$ and $\bar{U} = U \cap J_\lambda^{\bar{U}} = U$. □ (6)

(7) $\pi: J_\lambda^U \prec J_\lambda^U$ with some critical point $\alpha > \mu$.

Set $D := \{x \subseteq \alpha \mid x \in L[U] \text{ and } \alpha \in \pi(x)\}$. D is an ultrafilter on $\mathfrak{P}(\alpha) \cap L[U]$.

(8) $({}^\alpha L[U] \cap L[U])/D$ is well-founded.

Proof. Assume not. Then as in the proof of 2.8 we can show that there are $f_0, f_1, \ldots \in J_\lambda^U$ such that $\{\nu < \alpha \mid f_{i+1}(\nu) \in f_i(\nu)\} \in D$ for $i < \omega$. Then as in 2.9, $\pi(f_{i+1})(\alpha) \in \pi(f_i)(\alpha)$ for $i < \omega$. Contradiction! □ (8)

Hence the canonical embedding from $L[U]$ into $({}^\alpha L[U] \cap L[U])/D$ yields an elementary map $\tilde{\pi}: L[U] \to L[U]$ with critical point $\alpha > \mu$. So $0^\dagger$ exists. □

5. The weak Chang Conjecture

In this paragraph we give two equivalences of wCC(ρ^+). One, wCC*(ρ^+), will be used in the proofs of Theorems B and D; wCC*(ρ^+) is an apparent strengthening of wCC(ρ^+) which does not seem interesting in itself. The other equivalence, 5.1(3), is a statement about the ranks of functions which was also considered by Shelah [18]. The equivalence between wCC(ρ^+) and 5.1(2) has already been proved by Galvin (see [18], Section 35]).

Definition. Let $\lambda = \rho^+$ be successor cardinal. Let wCC*(λ) be the assertion: For every transitive structure $\mathfrak{A} = \langle A, \in, \ldots \rangle$ such that the language of $\mathfrak{A}$ is countable and $\lambda^+ \subseteq A \subseteq H_{\lambda^+}$, and for every $\xi < \lambda$ there exists an elementary map $\pi: \bar{\mathfrak{A}} \to \mathfrak{A}$ with the properties:

(1) $\bar{\mathfrak{A}}$ is transitive.

(2) There is $\alpha \in (\xi, \lambda)$ such that $\pi \upharpoonright \alpha = \mathrm{id}$ and $\pi(\alpha) = \lambda$. (So α is the critical point of π.)

(3) For every $\beta<\lambda$ there exists an elementary map $\pi':\mathfrak{B}\to\mathfrak{A}$ such that $\mathfrak{B}$ is transitive, $\bar{\mathfrak{A}}\prec\mathfrak{B}$, $\pi'\upharpoonright\bar{\mathfrak{B}}=\pi$, and $\mathfrak{B}\cap\mathrm{On}>\beta$.

So $\mathrm{wCC}^*(\lambda)$ is similar to $\mathrm{wCC}(\lambda)$ but $\mathrm{wCC}^*(\lambda)$ demands that the various substructures of $\mathfrak{A}$ required by $\mathrm{wCC}(\lambda)$ grow nicely out of one elementary substructure of $\mathfrak{A}$, namely out of $\pi''\bar{\mathfrak{A}}$. Also the critical point of π can be arbitrarily high below λ.

Definition. Let λ be a regular uncountable cardinal. For $f, g:\lambda\to\lambda$ define: $f<^* g$ iff $\{\nu<\lambda \mid f(\nu)<g(\nu)\}$ contains a closed unbounded subset of λ. $<^*$ is a well-founded partial order. For $f:\lambda\to\lambda$ define the *rank* of f, $\|f\|$, by

$$\|f\|=\sup\{\|g\|+1 \mid g<^* f\}\in\mathrm{On}.$$

Theorem 5.1. *Let $\lambda=\rho^+$ be a successor cardinal. Then the following are equivalent:*

(1) $\mathrm{wCC}(\lambda)$,

(2) $\forall f:\lambda\to\lambda\ \|f\|<\lambda^+$,

(3) $\mathrm{wCC}^*(\lambda)$.

Proof. (1)$\to$(2). Assume there is $f:\lambda\to\lambda$ with $\|f\|=\lambda^+$. It is well known that there is a sequence $\langle f_i \mid i<\lambda^+\rangle$ of functions from λ into λ such that $i<j<\lambda^+\to f_i<^* f_j<^* f$. For $i<j<\lambda^+$ there are closed unbounded sets $C_{ij}, D_i\subseteq\lambda$ such that $\nu\in C_{ij}\to f_i(\nu)<f_j(\nu)$ and $\nu\in D_i\to f_i(\nu)<f(\nu)$. We code the f_i, C_{ij}, D_i into relations on λ^+: Let

$$F=\{\langle\xi,\nu,i\rangle \mid f_i(\nu)=\xi\},$$

$$C=\{\langle\nu,i,j\rangle \mid i<j \text{ and } \nu\in C_{ij}\},$$

$$D=\{\langle\nu,i\rangle \mid \nu\in D_i\}.$$

Apply $\mathrm{wCC}(\lambda)$ to the structure $\mathfrak{A}=\langle\lambda^+,\in,f,F,C,D\rangle$. So there is $\alpha<\lambda$ such that for all $\beta<\lambda$ there is $X\prec\mathfrak{A}$ with $X\cap\lambda\subseteq\alpha$ and $\mathrm{otp}(X)>\beta$. Let $\beta=\sup\{f(\nu) \mid \nu\leqslant\alpha\}<\lambda$, and take $X\prec\mathfrak{A}$ with $X\cap\lambda\subseteq\alpha$, $\mathrm{otp}(X)>\beta$. Let $\bar{\alpha}=\sup(X\cap\lambda)\leqslant\alpha$. Since $X\prec\mathfrak{A}$, we have that for $i,j\in X$, $i<j$: C_{ij} and D_i are cofinal in $\bar{\alpha}$, and so $\bar{\alpha}\in C_{ij}$ and $\bar{\alpha}\in D_i$. Then $i\mapsto f_i(\bar{\alpha})$ is an order preserving map from X into $f(\bar{\alpha})$. But this is impossible since $\mathrm{otp}(X)>\beta\geqslant f(\bar{\alpha})$.

(2)$\to$(3). Let $\mathfrak{A}=\langle A,\in,\ldots\rangle$ be a transitive structure with a countable language and $\lambda^+\subseteq A\subseteq H_{\lambda^+}$. We may also assume that $\mathrm{card}(A)=\lambda^+$; let $h:\lambda^+\to A$. For every $\tau<\lambda^+$ take a surjective map $f_\tau:\lambda\to\tau$. Put these maps together in $F:\lambda\times\lambda^+\to\lambda^+$, $f(\xi,\tau)=f_\tau(\xi)$. We may as well assume that h, F are already functions of $\mathfrak{A}$, i.e. $\mathfrak{A}=\langle A,\in,h,F,\ldots\rangle$.

The following system of structures and embeddings is modelled after the one in Ketonen [13]. Let

$$E:=\{\tau<\lambda^+ \mid h''\tau \text{ is transitive and } h''\tau\prec\mathfrak{A} \text{ and } (h''\tau)\cap\mathrm{On}=\tau\}.$$

E is closed unbounded in λ^+. (Note that $A \subseteq H_{\lambda^+}$.) For $\tau \in E$ set $\mathfrak{A}^\tau = \mathfrak{A} \upharpoonright (h''\tau)$. For $\tau \in E$ let

$$C_\tau := \{\alpha < \lambda \mid (h \circ f''_\tau \alpha) \prec \mathfrak{A}^\tau \text{ and } (h \circ f''_\tau \alpha) \cap \mathrm{On} = \alpha\}.$$

Every such C_τ is closed unbounded in λ. Set $\mathfrak{A}^\tau_\alpha = \mathfrak{A}^\tau \mid (h \circ f''_\tau \alpha)$, $(\tau \in E,\ \alpha \in C_\tau)$.

(a) Let $\sigma, \tau \in E$, $\alpha \in C_\sigma \cap C_\tau$, and $\sigma \in \mathfrak{A}^\tau_\alpha$.
Then $\mathfrak{A}^\sigma_\alpha = \mathfrak{A}^\tau_\alpha \cap \mathfrak{A}^\sigma$.

Proof. Let $x \in \mathfrak{A}^\sigma_\alpha$. $x = h \circ f_\sigma(\xi)$ for some $\xi < \alpha$. x is definable in $\mathfrak{A}$ from the parameters ξ and σ. $\xi, \sigma \in \mathfrak{A}^\tau_\alpha$ and $\mathfrak{A}^\tau_\alpha \prec \mathfrak{A}$. Hence $x \in \mathfrak{A}^\tau_\alpha$. Conversely, let $x \in \mathfrak{A}^\tau_\alpha \cap \mathfrak{A}^\sigma$. $x = h(\eta)$ for some $\eta < \sigma$. $\mathfrak{A}^\tau_\alpha \models \exists \xi < \lambda\ \eta = f_\sigma(\xi)$, and since $\mathfrak{A}^\tau_\alpha \cap \lambda = \alpha$ there is $\xi < \alpha$ with $\eta = f_\sigma(\xi)$. Then $x = h \circ f_\sigma(\xi) \in \mathfrak{A}^\sigma_\alpha$. $\square$ (a)

In the situation of (a), $\mathfrak{A}^\sigma_\alpha$ is an $\in$-initial segment of $\mathfrak{A}^\tau_\alpha$, since $\mathfrak{A}^\sigma$ is transitive.

For $\tau \in E$, $\alpha \in C_\tau$, let $\pi^\tau_\alpha : \bar{\mathfrak{A}}^\tau_\alpha \cong \mathfrak{A}^\tau_\alpha \prec \mathfrak{A}^\tau \prec \mathfrak{A}$, where $\bar{\mathfrak{A}}^\tau_\alpha$ is transitive. By the remark following the proof of (a), we get immediately:

(b) Let $\sigma, \tau \in E$, $\alpha \in C_\sigma \cap C_\tau$, and $\sigma \in \mathfrak{A}^\tau_\alpha$.
Then $\pi^\tau_\alpha \upharpoonright \bar{\mathfrak{A}}^\sigma_\alpha = \pi^\sigma_\alpha$ and $\bar{\mathfrak{A}}^\sigma_\alpha \prec \bar{\mathfrak{A}}^\tau_\alpha$.

Of course, $\mathrm{card}(\bar{\mathfrak{A}}^\tau_\alpha) = \mathrm{card}(\alpha) < \lambda$. So for every $\tau \in E$ we can define a function $g_\tau : \lambda \to \lambda$ by

$$g_\tau(\alpha) = \begin{cases} \bar{\mathfrak{A}}^\tau_\alpha \cap \mathrm{On}, & \text{if } \alpha \in C_\tau, \\ 0, & \text{else.} \end{cases}$$

(c) If $\sigma, \tau \in E$, $\sigma < \tau$, then $g_\sigma <^* g_\tau$.

Proof. $\sigma = h \circ f_\tau(\nu)$ for some $\nu < \lambda$. $C := (C_\sigma \cap C_\tau) - (\nu + 1)$ is closed unbounded in λ. $\alpha \in C$ implies that $\sigma \in \mathfrak{A}^\tau_\alpha$. Then, by (a), $g_\sigma(\alpha) = \mathrm{otp}(\mathfrak{A}^\sigma_\alpha \cap \mathrm{On}) < \mathrm{otp}(\mathfrak{A}^\tau_\alpha \cap \mathrm{On}) = g_\tau(\alpha)$. $\square$ (c)

(d) For every $g : \lambda \to \lambda$ there is $\tau \in E$ such that
$\{\alpha \in C_\tau \mid g(\alpha) < g_\tau(\alpha)\}$ is stationary in λ.

Proof. If not, then $\tau \in E \to g_\tau \leq^* g$. Then, by (c), $\|g\| \geq \lambda^+$. Contradicting our hypothesis (2). $\square$ (d)

Now fix $\sigma \in E$. Define $g : C_\sigma \to (\lambda + 1)$ by

$$g(\alpha) = \sup\{\bar{\mathfrak{A}}^\tau_\alpha \cap \mathrm{On} \mid \alpha \in C_\tau \text{ and } \sigma \in \mathfrak{A}^\tau_\alpha\} \leq \lambda.$$

(e) $g(\alpha) = \lambda$ for cofinally many $\alpha < \lambda$.

Proof. Assume not. By (d), there is $\tau \in E$, $\sigma < \tau$, such that $S := \{\alpha \in C_\sigma \cap C_\tau \mid g(\alpha) < g_\tau(\alpha)\}$ is stationary in λ. Take $\alpha \in S$ such that $\sigma \in \mathfrak{A}^\tau_\alpha$. Then $g_\tau(\alpha) = \bar{\mathfrak{A}}^\tau_\alpha \cap \mathrm{On} \leq g(\alpha) < g_\tau(\alpha)$. Contradiction! $\square$ (e)

Now let $\xi < \lambda$ be given; we check $\mathrm{wCC}^*(\lambda)$ for $\mathfrak{A}$ and ξ. By (e), take $\alpha > \xi$ such that $g(\alpha) = \lambda$. We show that the elementary map $\pi^\sigma_\alpha : \bar{\mathfrak{A}}^\sigma_\alpha \to \mathfrak{A}$ satisfies $\mathrm{wCC}^*(\lambda)$. Of course $\pi \upharpoonright \alpha = \mathrm{id}$ and $\pi(\alpha) = \lambda$. If $\beta < \lambda$ is given then, since $g(\alpha) = \lambda$, we find $\tau \in E$

with $\alpha \in C_\tau$ and $\sigma \in \mathfrak{A}_\alpha^\tau$ such that $\bar{\mathfrak{A}}_\alpha^\tau \cap \mathrm{On} > \beta$. $\pi_\alpha^\tau : \bar{\mathfrak{A}}_\alpha^\tau \to \mathfrak{A}$ is elementary, and by (b), $\pi_\alpha \upharpoonright \bar{\mathfrak{A}}_\alpha^\sigma = \pi_\alpha^\sigma$, $\bar{\mathfrak{A}}_\alpha^\sigma \prec \bar{\mathfrak{A}}_\alpha^\tau$.

(3) $\to$ (1) is trivial. □

6. Proof of Theorem B1

Theorem B1. *Let $\lambda = \rho^+ \geqslant \omega_2$ be a successor cardinal and assume* wCC(λ). *Then there is an inner model with a measurable cardinal $<\lambda^+$.*

Proof. Assume not. Then, by 2.3, K_{λ^+} covers V_{λ^+}. Take $G: \lambda \times \lambda^+ \to \lambda^+$ such that if $\tau < \lambda^+$ and $\mathrm{cof}(\tau) = \lambda$, then the function $\xi \mapsto G(\xi, \tau)$ maps λ monotone cofinally into τ. Let $\mathfrak{A} := \langle K_{\lambda^+}, G\rangle$. By 2.4(2),

$$\tau \in (\lambda, \lambda^+) \to \mathfrak{A} \vDash \text{“}\xi \mapsto G(\xi, \tau) \text{ does not map } \lambda \text{ monotone cofinally into } \to \tau \text{ is singular”}.$$

Let $(*) = (*)_\lambda$ be the property: For every $\xi < \lambda$ there is an elementary map $\pi: \bar{K} \to K_{\lambda^+}$ such that:

(1) $\bar{K}$ is transitive.

(2) There is $\alpha \in (\xi, \lambda)$ such that $\pi \upharpoonright \alpha = \mathrm{id}$ and $\pi(\alpha) = \lambda$.

(3) For every iterable premouse M with $\mathrm{card}(M) < \lambda$ there is an elementary map $\pi' : K' \to K_{\lambda^+}$ such that $\bar{K} \prec K'$, $\pi = \pi' \upharpoonright \bar{K}$, and $\mathrm{lp}(M) \subseteq K'$.

Claim. $(*)_\lambda$ holds.

Proof. Let $\xi < \lambda$ and apply wCC*(λ) to $\langle K_{\lambda^+}, G\rangle$ and ξ. So there is an elementary embedding $\pi: \langle \bar{K}, \bar{G}\rangle \to \langle K_{\lambda^+}, G\rangle$ such that (1)–(3) in the definition of wCC*(λ) hold.

Let $\alpha \in (\xi, \lambda)$ be the critical point of π. Let M be an iterable premouse with $\mathrm{card}(M) < \lambda$. For $i < \infty$ let M_i at κ_i be the i-th iterate of M. Without loss of generality assume that $\kappa_0 \in (\alpha, \lambda)$. Put $\beta = \kappa_{\omega_1}$. Since $\lambda \geqslant \omega_2$ we have $\beta < \lambda$. Apply (3) of the definition of wCC*(λ) with this β: There is $\pi' : \langle K', G'\rangle \to \langle K_{\lambda^+}, G\rangle$ such that

$$\langle \bar{K}, \bar{G}\rangle \prec \langle K', G'\rangle, \qquad \pi = \pi' \upharpoonright \bar{K} \quad \text{and} \quad K' \cap \mathrm{On} > \beta.$$

Choose either $\tilde{\kappa} = \kappa_\omega$, $\tilde{M} = M_\omega$ or $\tilde{\kappa} = \kappa_{\omega_1}$, $\tilde{M} = M_{\omega_1}$ making sure that $\mathrm{cof}(\tilde{\kappa}) \neq \mathrm{cof}(\alpha)$. Then

$$\langle K', G'\rangle \vDash \text{“}\xi \mapsto G'(\xi, \tilde{\kappa}) \text{ does not map } \alpha \text{ monotone cofinally into } \tilde{\kappa}\text{”}.$$

$$\langle K_{\lambda^+}, G\rangle \vDash \text{“}\xi \mapsto G(\xi, \pi'(\tilde{\kappa})) \text{ does not map } \lambda \text{ montone cofinally into } \pi'(\tilde{\kappa})\text{”}.$$

$\pi'(\tilde{\kappa}) \in (\lambda, \lambda^+)$ and so $\langle K_{\lambda^+}, G\rangle \vDash$ “$\pi'(\tilde{\kappa})$ is singular”. Then $\langle K', G'\rangle \vDash$ “$\tilde{\kappa}$ is singular” and by 2.6, $\mathrm{lp}(M) \subseteq \mathrm{lp}(\tilde{M}) \subseteq K'$. □ (Claim)

Theorem B1 follows immediately from:

Lemma 6.1. $(*)_\lambda$, $\lambda > \omega_1$ *implies that there is an inner model with a measurable cardinal* $< \lambda$.

Note that this does not mean that we can yet strengthen the conclusion of Theorem B1 to "there is an inner model with a measurable $<\lambda$", since in the proof of $(*)$ we needed that K_{λ^+} covers V_{λ^+}. The proof of Lemma 6.1 is similar to the proof of Ketonen's theorem that a non-regular ultrafilter over ω_1 implies the existence of $0^{\#}$. (See e.g. Jech [11, p. 489].)

Proof of Lemma 6.1. If λ is inaccessible in K, put $\xi = 0$. Otherwise let ξ be the cardinal in K such that $\lambda = (\xi^+)^K$. Apply $(*) = (*)_\lambda$ to the structure K_{λ^+} and ξ. Let $\pi : \bar{K} \to K_{\lambda^+}$ be an elementary map satisfying (1)–(3) of $(*)$. Let $\alpha \in (\xi, \lambda)$ be the critical point of π.

(a) λ is inaccessible in K.

Proof. Otherwise $\lambda = (\xi^+)^K$. Let M be an iterable premouse with $\mathrm{card}(M) < \lambda$ such that $\mathrm{lp}(M)$ contains a surjective map $f : \xi \to \alpha$. By $(*)$ there is an elementary map $\pi' : K' \to K_{\lambda^+}$ such that K' is transitive, $\bar{K} \prec K'$, $\pi = \pi' \upharpoonright \bar{K}$, and $f \in \mathrm{lp}(M) \subseteq K'$. But then $\pi'(f)$ is a map from ξ onto λ, contrary to λ being a cardinal. $\square$ (a)

(b) $\bar{K} \subseteq K$.

Proof. Let $x \in \bar{K}$. If $x \in L \cap \bar{K} = (L)^{\bar{K}}$, then $x \in K$. So assume $x \notin (L)^{\bar{K}}$. Since $\bar{K} \models V = K$, there is $M \in \bar{K}$ such that

$$\bar{K} \models \text{"}M \text{ is an iterable premouse and } x \in \mathrm{lp}(M)\text{"}.$$

$K_{\lambda^+} \models$ "$\pi(M)$ is an iterable premouse", and by 1.16, $\pi(M)$ is an iterable premouse in the universe. $\pi \upharpoonright M : M \to \pi(M)$ is an elementary map. By 1.17, M is an iterable premouse. Hence $x \in K$. $\square$ (b)

(c) $\mathfrak{P}(\alpha) \cap K \in \bar{K}$.

Proof. By (a) there exists an iterable premouse M with $\mathrm{card}(M) < \lambda$ such that $\mathfrak{P}(\alpha) \cap K \in \mathrm{lp}(M)$. By $(*)$ there is an elementary $\pi' : K' \to K_{\lambda^+}$ such that K' is transitive, $\bar{K} \prec K'$, $\pi = \pi' \upharpoonright \bar{K}$ and $\mathfrak{P}(\alpha) \cap K \in \mathrm{lp}(M) \subseteq K'$. As in (b) we have $K' \subseteq K$. $\mathfrak{P}(\alpha) \cap K = (\mathfrak{P}(\alpha))^{K'}$, and $K' \models$ "$\mathfrak{P}(\alpha)$ exists". Then $\bar{K} \models$ "$\mathfrak{P}(\alpha)$ exists", $(\mathfrak{P}(\alpha))^{K'} = (\mathfrak{P}(\alpha))^{\bar{K}}$, and $\mathfrak{P}(\alpha) \cap K = (\mathfrak{P}(\alpha))^{\bar{K}} \in \bar{K}$. $\square$ (c)

It follows from (c) that $U := \{x \subseteq \alpha \mid x \in K \text{ and } \alpha \in \pi(x)\}$ is an ultrafilter on $\mathfrak{P}(\alpha) \cap K$.

(d) $({}^{\alpha}K \cap K)/U$ is well-founded.

Proof. Assume not. By 2.8 there are $f_0, f_1, \ldots \in K_\lambda$ such that for $i < \omega$: $\{\nu < \alpha \mid f_{i+1}(\nu) \in f_i(\nu)\} \in U$. Let M be an iterable premouse such that $f_0, f_1, \ldots \in \mathrm{lp}(M)$ and $\mathrm{card}(M) < \lambda$. By $(*)$, there is $\pi' : K' \to K_{\lambda^+}$ such that K' is transitive, $\bar{K} \prec K'$, $\pi = \pi' \upharpoonright \bar{K}$, and $\mathrm{lp}(M) \subseteq K'$. For $i < \omega$:

$$\alpha \in \pi(\{\nu \in \alpha \mid f_{i+1}(\nu) \in f_i(\nu)\}) = \{\nu < \lambda \mid \pi'(f_{i+1})(\nu) \in \pi'(f_i)(\nu)\}.$$

So $\pi'(f_0)(\alpha) \ni \pi'(f_1)(\alpha) \ni \cdots$ Contradiction! $\square$ (d)

The canonical embedding from K into $({}^{\alpha}K \cap K)/U$ yields an elementary map $\tilde{\pi} : K \to K$ with critical point α(see 1.4). By 1.5, there is an inner model with a measurable cardinal $<\lambda$. $\square$

We remark that for $\lambda=\omega_1$ the proof of Lemma 6.1 goes through unchanged with the exception of the last sentence. Hence 1.5 yields that there is an inner model with a measurable cardinal $\leqslant\lambda=\omega_1$.

Lemma 6.2. *If $(*)_{\omega_1}$ holds, then there is an inner model with a measurable cardinal $\leqslant\omega_1$.*

This will be used in the next section.

7. Proof of Theorem B

To prove Theorem B, it is now, by Section 6, enough to show:

Theorem B2. *Assume $\lambda=\rho^+$ is a successor cardinal and* wCC(λ) *holds. Assume further that there is an inner model with a measurable cardinal $<\lambda^+$. Then $0^\dagger$ exists.*

Proof. Let μ be the smallest ordinal which is measurable in an inner model. Take U such that $L[U]\models$"U is a normal ultrafilter on μ" and $U\in L[U]$.

(a) $(\mu^+)^{L[U]}<\lambda^+$.

Proof. Assume $(\mu^+)^{L[U]}=\lambda^+$. By 1.6, $\mathfrak{P}(\mu)\cap K=\mathfrak{P}(\mu)\cap L[U]$, and so $(\mu^+)^{L[U]}=(\mu^+)^K=\lambda^+$. Hence

(i) $\tau\in(\mu,\lambda^+)\to K_{\lambda^+}\models$"$\tau$ is singular".

(ii) $(*)_\lambda$, as defined in Section 6, holds.

Proof. Let $\xi<\lambda$. We apply wCC*(λ) to K_{λ^+} and ξ. Let $\pi:\bar K\to K_{\lambda^+}$ be an elementary map satisfying (1)–(3) of wCC*(λ). Let $\alpha\in(\xi,\lambda)$ be the critical point of π. Let $\bar\mu=\pi^{-1}(\mu)$. We show (3) of $(*)_\lambda$:Let M be an iterable premouse with card(M)$<\lambda$. Let κ be the measurable of M. Without loss of generality assume that $\bar\mu<\kappa<\lambda$. By (3) of wCC*(λ) there is $\pi':K'\to K_{\lambda^+}$ such that K' is transitive, $\pi=\pi'\restriction\bar K$, and $K'\cap\mathrm{On}>\kappa$. $K_{\lambda^+}\models$"μ is the greatest cardinal". Hence $K'\models$"$\bar\mu$ is the greatest cardinal", and $K'\models$"κ is singular". By 2.6, lp(M)$\subseteq K'$. $\square$ (ii)

By Lemma 6.1 and Lemma 6.2 there is an inner model with a measurable cardinal $\leqslant\lambda$. Hence $\mu=\lambda$.

Apply wCC(λ) to the structure K_{λ^+}: There is $\alpha<\lambda$ such for all $\beta<\lambda$ there is $X\prec K_{\lambda^+}$ with $X\cap\lambda\subseteq\alpha$ and otp($X\cap\lambda^+$)$>\beta$. Choose $\beta=(\alpha^+)^K$. Since μ is measurable in $L[U]$ it is inaccessible in K; hence $\beta<\lambda$. There is $X\prec K_{\lambda^+}$ with $X\cap\lambda\subseteq\alpha$ and otp($X\cap\lambda^+$)$>\beta$. Let $\pi:\bar K\cong X\prec K_{\lambda^+}$, where $\bar K$ is transitive. $\pi^{-1}(\lambda)\leqslant\alpha$ and $\bar K\cap\mathrm{On}>\beta$. The proof of (b) in the proof of Lemma 6.1 goes through word by word, and so $\bar K\subseteq K$. Since β is regular in K, $\bar K\models$"β is regular". So $K_{\lambda^+}\models$"$\pi(\beta)$ is regular", and $\lambda<\pi(\beta)<\lambda^+$. This contradicts (i) above. $\square$ (a)

By (a), $U\in L_{\lambda^+}[U]$. Pick $\xi<\lambda$ such that $\xi>\mu$, if $\mu<\lambda$, and such that $\xi>\eta$ if

$L_{\lambda^+}[U] \models$ "$\lambda = \eta^+$ and η is a cardinal". Apply wCC*(λ) to $L_{\lambda^+}[U]$ and ξ: There is an elementary embedding $\pi: \bar{A} \to L_{\lambda^+}[U]$ such that (1)–(3) of wCC*(λ) hold. Let $\alpha \in (\xi, \lambda)$ be the critical point of π. The condensation lemma for relative constructibility shows that $\bar{A} = L_\theta[\bar{U}]$ for some θ, where $\bar{U} = \pi^{-1}(U)$.

(b) $\mu < \lambda$.

Proof. Assume $\mu \geq \lambda$. Let $\bar{\mu} = \pi^{-1}(\mu)$. $\bar{\mu} < \mu$, and by the minimality of $\mu: L[\bar{U}] \models$ "$\bar{U}$ is not a normal ultrafilter on $\bar{\mu}$". A condensation argument for the $L_\nu[\bar{U}]$-hierarchy shows that already for some $\beta < \lambda$: $L_\beta[\bar{U}] \models$ "$\bar{U}$ is not a normal ultrafilter on $\bar{\mu}$". By (3) of wCC*(λ) there is an elementary map $\pi': A' \to L_{\lambda^+}[U]$ such that A' is transitive, $L_\theta[\bar{U}] \prec A'$, $\pi = \pi' \restriction L_\theta[\bar{U}]$ and $A' \cap \mathrm{On} > \beta$. Another condensation argument shows that $A' = L_{\theta'}[\bar{U}]$ for some $\theta' > \beta$. Since $\theta' > \beta$, $L_{\theta'}[\bar{U}] \models$ "$\bar{U}$ is not a normal ultrafilter on $\bar{\mu}$". π' is elementary, and $L_{\lambda^+}[U] \models$ "U is not a normal ultrafilter on μ". Contradiction! □ (b)

By (b), $\alpha > \xi > \mu$. So $\bar{\mu} = \mu$ and $\pi \restriction \mu = \mathrm{id} \restriction \mu$. Moreover $L_\theta[\bar{U}] = L_\theta[U]$.

(c) λ is inaccessible in $L[U]$.

Proof. Assume not. There is η such that η is a cardinal in $L[U]$ and $\lambda = (\eta^+)^{L[U]}$. Since $\eta < \alpha < \lambda$, $L[U] \models$ "α is singular." Take $\beta < \lambda$ such that $L_\beta[U] \models$ "α is singular". By (3) of wCC*(λ) there is an elementary map $\pi': L_{\theta'}[U] \to L_{\lambda^+}[U]$ such that $L_\theta[U] \prec L_{\theta'}[U]$, $\pi = \pi' \restriction L_\theta[U]$ and $\theta' > \beta$. $L_{\theta'}[U] \models$ "α is singular", and since π' is elementary, $L_{\lambda^+}[U] \models$ "λ is singular". Contradiction! □ (c)

(d) $\mathfrak{P}(\alpha) \cap L[U] \in L_\theta[U]$.

Proof. By (c), $\mathfrak{P}(\alpha) \cap L[U] \in L_\beta[U]$ for some $\beta < \lambda$. By (3) of wCC*(λ) there is $\pi': L_{\theta'}[U] \to L_{\lambda^+}[U]$ with $L_\theta[U] \prec L_{\theta'}[U]$ and $\mathfrak{P}(\alpha) \cap L[U] \in L_{\theta'}[U]$. But then $\mathfrak{P}(\alpha) \cap L[U] \in L_\theta[U]$. □ (d)

Let $D = \{x \subseteq \alpha \mid x \in L[U] \text{ and } \alpha \in \pi(x)\}$. By (d), D is an ultrafilter on $\mathfrak{P}(\alpha) \cap L[U]$.

(e) $({}^\alpha L[U] \cap L[U])/D$ is well-founded.

Proof. Assume not. Then there is a sequence $f_0, f_1, \ldots$, which is descending in $({}^\alpha L[U] \cap L[U])/D$. Using condensation arguments similar to those in the proof of 2.8, we may assume that $f_0, f_1, \ldots \in L_\beta[U]$ for some $\beta < \lambda$. By (3) of wCC*(λ), there is an elementary embedding $\pi': L_{\theta'}[U] \to L_{\lambda^+}[U]$ such that $L_\theta[U] \prec L_{\theta'}[U]$, $\pi = \pi' \restriction L_\theta[U]$, and $\theta' > \beta$. $f_0, f_1, \ldots \in L_{\theta'}[U]$. $\{\nu < \alpha \mid f_{i+1}(\nu) \in f_i(\nu)\} \in D$, for $i < \omega$. So

$$\alpha \in \pi(\{\nu < \alpha \mid f_{i+1}(\nu) \in f_i(\nu)\}) = \pi'(\{\nu < \alpha \mid f_{i+1}(\nu) \in f_i(\nu)\}) \\ = \{\nu < \lambda \mid \pi'(f_{i+1})(\nu) \in \pi'(f_i)(\nu)\},$$

for $i < \omega$. In other words $\cdots \in \pi'(f_1)(\alpha) \in \pi'(f_0)(\alpha)$. Contradiction! □ (e)

Now the canonical embedding of $L[U]$ into its ultrapower by D yields a nontrivial elementary embedding of $L[U]$ into $L[U]$ with critical point $> \mu$. So $0^\dagger$ exists. □

8. Proof of Theorem C

We first give another definition concerning the partition relation defined in the introduction.

Definition. Let $\omega\tau = \tau$, $\tau \neq 0$. κ is called *$<\tau$-Erdös* iff for all regressive functions $f : [C]^{<\omega} \to \kappa$, C club in κ, there is a homogeneous sequence $\langle X_\alpha \mid \alpha < \tau \rangle$.

It is well known that for limit ordinals α the least κ which satisfies $\kappa \to (\alpha)_2^{<\omega}$ is α-Erdös. On the other hand Baumgartner and Galvin have shown (see [2]) that the least κ which satisfies $\kappa \to (<\omega_1)_2^{<\omega}$ is singular. A straightforward generalization of their argument shows, that the least κ which is almost $<\omega_1$-Erdös is not Mahlo. But it is easy to see that every $<\tau$-Erdös cardinal is Mahlo.

Silver proved the consistency of Chang's conjecture starting with an ω_1-Erdös cardinal. A rather straightforward adaption of his method gives the consistency of $\mathrm{wCC}(\omega_1)$ starting with a $<\omega_1$-Erdös cardinal. For the sharper result presented below we have to be slightly more careful.

We first mention two simple facts.

Fact 1. *Let κ be almost $<\tau$-Erdös. Then κ is strongly inaccessible.*

Fact 2. *For $0 < \nu < \omega_2$ let g_ν map ω_1 onto ν and let $g_0 = \emptyset$. Define $f_\nu : \omega_1 \to \omega_1$ by $f_\nu(\alpha) = \mathrm{otp}(g_\nu''\alpha)$. Let $f : \omega_1 \to \mathrm{On}$. Then*

$$\|f\| \geq \omega_2 \quad \textit{iff} \quad \textit{for all } \nu < \omega_2\text{: } f_\nu <^* f.$$

A proof of Fact 2 can be found in [1].

Theorem C. *Let M be a countable transitive model of* ZFC *and let κ be almost $<\omega_1$-Erdös in M. Then there is a generic extension N of M such that $N \models \mathrm{wCC}(\omega_1)$.*

Proof. By standard methods one can show that κ remains almost $<\tau$-*Erdös* ($\tau = \omega_1^M$) in any generic extension obtained by a set of conditions Q s.t. $|Q| < \kappa$. So we may assume w.l.o.g. that MA_{ω_1} is true in M. We work in M from now on. Let $\mathbf{P}$ be the Silver collapse for making κ to ω_2. So the elements of P are functions $p : A \to \kappa$ s.t.

(i) $A \subseteq \omega_1 \times \kappa$, $p(\alpha, \beta) < \beta$,

(ii) $|\{\beta \mid \exists\alpha \langle \alpha, \beta \rangle \in \mathrm{dom}\, p\}| \leq \omega_1$,

(iii) $\sup\{\alpha \mid \exists\beta \langle \alpha, \beta \rangle \in \mathrm{dom}\, p\} < \omega_1$,

and we have $p \leq q$ iff $p \supseteq q$.

For $B \subseteq \kappa$ set

$$P(B) = \{p \in P \mid \mathrm{dom}\, p \subseteq \omega_1 \times B\}.$$

Clearly $\mathbf{P}$ is ω_1-closed and it is known that $\mathbf{P}$ satisfies the κ-a.c. We show that wCC(ω_1) is forced to be true. We verify the equivalent version given in Theorem 5.1(2).

So it suffices to show:

Claim. Let $p \Vdash$"$\dot{f}: \omega_1 \to \kappa$ and $\|\dot{f}\| \geqslant \kappa$". Then there is some $\xi < \omega_1$ and $q \leqslant p$ such that $q \Vdash \dot{f}(\xi) \geqslant \omega_1$.

Let $\dot{G} = \langle \dot{g}_\nu \mid \nu < \kappa \rangle$ be the 'canonical' generic sequence of collapsing maps adjoined by $\mathbf{P}$. Since $\mathbf{P}$ satisfies the κ-a.c. there is some $\rho < \kappa$ such that $\dot{f} \in M[\dot{G} \upharpoonright \rho]$. We may assume that $p \in P(\rho)$, too.

Now define $\dot{f}_\nu: \omega_1 \to \omega_1$ by $\dot{f}_\nu(\alpha) = \mathrm{otp}(\dot{g}_\nu''\alpha)$. So we have $\dot{f}_\nu \in M[\dot{g}_\nu]$.

In M define $h_\nu: \omega_1 \times P \to \omega_1$ by $h_\nu(\alpha, q) = \mathrm{otp}\{q(\beta, \nu) \mid \beta < \alpha\}$. We clearly have

(1) $\quad q \Vdash \dot{f}_\nu(\alpha) \geqslant h_\nu(\alpha, q)$.

Applying Fact 2 we get

(2) $\quad p \Vdash \forall \nu < \omega_2 \ \dot{f}_\nu <^* \dot{f}$.

We now strengthen this to

(3) $\quad p \Vdash \forall \nu < \omega_2 \ M[\dot{G} \upharpoonright \rho \cup \{\nu\}] \vDash \dot{f}_\nu <^* \dot{f}$.

Proof. Assume not. So let G be $\mathbf{P}$-generic over M such that $p \in G$ and the statement is false in $M[G]$. Let f_ν, f denote the G-interpretation of $\dot{f}_\nu, \dot{f}$. So there is some $\nu < \omega_2$ s.t.

$$E = \{\alpha < \omega_1 \mid f(\alpha) \leqslant f_\nu(\alpha)\}$$

is stationary in $M[G \upharpoonright \rho \cup \{\nu\}]$. Set $A = \rho \cup \{\nu\}$. So we have $M[G] = M[G \upharpoonright A][G{<}\kappa - A]$. By the product theorem $G \upharpoonright \kappa - A$ is $P(\kappa - A)$-generic over $M_1 = M[G \upharpoonright A]$. But $P(\kappa - A)$ is ω_1-closed in M_1, since M and M_1 have the same ω-sequences from $P(\kappa - A)$. So by a well-known fact E remains stationary in $M_1[G \upharpoonright \kappa - A] = M[G]$. This contradicts (2). $\square$ (3)

So there is a sequence $\langle \dot{C}_\nu \mid \nu < \kappa \rangle$ such that $\dot{C}_\nu \in M[\dot{G} \upharpoonright \rho \cup \{\nu\}]$ and

(4) $\quad p \Vdash \forall \nu < \omega_2 \, (\dot{C}_\nu$ club in ω_1 and $\forall \alpha \,\dot{\in}\, \dot{C}_\nu \, \dot{f}_\nu(\alpha) \,\dot{<}\, \dot{f}(\alpha))$.

Hence we especially have

(5) $\quad$ Let $\nu < \kappa$ and $\alpha < \omega_1$. Set
$D = \{q \in P(\rho \cup \{\nu\}) \mid \exists \gamma > \alpha \ q \Vdash \gamma \in \dot{C}_\nu\}$.
Then D is dense in $P(\rho \cup \{\nu\})$ below p.

Now define a relation $R \subseteq P \times \omega_1 \times \kappa$ by

$$\langle q, \alpha, \nu \rangle \in R \quad \text{iff} \quad q \Vdash \alpha \in \dot{C}_\nu.$$

Choose some $A \subseteq \kappa$ s.t. $V_\kappa = L_\kappa[A]$ and let $\mathfrak{A} = \langle L_\kappa[A], \in, A, \{p, \rho\}, P, R \rangle$. Note that $\mathfrak{A}$ has definable Skolem functions. Since κ is almost $<\omega_1$-Erdös we immediately get

(6) There is a sequence $\langle I_\alpha \mid \alpha<\omega_1\rangle$ such that
(i) $I_\alpha \subseteq \kappa-\rho$, $\quad$ otp $I_\alpha=\omega(1+\alpha)$,
(ii) for all $\alpha, \beta<\omega_1 \quad H_\alpha \cap (P(\rho)\cup\omega_1)=H_\beta\cap(P(\rho)\cup\omega_1)$ where $H_\alpha=$ the Skolem hull of I_α in $\mathfrak{A}$

An easy argument shows that we also may assume w.l.o.g. that

(iii) for all $\alpha<\beta<\omega_1 \quad I_\alpha\cap I_\beta=\emptyset$.

Now set $Q_\alpha=H_\alpha\cap\{q\in P(\rho\cup I_\alpha)\mid q\leqslant p\}$ and $\bar{Q}=Q_0\cap P(\rho)$. Note that $\bar{Q}=Q_\alpha\cap P(\rho)$ for all $\alpha<\omega_1$ by (6) (iii). Let Q be the closure of $\bigcup_{\alpha<\omega_1} Q_\alpha$ under finite unions of compatible elements. We clearly have

(7) Let $D\subseteq Q_\alpha$ be predense in Q_α. Then D is predense in Q.

A simple Δ-system argument yields

(8) Q satisfies the countable antichain condition.

Now let $\xi=\omega_1\cap H_0$. So $\xi<\omega_1$ and we have $\xi=\omega_1\cap H_\alpha$ for all $\alpha<\omega_1$ by (6) (ii). Obviously, we have

(9) For all $q\in Q \quad \operatorname{dom} q\subseteq \xi\times\kappa$.

Now let $\xi=\sup_n \xi_n$ where $\xi_n<\xi$. For $n<\omega$ and $\tau\in I_\alpha$ set

$$D(n,\tau)=\{q\in Q_\alpha \mid \exists\gamma\in\xi-\xi_n \;\; q\Vdash\gamma\in\dot{C}_\tau\}$$

It follows from (5) that

(10) Let $n<\omega, \tau\in I_\alpha$. Then $D(n,\tau)$ is dense in Q_α (hence predense in Q by (7)).

For $\bar\tau,\tau\in I_\alpha$, $\bar\tau<\tau$, set $\bar{D}(\bar\tau,\tau)=\{q\in Q_\alpha \mid \exists\gamma\; q(\gamma,\tau)=\bar\tau\}$. Clearly, $\bar{D}(\bar\tau,\tau)$ is dense in Q_α, hence predense in Q. Now let

$$\mathcal{F}=\{D(n,\tau)\mid n<\omega, \tau\in I_\alpha\}\cup\{\bar{D}(\bar\tau,\tau)\mid \bar\tau,\tau\in I_\alpha, \bar\tau<\tau\}.$$

By MA_{ω_1} there is some filter $\bar{G}\subseteq Q$ which meets all $D\in\mathcal{F}$. Eventually, we set $q=\bigcup\bar{G}$. Then $q\in P$, since $|q|\leqslant\omega_1$ and $\operatorname{dom} q\subseteq\xi\times\kappa$ by (9). We now show that q satisfies our crucial claim. Clearly, $q\leqslant p$. So it suffices to show:

(11) $q\Vdash\dot{f}(\xi)\geqslant\omega_1$.

Proof. Let $\tau\in I_\alpha$ for some $\alpha<\omega_1$. Since $\bar{G}$ meets $D(n,\tau)$ for all $n<\omega$ we get by (4) that $q\Vdash\xi\in C_\tau$. So by the other part of (4) we only have to show that for all $\delta<\omega_1$ there is some $\tau\in\bigcup_{\alpha<\omega_1} I_\alpha$ such that $q\Vdash\dot{f}_\tau(\xi)\geqslant\delta$. So let $\delta<\omega_1$. Let $\delta<\alpha<\omega_1$ and choose $\tau\in I_\alpha$ such that $\operatorname{otp}(I_\alpha\cap\tau)\geqslant\delta$. Since $\bar{G}$ meets all the $\bar{D}(\bar\tau,\tau)$ $(\bar\tau\in I_\alpha\cap\tau)$ we have $h_\tau(\xi,q)\geqslant\delta$. But then $q\Vdash f_\tau(\xi)\geqslant\delta$ by (1). $\square$

9. Proof of Theorem D

The fundamental result about the relationship between partition cardinals and K is Jensen's indiscernibles lemma (see [9]). To state this we first need a definition.

Definition. (a) Let $\mathfrak{A} = \langle L_\kappa[A], \in, A, \ldots \rangle$ where $A \subseteq \kappa$ and set $\mathfrak{A}_\beta = \mathfrak{A} \restriction L_\beta[A]$ for $\beta < \kappa$. $I \subseteq \kappa$ is a *good* set of indiscernibles for $\mathfrak{A}$ (or *good for* $\mathfrak{A}$) iff for all $\gamma \in I$:

(G1) $\mathfrak{A}_\gamma \prec \mathfrak{A}$,

(G2) $I - \gamma$ is a set of indiscernibles for $\langle \mathfrak{A}, (\xi)_{\xi<\gamma} \rangle$.

(b) Let $\mathfrak{A} = \langle L_\kappa[A], \in, A, B_0, \ldots, B_n \rangle$ where $A \subseteq \kappa$ and $B_i \subseteq L_\kappa[A]$. Then $\mathfrak{A}$ is *amenable* iff $B_i \cap x \in L_\kappa[A]$ for all $x \in L_\kappa[A]$ and $i \leq n$.

Jensen's Indiscernibles Lemma. *Let $\mathfrak{A}$ be amenable such that $\mathfrak{A} \models V = K$. Let I be good for $\mathfrak{A}$ such that* $\mathrm{cf}(\mathrm{otp}(I)) > \omega$. *Then there is $I' \in K$ such that I' is good for $\mathfrak{A}$ and $I \subseteq I'$.*

Actually, in [9] slightly stronger assumptions about $\mathfrak{A}$ are made. Namely it is stated that $\mathfrak{A} = \langle K_\kappa, \ldots \rangle$. But the proof given there shows that only $\mathfrak{A} \models V = K$ is needed. As an easy consequence we get that for $\mathrm{cf}(\tau) > \omega_1$ every $<\tau$-Erdös cardinal is $<\tau$-Erdös in K. But the interesting case for us is $\tau = \omega_1$. Here Jensen helped us by showing: Let κ be $<\omega_1$-Erdös. Set $\tau = \omega_1^K$. Then κ is $<\tau$-Erdös in K.

Of course, by the remarks made in the last section this does not immediately give the analogous result for almost $<\omega_1$-Erdös cardinals. But the proof below shows that it is true.

Theorem D. *Assume* $\mathrm{wCC}(\omega_1)$. *Let $\kappa = \omega_2$ and $\tau = \omega_1^K$. Then κ is almost $<\tau$-Erdös in K.*

Proof. We distinguish two cases.

Case 1: $(*)_{\omega_1}$ *holds.*

Set $\rho = \omega_1$. By Lemma 6.2 there is an inner model with a measurable cardinal $\leq \rho$. But then every cardinal bigger than $(\rho^+)^K$ is measurable in an inner model. But the arguments of Section 6, implicitly contain a proof that κ is inaccessible in K (since $\mathrm{wCC}(\omega_1)$ holds). So κ is measurable in an inner model. Hence κ is even Ramsey in K. $\square$ (Case 1)

Case 2: $(*)_{\omega_1}$ *does not hold.*

We first show:

Claim 1: Let $g \in K$ such that $g : [\kappa]^{<\omega} \to \eta$ where $\eta < \kappa$. Then there exists a sequence $\langle X_\alpha \mid \alpha < \omega_1 \rangle$ (in V) which is homogeneous for g.

Proof. Let $\xi < \omega_1$ such that $(*)_{\omega_1}$ fails for ξ. Let $\mathfrak{A} = \langle K_\kappa, \in, g, \{\eta\} \rangle$. By Theorem

5.1, wCC*(ω_1) holds. So there is some elementary map $\sigma:\bar{\mathfrak{A}}\to\mathfrak{A}$ such that

(i) $\bar{\mathfrak{A}}$ is transitive (and countable).

(ii) There is $\alpha\in(\xi,\omega_1)$ such that $\sigma(\alpha)=\omega_1$, $\sigma\restriction\alpha=\mathrm{id}\restriction\alpha$.

(iii) For every $\beta<\omega_1$ there exists an elementary map $\sigma_\beta:\mathfrak{B}_\beta\to\mathfrak{A}$ such that $\mathfrak{B}_\beta$ is transitive, $\bar{\mathfrak{A}}<\mathfrak{B}_\beta$, $\sigma_\beta\restriction\bar{\mathfrak{A}}=\sigma$, and $\mathfrak{B}_\beta\cap\mathrm{On}>\beta$.

Let $\sigma(\bar\eta)=\eta$ and $\mathfrak{B}_\beta=\langle B_\beta,\in,g_\beta,\{\bar\eta\}\rangle$ for $\beta<\omega_1$. Since $(*)_{\omega_1}$ fails for ξ, there is some countable iterable premouse M such that $\mathrm{lp}(M)\not\subset\mathfrak{B}_\beta$ for all $\beta<\omega_1$. Clearly we may assume w.l.o.g. that M is a premouse at a $\gamma>\bar\eta$. Now let $\langle M_i,\pi_{ij},\gamma_i,U_i\rangle_{i\le j<\omega_1}$ be the ω_1-iteration of M, hence $\omega_1=\sup\{\gamma_i\mid i<\omega\}$. For $i<\omega_1$ set $\bar g_i=g_{\gamma_i}\restriction[\gamma_i]^{<\omega}$. We first show:

(1) $\quad\bar g_i\in M_i$.

Proof. Set $\gamma=\gamma_i$, $\bar g=\bar g_i$, $\bar M=M_i$, $\bar\sigma=\sigma_\gamma$. By considering $\bar\sigma$ we see that $\bar g\in\mathfrak{B}_\gamma$ and $\mathfrak{B}_\gamma\models V=K$. Hence there is some $N\in\mathfrak{B}_\gamma$ such that $\bar g\in\mathrm{lp}(N)$ and $\mathfrak{B}_\gamma\models$"$N$ is an iterable premouse". Applying $\bar\sigma$ again we see that N is really iterable. So by the results in Section 1 we only have to show that $N_{\omega_1}\subseteq\bar M_{\omega_1}$. But this is clear since otherwise we would get $\bar M_{\omega_1}\subseteq N_{\omega_1}$, hence $\mathrm{lp}(M)\subseteq N\subseteq\mathfrak{B}_\gamma$. □ (1)

Now by 1.14(1) for each $j<\omega_1$ there are some $x_j\in M$, and $\vec\rho_j\in\{\gamma_i\mid i<j\}^{<\omega}$ such that $\bar g_j=\pi_{0j}(x_j)(\vec\rho_j)$. By Fodor there is some stationary $E\subseteq\omega_1$ such that for all $j\in E$ $\langle x_j,\vec\rho_j\rangle$ is constant, say $\langle x,\vec\rho\rangle$. Now set $C_j=\{\gamma_i\mid i<j,\ \gamma_i>\max\vec\rho\}$. By 1.14(3) we get

(2) $\quad$ There is a sequence $\langle\bar\delta_n\mid n<\omega\rangle$, $\bar\delta_n<\bar\eta$, such that $\forall j\in E\ \forall n<\omega\ \bar g_j''[C_j]^n=\{\bar\delta_n\}$.

Now set $Y_j=\sigma_{\gamma_j}''C_j$ and $\delta_n=\sigma(\bar\delta_n)$. Note that by (iii) $\delta_n=\sigma_\beta(\bar\delta_n)$ for all $\beta<\omega_1$. Hence $\langle Y_j\mid j\in E\rangle$ essentially gives us Claim 1. □ (Claim 1)

We now show that κ is almost $<\tau$-Erdös in K. Clearly κ is regular in K. Now let $f\in K$ such that $f:[\kappa]^{<\omega}\to\lambda$ where $\lambda<\kappa$ is regular in K. We have to show that there is a sequence $\langle X_\alpha\mid\alpha<\tau\rangle\subset K$ which is homogeneous. Consider the amenable structure $\mathfrak{A}=\langle K_\kappa,\in,D,f\rangle$ where $D\subseteq\kappa$ is such that $K_\rho=L_\rho[D]$ for all K-cardinals $\rho\leqslant\kappa$. By Claim 1 we get:

(3) $\quad$ There is a sequence $\langle I_\alpha\mid\alpha<\omega_1\rangle$ such that $\mathrm{otp}(I_\alpha)=\omega(1+\alpha)$ and

(i) I_α is a set of indiscernibles for $\mathfrak{A}$,

(ii) if $\vec\gamma\in[I_\alpha]^n$, $\vec\delta\in[I_\beta]^n$, $\vec\rho<\lambda$, then

$$\mathfrak{A}\models\phi(\vec\rho,\vec\gamma)\leftrightarrow\phi(\vec\rho,\vec\delta)\quad\text{for all formulae }\phi.$$

Now choose the sequence $\langle I_\alpha\mid\alpha<\omega_1\rangle$ such that $\min I_0$ is minimal for sequences with the above properties. Then standard indiscernibility arguments show that we also have

(iii) $\quad I_\alpha-\gamma$ is a set of indiscernibles for $\langle\mathfrak{A},(\xi)_{\xi<\gamma}\rangle$,

(iv) $\quad\gamma\in I_\alpha\to\gamma$ is inaccessible in $\mathfrak{A}$.

The sequence $\langle I_\alpha\mid\alpha<\omega_1\rangle$ is clearly homogeneous for f. So let $a=\mathrm{tp}_f(I_0)$. It suffices to show:

(4) $\quad a \in K$.

To see this, we apply a well-known argument due to Silver. Namely, there is a sequence $\langle R_\alpha \mid \alpha < \tau = \omega_1^K \rangle \in K$ such that the statement "f has a homogeneous set X of order type α such that $\mathrm{tp}_f(X) = a$" is equivalent to "R_α is well-founded".

So it remains to prove (4). For this let $\mathscr{H}_\alpha$ be the Skolem hull of $\lambda \cup I_\alpha$ in $\mathfrak{A}$ and let $\bar{\mathscr{H}}_\alpha = \mathscr{H}_\alpha \cap L_{\rho_\alpha}[D]$, where $\rho_\alpha = \sup I_\alpha$. Then let $\pi_\alpha : \bar{\mathscr{H}}_\alpha \xrightarrow{\sim} \bar{\mathscr{H}}_\alpha$ where $\bar{\mathscr{H}}_\alpha$ is transitive. As a consequence of (3) (i)–(iv) we get:

(5) (a) $\alpha < \beta \to \pi''_\alpha I_\alpha$ is an initial segment of $\pi''_\beta I_\beta$,
(b) $\alpha < \beta \to \bar{\mathscr{H}}_\alpha < \bar{\mathscr{H}}_\beta$.

So let $\bar{\mathfrak{A}} = \langle \bar{K}, \in, \bar{D}, \bar{f} \rangle = \bigcup_{\alpha < \omega_1} \bar{\mathscr{H}}_\alpha$ and $\bar{I} = \bigcup_{\alpha < \omega_1} \pi''_\alpha I_\alpha$. Then we get:
(c) $\bar{\mathfrak{A}}$ is amenable, $\quad \bar{\mathfrak{A}} \models V = K$, $\quad \mathrm{otp}(\bar{I}) = \omega_1$,
and $\bar{I}$ is a good set of indiscernibles for $\bar{\mathfrak{A}}$.

So we can apply Jensen's indiscernibles lemma. Hence there is some $I' \in K$ such that $I' \supseteq \bar{I}$ and I' is good for $\bar{\mathfrak{A}}$. So we have $\mathrm{tp}_{\bar{f}}(I') \in K$. But since $\pi_0 \restriction \lambda = \mathrm{id} \restriction \lambda$ we have $a = \mathrm{tp}_f(I_0) = \mathrm{tp}_{\bar{f}}(\bar{I}) = \mathrm{tp}_{\bar{f}}(I')$. $\square$

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