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EXTENDERS, EMBEDDING NORMAL FORMS, AND THE MARTIN-STEEL-THEOREM

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Abstract. We propose a simple notion of "extender" for coding large elementary embeddings of models of set theory. As an application we present a self-contained proof of the theorem by D. Martin and J. Steel that infinitely many Woodin cardinals imply the determinacy of every projective set.

§1. Introduction. Many large cardinals can be characterized in terms of elementary embeddings between transitive models of set theory. A "model-theoretic" approach is usually more elegant and efficient than some equivalent combinatorial version. To work with such embeddings within set theory one has to code sufficiently many of them by *sets*. Devices like *normal measures, hypermeasures* and *extenders* have been introduced for this purpose (see [12], [8]).

In the present article we take up a suggestion of S. D. Friedman [4] whereby elementary maps can often be coded by an initial segment of the map itself; such initial segments will be termed "extenders". One obvious advantage of this method lies in the fact that an important part of the coded map need not be "computed" from the code but plainly *is* the code. Let us illustrate this idea in the case of a Scott-type ultrapower

$$\pi\colon V \longrightarrow \mathrm{Ult}(V, U),$$

where U is a normal ultrafilter on a measurable cardinal κ. The filter U can be defined from π by:

$$x \in U \longleftrightarrow x \subseteq \mathscr{P}(\kappa) \wedge \kappa \in \pi(x).$$

Therefore, U can be reconstructed from $\pi \restriction H_{\kappa^+}$ and we can take $\pi \restriction H_{\kappa^+}$ as an extender coding π.

The usual extender theory can be carried over smoothly to the new setting. We apply the theory to give a self-contained proof of the famous Martin-Steel-theorem [10]:

THEOREM. *If there are infinitely many Woodin-cardinals then projective determinacy* (PD) *holds.*

Indeed we show:

THEOREM. (a) *If $a^\sharp$ exists then every $A \subseteq \mathbb{R}$ which is Π^1_1 in the parameter a is determined.*

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(b) *Let $\delta_n < \delta_{n-1} < \cdots < \delta_1$ be Woodin-cardinals, $n \geq 1$, and assume that $V_{\delta_1}^{\sharp}$ exists. Then every $\mathbf{\Pi}^1_{n+1}$-set $A \subseteq \mathbb{R}$ is determined.*

Part (a) is the classical theorem of Martin [9]; (b) slightly strengthens a result from [10], which of course could also be proved by the methods of [10].

We continue to emphasize the use of models and embeddings in contrast with combinatorial methods. The determinacy of a set A of reals is shown by representing A in an *embedding normal form* (ENF) which is a system of models and embeddings indexed by the tree ${}^{<\omega}\omega$ of finite sequences of natural numbers. ENFs are considered in [10]: every set of reals which is the projection of a homogeneous tree possesses an ENF. The converse is false in the context of general ENFs but becomes true if the notion of embedding normal form is strengthened by stipulating a certain degree of closure of the models of the system, like, e.g., $(2^{\aleph_0})^+$ closure. This was observed by Katrin Windßus and proved in her diplom thesis at the University of Bonn [14] which also contains some simplifications of the original Martin-Steel argument. The result of Windßus initiated my project of understanding the Martin-Steel-theorem in terms of elementary embeddings, without that insight this article would not have been written.

In our paper we identify the notion of an *embedding normal form with witnesses* (ENFW) where the closure property is weakened to requiring that *witnesses*, i.e., certain $(2^{\aleph_0})^+$-sequences of ordinals, exist in the models. We shall obtain the required ENFWs directly from branches of iteration trees which also consist of models and elementary embeddings, so that we are able to work "model-theoretically" throughout.

Our paper is structured as follows: In §2, we introduce extenders and develop the basic theory. In §3, strong cardinals and Woodin-cardinals are characterized. In §4, we consider trees of models of set theory connected by elementary embeddings and prove some properties which apply to embedding normal forms and iteration trees alike. In §5, we show that a set having an embedding normal form with witnesses is determined. In §6, ENFWs for $\mathbf{\Pi}^1_1$-sets are obtained from measurable cardinals and from "sharps". In §7, we define iteration trees and give a short proof of a special case of "Steel's Lemma" (Theorem 5.6 of [10]) about the existence of wellfounded branches which is at the core of the projective determinacy proof. §8 explains a method for the construction of alternating iteration trees. This is used in §9 in the inductive argument of the Martin-Steel proof, by which—in our scenario—ENFWs for $\mathbf{\Pi}^1_{n+1}$-sets are obtained from ENFWs for $\mathbf{\Pi}^1_n$-sets in the presence of Woodin cardinals.

§2. Extenders. Let us study elementary maps between transitive $\in$-models of set theory. The following axiom systems will be used: ZF denotes full Zermelo-Fraenkel set theory, ZF^- is ZF except the powerset axiom. ZFC and ZFC^- are the extensions of ZF and ZF^-, respectively, by the axiom of choice in the form

$$\forall x\, \exists f\, \exists \alpha \colon f : \alpha \longleftrightarrow x.$$

The *Skolem principle* (SP) is the schema: for all $\in$-formulae $\varphi(x, y, \vec{z})$ postulate

$$\forall \vec{z}\, \forall a\, \exists f\, \forall x \in a\, (\exists y\, \varphi(x, y, \vec{z}) \longleftrightarrow \varphi(x, f(x), \vec{z})).$$

This principle is of particular interest for ultrapower-like constructions and follows from ZFC. All axiom systems and other model-theoretic notions are taken to be schemes when dealing with classes and as the corresponding Gödel-sets when we work with set-sized structures.

A non-trivial elementary map $E\colon (A,\in) \to (B,\in)$ between transitive models of set theory can be seen as an "extension" of A via the map E since, obviously, $B \supsetneq E''A$. Trivially, B is generated over $E''A$ by some generators from B. If κ is the *critical point* of E, i.e., $E\restriction\kappa = \mathrm{id}$ and $E(\kappa) > \kappa$, we want to consider generators between κ and $E(\kappa)$. Setting $S := H_\kappa^A$ and $T := E(S) = H_{E(\kappa)}^B$, we could say that E "extends" S to a larger set T of generators. The following definition will be satisfied:

DEFINITION 2.1. Let $E\colon A \to B$ be an elementary map where A and B are transitive $\in$-models of ZFC$^-$. Let $S \in A$, $T \in B$. Then E *extends* S to T if:
(a) S is a transitive $\in$-model of ZFC;
(b) $E\restriction S = \mathrm{id}$;
(c) $E(S) = T \neq S$.
Then, if E is a set, we call E an *extender* from S to T; S is called the *source* of E, T is the *target* of E. The *critical point* of E is $\mathrm{crit}(E) = S \cap \mathrm{On}$, and we also say that E is *at* κ. If M is a transitive class E is said to be an *extender on* M if $S \in M$ and $(H_{\kappa^+})^M \subseteq A = \mathrm{dom}(E)$.

We usually take letters E, F, ... for extenders and write $E\colon S \prec T$ to express that E is an extender from S to T. The following theorem shows that extenders code elementary maps which may be class-sized.

THEOREM 2.2. *Let $E\colon S \prec T$ be an extender on M where M is a transitive $\in$-model of* ZF$^-$ + SP. *Then there is an elementary embedding*

$$\pi\colon (M,\in) \to (N,\in')$$

such that

$$\pi\restriction(H_{\kappa^+})^M = E\restriction(H_{\kappa^+})^M.$$

The proof of the theorem will occupy the rest of this section. The *extension* $N = \mathrm{Ext}(M;E)$ of M by E will be explicitly defined by an ultrapower-like construction which also has some similarities with the upward-mapping techniques of [1].

First define a structure $(\tilde{N},\sim,\tilde{\in})$ with $\sim$ interpreting equality and $\tilde{\in}$ interpreting the $\in$-symbol:

$$\tilde{N} := \{\, (f,a) \mid f\colon S \to M,\ f \in M,\ a \in T \,\}$$
$$(f,a) \sim (g,b) :\longleftrightarrow (a,b) \in E\{\, (u,v) \in S \times S \mid f(u) = g(v) \,\}$$
$$(f,a) \mathrel{\tilde{\in}} (g,b) :\longleftrightarrow (a,b) \in E\{\, (u,v) \in S \times S \mid f(u) \in g(v) \,\}.$$

This structure satisfies a version of Łoś's theorem:

LEMMA 2.3. *Let $\varphi(v_1,\dots,v_n)$ be an $\in$-formula and $(f_1,a_1), \dots, (f_n,a_n) \in \tilde{N}$. Then*

$$(\tilde{N},\sim,\tilde{\in}) \models \varphi((f_1,a_1),\dots,(f_n,a_n))$$

if and only if

$$(a_1,\dots,a_n) \in E\{\, (u_1,\dots,u_n) \in S^n \mid (M,=,\in) \models \varphi(f_1(u_1),\dots,f_n(u_n)) \,\}.$$

PROOF. By induction on the complexity of φ.

Let $\varphi \equiv v_i = v_j$. Then

$$(\tilde{N}, \sim \tilde{\in}) \models v_i = v_j((f_1, a_1), \ldots, (f_n, a_n)),$$

if and only if

$$(f_i, a_i) \sim (f_j, a_j),$$

if and only if

$$(a_i, a_j) \in E\{ (u_i, u_j) \in S^2 \mid f_i(u_i) = f_j(u_j) \},$$

by definition, if and only if

$$(a_1, \ldots, a_n) \in E\{ (u_1, \ldots, u_n) \in S^n \mid (M, =, \in) \models v_i = v_j(f_1(u_1), \ldots, f_n(u_n)) \},$$

since E is an elementary map.

The case $\varphi \equiv v_i \in v_j$ is treated entirely similar.

Next let $\varphi \equiv \varphi_1 \wedge \varphi_2$, where φ_1 and φ_2 satisfy the lemma.

$$(\tilde{N}, \sim, \tilde{\in}) \models \varphi_1 \wedge \varphi_2((f_1, a_1), \ldots),$$

if and only if

$$(\tilde{N}, \sim, \tilde{\in}) \models \varphi_1((f_1, a_1), \ldots) \quad \text{and} \quad (\tilde{N}, \sim, \tilde{\in}) \models \varphi_2((f_1, a_1), \ldots),$$

if and only if

$$(a_1, \ldots, a_n) \in E\{ (u_1, \ldots, u_n) \in S^n \mid (M, =, \in) \models \varphi_1(f_1(u_1), \ldots) \}$$

and

$$(a_1, \ldots, a_n) \in E\{ (u_1, \ldots, u_n) \in S^n \mid (M, =, \in) \models \varphi_2(f_1(u_1), \ldots) \},$$

by the inductive hypothesis, if and only if

$$(a_1, \ldots, a_n) \in E\{ (u_1, \ldots, u_n) \in S^n \mid (M, =, \in) \models \varphi_1 \wedge \varphi_2(f_1(u_1), \ldots) \},$$

since the elementary map E preserves intersections.

The other propositional case $\varphi \equiv \neg\psi$ is treated analogously.

Finally, consider $\varphi \equiv \exists v_0\, \psi$ where ψ satisfies the lemma.

If $(\tilde{N}, \sim, \tilde{\in}) \models \exists v_0\, \psi((f_1, a_1), \ldots, (f_n, a_n))$, then

$$(\tilde{N}, \sim, \tilde{\in}) \models \psi((f_0, a_0), (f_1, a_1), \ldots, (f_n, a_n)),$$

for some $(f_0, a_0) \in \tilde{N}$, then

$$(a_0, \ldots, a_n) \in E\{ (u_0, \ldots, u_n) \in S^{n+1} \mid (M, =, \in) \models \psi(f_0(u_0), \ldots, f_n(u_n)) \},$$

by the inductive hypothesis, then

$$(a_1, \ldots, a_n) \in E\{ (u_1, \ldots, u_n) \in S^n \mid (M, =, \in) \models \exists v_0\, \psi(f_1(u_1), \ldots f_n(u_n)) \},$$

since E is an elementary map.

Conversely assume

$$(a_1, \ldots, a_n) \in E\{ (u_1, \ldots, u_n) \in S^n \mid (M, =, \in) \models \exists v_0 \psi(v_0, f_1(u_1), \ldots, f_n(u_n)) \}.$$

By the Skolem principle SP there exists $f_0 \colon S^n \to M,\ f_0 \in M$ so that:

$$(M, =, \in) \models \forall (u_1, \dots, u_n) \in S^n\ (\exists v_0\ \psi(v_0, f_1(u_1), \dots, f_n(u_n)) \longrightarrow \psi(f_0(u_1, \dots, u_n), f_1(u_1), \dots, f_n(u_n))).$$

Then

$$(a_1, \dots, a_n) \in E\{\, (u_1, \dots, u_n) \in S^n \mid (M, =, \in) \models \psi(f_0(u_1, \dots, u_n), f_1(u_1), \dots, f_n(u_n)) \,\},$$

and by the elementarily of E:

$$((a_1, \dots, a_n), a_1, \dots, a_n) \in E\{\, (u_0, u_1, \dots, u_n) \in S^{n+1} \mid (M, =, \in) \models \psi(f_0(u_0), f_1(u_1), \dots, f_n(u_n)) \,\}.$$

By induction hypothesis,

$$(\tilde{N}, \sim, \tilde{\in}) \models \psi((f_0, (a_1, \dots, a_n)), (f_1, a_1), \dots, (f_n, a_n))$$

and

$$(\tilde{N}, \sim, \tilde{\in}) \models \exists v_0\ \psi(v_0, (f_1, a_1), \dots, (f_n, a_n)). \qquad \dashv$$

By this lemma, the equality axioms transfer from $(M, =, \in)$ to $(\tilde{N}, \sim, \tilde{\in})$ and we can form the quotient $(\tilde{N}/{\sim}, =, \tilde{\in}/{\sim})$ by the congruence relation $\sim$; here we restrict the equivalence class of some $(f, a) \in \tilde{N}$ to the set of its rank-minimal members ("Scott's trick", see [3, p. 179]):

$$(f, a)_\sim := \{\, (g, b) \in \tilde{N} \mid (g, b) \sim (f, a) \wedge \forall (h, c) \in \tilde{N}\ ((h, c) \sim (f, a) \longrightarrow \mathrm{rk}(g, b) \leq \mathrm{rk}(h, c)) \,\}.$$

LEMMA 2.4. *The relation $\tilde{\in}/{\sim}$ is set-like, i.e., if $(g, b)_\sim \in \tilde{N}/{\sim}$ then*

$$\{\, (f, a)_\sim \mid (f, a)_\sim\ \tilde{\in}/{\sim}\ (g, b)_\sim \,\} \in V.$$

PROOF. If $(f, a)_\sim\ \tilde{\in}/{\sim}\ (g, b)_\sim$ we may assume that

$$\forall u \in S\ \exists v \in S\ f(u) \in g(v),$$

and this implies $\mathrm{rk}(f) \leq \mathrm{rk}(g)$. Hence

$$\{\, (f, a)_\sim \mid (f, a)_\sim\ \tilde{\in}/{\sim}\ (g, b)_\sim \,\} \subseteq \{\, (f, a)_\sim \mid \mathrm{rk}(f) \leq \mathrm{rk}(g) \wedge a \in T \,\} \in V. \qquad \dashv$$

By Lemma 2.3, the axiom of extensionality also transfers from $(M, =, \in)$ to $(\tilde{N}, \sim, \tilde{\in})$ and $(\tilde{N}/{\sim}, =, \tilde{\in}/{\sim})$. Let $\sigma \colon \mathrm{wfp}(\tilde{N}/{\sim}, \tilde{\in}/{\sim}) \cong N^*$ be the Mostowski transitivisation map on the wellfounded part of $(\tilde{N}/{\sim}, \tilde{\in}/{\sim})$. We can now define the desired structure $(N, \in')$:

For $(f, a) \in \tilde{N}$ let

$$[f, a] := \begin{cases} \sigma((f, a)_\sim), & \text{if } (f, a)_\sim \in \mathrm{wfp}(\tilde{N}/{\sim}, \tilde{\in}/{\sim}); \\ ((f, a)_\sim, N^*), & \text{else.} \end{cases}$$

Note that the second clause only applies if $(\tilde{N}/{\sim}, \tilde{\in}/{\sim})$ is not wellfounded; In that case, N^* is a set and the formation of the ordered pair $((f,a)_\sim, N^*)$ ensures that $[f,a]$ is not an element of N^*. Then let

$$N := \{\, [f,a] \mid (f,a) \in \tilde{N} \,\}$$

and

$$[f,a] \in' [g,b] :\longleftrightarrow (f,a) \mathrel{\tilde{\in}} (g,b).$$

Obviously the Łoś property Lemma 2.3 carries over to $(N, \in')$:

Lemma 2.5. *Let $\varphi(v_1, \dots, v_n)$ be an $\in$-formula and $[f_1, a_1], \dots, [f_n, a_n] \in N$. Then*

$$(N, \in') \models \varphi([f_1, a_1], \dots, [f_n, a_n])$$

if and only if

$$(a_1, \dots, a_n) \in E\{\, (u_1, \dots, u_n) \in S^n \mid (M, \in) \models \varphi(f_1(u_1), \dots, f_n(u_n)) \,\}.$$

Next we embed M into N and examine how the embedding relates to E. For $x \in M$ let $\text{const}_x \in M$ be the constant function $\text{const}_x \colon S \to \{x\}$. Define $\pi \colon M \to N$ by $\pi(x) := [\text{const}_x, 0]$.

Lemma 2.6. *$\pi \colon (M, \in) \to (N, \in')$ is elementary.*

Proof. Let $\varphi(v_1, \dots, v_n)$ be an $\in$-formula and $x_1, \dots, x_n \in M$.

$$(M, \in) \models \varphi(x_1, \dots, x_n)$$

if and only if

$$(0, \dots, 0) \in \{\, (u_1, \dots, u_n) \in S^n \mid (M, \in) \models \varphi(\text{const}_{x_1}(u_1), \dots, \text{const}_{x_n}(u_n)) \,\}$$

if and only if

$$(0, \dots, 0) \in E\{\, (u_1, \dots, u_n) \in S^n \mid (M, \in) \models \varphi(\text{const}_{x_1}(u_1), \dots, \text{const}_{x_n}(u_n)) \,\}$$

if and only if

$$(N, \in') \models \varphi([\text{const}_{x_1}, 0], \dots, [\text{const}_{x_n}, 0]),$$

by Lemma 2.5, if and only if

$$(N, \in) \models \varphi(\pi(x_1), \dots, \pi(x_n)). \qquad \dashv$$

We use the identity function $I = \text{id} \restriction S$, $I \in M$ to locate the generators $a \in T$ in the model N:

Lemma 2.7. *For all $a \in T$, $(I, a)_\sim$ is in the wellfounded part of $\tilde{\in}/{\sim}$ and $[I, a] = a$.*

Proof. By $\in$-induction on $a \in T$; assume that the lemma holds for all $b \in a$.

(1) If $(f, c) \mathrel{\tilde{\in}} (I, a)$ then $(f, c) \sim (I, b)$ for some $b \in a$.

Proof. $(c, a) \in E\{\, (w, u) \in S^2 \mid f(w) \in I(u) = u \,\}$. Define $f' \colon S \to S$, $f' \in M$ by

$$f'(w) = f(w),$$

if $f(w) \in S$, and $f'(w) = 0$, else.

Then

$$(c, a) \in E\{\, (w, u) \in S^2 \mid f'(w) = f(w) \,\},$$

i.e., $(f',c) \sim (f,c)$, and

$$(c,a) \in E\{\,(w,u) \in S^2 \mid f'(w) \in u\,\}.$$

Since $f' \in (H_{\kappa^+})^M \subseteq \operatorname{dom}(E)$, we can pull E inside the set brackets:

$$(c,a) \in \{\,(w,u) \in T^2 \mid E(f')(w) \in u\,\},$$

and so $E(f')(c) \in a$. Set $b = E(f'(c))$. Then

$$(c,b) \in \{\,(w,v) \in T^2 \mid E(f')(w) = v\,\} = E\{\,(w,v) \in S^2 \mid f'(w) = I(v) = v\,\},$$

and so $(f,c) \sim (f',c) \sim (I,b)$, where $b \in a$. ⊣ (1)

(2) $(I,a)_\sim$ is in the wellfounded part of $\tilde{\in}/\sim$.

Proof. By (1), every $\tilde{\in}/\sim$ predecessor of $(I,a)_\sim$ is of the form $(I,b)_\sim$ for some $b \in a$. By induction hypothesis that $(I,b)_\sim$ is in the wellfounded part of $\tilde{\in}/\sim$ and so $(I,a)_\sim$ is in the wellfounded part of $\tilde{\in}/\sim$. ⊣ (2)

(3) $[I,a] \subseteq a$.

Proof. Let $x \in [I,a]$. Let $x = [f,c]$ where $(f,c) \in \tilde{N}$. By (1), $[f,c] = [I,b]$ for some $b \in a$. By the induction hypothesis

$$x = [f,c] = [I,b] = b \in a. \qquad ⊣ (3)$$

(4) $a \subseteq [I,a]$.

Proof. Let $b \in a$.

$$(b,a) \in \{\,(u,v) \in T^2 \mid u \in v\,\} = E\{\,(u,v) \in S^2 \mid u \in v\,\},$$

and so $[I,b] \in [I,a]$. By induction hypothesis,

$$b = [I,b] \in [I,a].$$ ⊣

Lemma 2.8. *If* $[f,a] \in N$ *then* $(N,\in') \models [f,a] = \pi(f)(a)$.

Proof.

$$\begin{aligned}
&\forall s \in S\colon f(s) = \operatorname{const}_f(0)(I(s))\\
&\quad\Longrightarrow \forall s \in S\colon (s,0,s) \in \{\,(u,v,w) \in S^3 \mid f(u) = \operatorname{const}_f(v)(I(w))\,\}\\
&\quad\Longrightarrow \forall s \in T\colon (s,0,s) \in E\{\,(u,v,w) \in S^3 \mid f(u) = \operatorname{const}_f(v)(I(w))\,\}\\
&\quad\Longrightarrow (a,0,a) \in E\{\,(u,v,w) \in S^3 \mid f(u) = \operatorname{const}_f(v)(I(w))\,\}\\
&\quad\Longrightarrow (N,\in') \models [f,a] = [\operatorname{const}_f,0]([I,a]) = \pi(f)(a),
\end{aligned}$$

by Lemma 2.7. ⊣

Lemma 2.9. $\pi{\restriction}(H_{\kappa^+})^M = E{\restriction}(H_{\kappa^+})^M$.

Proof. If $x \in (H_{\kappa^+})^M$, there is a transitive set z and a map $f\colon \kappa \longleftrightarrow z$, $f \in (H_{\kappa^+})^M$, and a relation $R \subseteq \kappa^2$, $R \in (H_{\kappa^+})^M$, such that $f\colon (\kappa,R) \cong (z,\in)$ is the Mostowski-collapse of the relation R, and $f(0) = x$. Apply π and E to this situation:

$(N,\in') \models \pi(f)\colon (\pi(\kappa),\pi(R)) \cong (\pi(z),\in')$ is the Mostowski-collapse of $\pi(R)$, and $\pi(f)(0) = \pi(x)$;

$E(f)\colon (E(\kappa), E(R)) \cong (E(z), \in)$ is the Mostowski-collapse of $E(R)$,

and $E(f)(0) = E(x)$.

So $\pi(x)$ and $E(x)$ are determined by $\pi(R)$ and $E(R)$, respectively, and the lemma will follow from:

(1) $\pi\restriction(\mathscr{P}(S) \cap M) = E\restriction(\mathscr{P}(S) \cap M)$.

Before this we show:

(2) If $X \in \mathscr{P}(S) \cap M$ then $\pi(X)$ is in the wellfounded part of $(N, \in')$; indeed $\pi(X) \subseteq T$.

PROOF. Let $[f, a] \in' \pi(X)$, i.e., $(f, a) \tilde{\in} (\text{const}_x, 0)$.

Then $a \in E\{ u \in S \mid f(u) \in X \}$, and as usual we may assume that $f : S \to X \subseteq S$ and $f \in (H_{\kappa^+})^M \subseteq \text{dom}(E)$. We can now pull E inside the abstraction term:

$$a \in \{ u \in T \mid E(f)(u) \in E(X) \},$$
$$E(f)(a) \in E(X) \subseteq E(S) = T.$$

Let $b = E(f)(a) \in T$. Then

$$(b, a) \in \{ (v, u) \in T^2 \mid v = E(f)(u) \} = E\{ (v, u) \in S^2 \mid I(v) = f(u) \}$$

and, by the Łoś property and Lemma 2.7:

$$[f, a] = [I, b] = b \in T.$$

So any $\in'$-predecessor of $\pi(X)$ is in T, which is in the wellfounded part of $(N, \in')$. Therefore $\pi(X)$ is in the wellfounded part of $(N, \in')$ and $\pi(X) \subseteq T$. ⊣ (2)

We can now prove (1): for $b \in T$:

$$b \in \pi(X) \longleftrightarrow [I, b] \in [\text{const}_X, 0]$$
$$\longleftrightarrow b \in E\{ u \in S \mid u \in X \} = E(X).$$ ⊣

The map $\pi\colon (M, \in) \to (N, \in')$ constructed so far is called the *extension* of M by E. This is often indicated by a subscript notation

$$\pi_E : (M, \in) \to_E (N, \in'),$$

and we also write $\pi_{M,E}$ for π_E and $\text{Ext}(M, E)$ to denote $(N, \in')$. Let us now summarize our results:

THEOREM 2.10. *The extension* $\pi_E : (M, \in) \to_E \text{Ext}(M, E)$ *of* M *by the extender* $E : S \prec T$ *satisfies*:

(a) $\pi_E : (M, \in) \to_E \text{Ext}(M, E)$ *is elementary and the wellfounded part of* $\text{Ext}(M, E)$ *is transitive*;

(b) $\pi_E\restriction(H_{\kappa^+})^M = E\restriction(H_{\kappa^+})^M$;

(c) $\text{Ext}(M, E) = \{ \pi_E(f)(a) \mid f \in M, f : S \to M, a \in T \}$, *where* $\pi_E(f)(a)$ *is computed within* $\text{Ext}(M, E)$ *as in Lemma* 2.8.

Moreover, (a)–(c) *determine the extension up to isomorphism*: *If* π^* *and* $(N^*, \in^*)$ *satisfy* (a)–(c) *in place of* π_E *and* $\text{Ext}(M, E)$, *respectively, there is an* $\in'$-$\in^*$-*isomorphism* $\sigma\colon \text{Ext}(M, E) \cong (N^*, \in^*)$ *such that* $\pi^* = \sigma \circ \pi_E$; σ *is the identity on the wellfounded part of* $\text{Ext}(M, E)$.

PROOF. It remains to check the isomorphism property.

Let $\varphi(v_1, \ldots, v_n)$ be an $\in$-formula and $\pi_E(f_1)(a_1), \ldots, \pi_E(f_n)(a_n) \in \mathrm{Ext}(M, E)$, $f_i \in M$, $f_i : S \to M$, $a_i \in T$. Then:

$$\begin{aligned}
&\mathrm{Ext}(M, E) \models \varphi(\pi_E(f_1)(a_1), \ldots, \pi_E(f_n)(a_n)) \\
&\quad \longleftrightarrow (a_1, \ldots, a_n) \in \{ (u_1, \ldots, u_n) \in T^n \mid \\
&\qquad\qquad\qquad\qquad \mathrm{Ext}(M, E) \models \varphi(\pi_E(f_1)(u_1), \ldots, \pi_E(f_n)(u_n)) \} \\
&\qquad = \pi_E \{ (u_1, \ldots, u_n) \in S^n \mid (M, \in) \models \varphi(f_1(u_1), \ldots, f_n(u_n)) \} \\
&\qquad = \pi^* \{ (u_1, \ldots, u_n) \in S^n \mid (M, \in) \models \varphi(f_1(u_1), \ldots, f_n(u_n)) \}, \\
&\qquad\qquad \text{since } \pi_E \restriction (H_{\kappa^+})^M = E \restriction (H_{\kappa^+})^M = \pi^* \restriction (H_{\kappa^+})^M, \\
&\qquad = \{ (u_1, \ldots, u_n) \in T^n \mid (N^*, \in^*) \models \varphi(\pi^*(f_1)(u_1), \ldots, \pi^*(f_n)(u_n)) \} \\
&\quad \longleftrightarrow (N^*, \in^*) \models \varphi(\pi^*(f_1)(a_1), \ldots, \pi^*(f_n)(a_n)).
\end{aligned}$$

This shows that

$$\pi_E(f)(a) \longmapsto \pi^*(f)(a)$$

defines an isomorphism $\sigma : \mathrm{Ext}(M, E) \cong (N^*, \in^*)$ with the required properties. ⊣

REMARKS.

1. The relationship between the above extenders and the Dodd-Jensen approach (see [2]) is roughly described as follows: If $E : S \prec T$ is an extender then for each $a \in T$,

$$E_a := \{ X \subseteq S \mid a \in E(X) \}$$

is an ultrafilter on S. The system $(E_a \mid a \in T)$ is the Dodd-Jensen extender corresponding to E. In it the various ultrafilters are connected via certain projection maps. Conversely, a Dodd-Jensen extender $(E_a \mid a \in T)$ with ultrafilters on S yields an extender $E : S \prec T$ by:

$$E(X) = \{ a \mid X \in E_a \}.$$

2. Our construction of $\mathrm{Ext}(M, E)$ is quite robust and allows for all sorts of variations. One could weaken the extender axioms by requiring Σ_0-elementarity for $E : A \to B$ instead of full elementarily. One could also work with $\bar{E} := E \restriction \mathscr{P}(S)$ and postulate:

$$(S, \in, (X \mid X \in \mathrm{dom}(\bar{E}))) \prec (T, \in, (\bar{E}(X) \mid X \in \mathrm{dom}(\bar{E}))).$$

3. For specific instances of the Łoś Theorem 2.3 or the transfer property Lemma 2.6, only a limited part of ZFC^- and the Skolem principle SP is required in M. This is important in inner model theory where extensions of weak structures are considered.

4. On the other hand, we can expand $(M, \in)$ to a structure $(M, \in, \vec{P})$ with extra predicates $\vec{P}$. If $(M, \in, \vec{P})$ satisfies enough set theory relative to $\vec{P}$, we can expand the extension in the obvious way:

$$\pi_E : (M, \in, \vec{P}) \Longrightarrow_E \mathrm{Ext}(M, E) = (N, \in', \vec{P}').$$

§3. Large cardinals. The formation of the extension $\mathrm{Ext}(M, E)$ attains large cardinal strength if $\mathrm{Ext}(M, E)$ is a *transitive* $\in$-model. We shall introduce a closure criterion for the wellfoundedness of $\mathrm{Ext}(M, E)$ and use it in a characterization of a couple of large cardinal axioms.

DEFINITION 3.1. Let E be an extender on $(M, \in)$, where $(M, \in)$ is a transitive model of ZFC^- and SP. Then $(M, \in)$ is called *extendable* by E if $\mathrm{Ext}(M, E)$ is wellfounded, i.e., transitive.

DEFINITION 3.2. A class X is *η-closed* if ${}^{\eta}X \subseteq X$ where ${}^{\eta}X = \{ f \mid f : \eta \to X \}$. An extender $E : S \prec T$ is *η-closed* if its target T is η-closed.

THEOREM 3.3. *Let M be a transitive η-closed model of* ZFC^- *and* SP. *Let $E : S \prec T$ be an η-closed extender on M such that $\omega \leq \eta \leq \mathrm{crit}(E)$. Then M is extendable by E and* $\mathrm{Ext}(M, E)$ *is η-closed.*

PROOF. Let $\pi_E : (M, \in) \to_E \mathrm{Ext}(M, E)$.

(1) If $([f_i, a_i] \mid i < \eta) \in {}^{\eta}\,\mathrm{Ext}(M, E)$ there is $[f, a] \in \mathrm{Ext}(M, E)$ such that for all $i < \eta$:

$$\mathrm{Ext}(M, E) \models [f, a](i) = [f_i, a_i].$$

PROOF. Define $f : S \to M$ by:

$$f(u) = \begin{cases} (f_i(u(i)) \mid i < \mathrm{dom}(u)), & \text{if } u \in S \text{ is a function, } \mathrm{dom}(u) \in \mathrm{On}, \\ & \quad \mathrm{dom}(u) \leq \eta; \\ 0, & \text{else.} \end{cases}$$

$f \in M$ because $(f_i \mid i < \eta) \in {}^{\eta}M \subseteq M$.
Let $a = (a_i \mid i < \eta)$; $a \in {}^{\eta}T \subseteq T$.
Now let $i < \eta$. $a(i) = a_i$ implies:

$$\begin{aligned} (a, a_i) \in \{ (u, v) \in T^2 \mid\ & u \text{ is a function} \wedge \mathrm{dom}(u) \in \mathrm{On} \\ & \wedge \mathrm{dom}(u) \leq \pi_E(\eta) \wedge i \in \mathrm{dom}(u) \wedge u(i) = v \} \\ = E\{ (u, v) \in S^2 \mid\ & u \text{ is a function} \wedge \mathrm{dom}(u) \in \mathrm{On} \\ & \wedge \mathrm{dom}(u) \leq \eta \wedge i \in \mathrm{dom}(u) \wedge u(i) = v \} \\ \subseteq E\{ (u, v) \in S^2 \mid\ & f(u)(i) = f_i(u(i)) = f_i(v) \}. \end{aligned}$$

By Lemma 2.5, $\mathrm{Ext}(M, E) \models [f, a](i) = [f_i, a_i]$. ⊣(1)

(2) $\mathrm{Ext}(M, E)$ is a transitive $\in$-model.

PROOF. Assume not. Considering Lemma 2.4, this is due to an infinite descending chain $([f_n, a_n] \mid n < \omega)$ in $\in'$: for $n < \omega$:

$$\mathrm{Ext}(M, E) \models [f_{n+1}, a_{n+1}] \in [f_n, a_n].$$

By (1), there is $[f, a] \in \mathrm{Ext}(M, E)$ such that for $n < \omega$:

$$\mathrm{Ext}(M, E) \models [f, a](n) = [f_n, a_n].$$

Then

$$\mathrm{Ext}(M, E) \models \forall n < \omega\ [f, a](n + 1) \in [f, a](n),$$

contradicting the axiom of foundation inside $\mathrm{Ext}(M, E)$. ⊣(2)

(3) ${}^{\eta}\operatorname{Ext}(M,E) \subseteq \operatorname{Ext}(M,E)$.

PROOF. Follows immediately from (2) and (1). ⊣

LEMMA 3.4. *Let* $\pi\colon (M,\in) \to (N,\in)$ *be an elementary map between transitive* ZFC-*models with* $\kappa = \operatorname{crit}(\pi)$. *Let* $S = H_\kappa^M$, $T = \pi(S)$. *Then* $E = \pi\restriction(H_{\kappa^+})^M$ *is an extender from* S *to* T *on* M *which is called the extender* induced *by* π. *If* N *is* η-*closed and* $\eta < \pi(\kappa)$ *then* E *is an* η-*closed extender.*

PROOF. ${}^{\eta}T \subseteq {}^{\eta}N \subseteq N$. Hence ${}^{\eta}T = ({}^{\eta}T)^N \subseteq T$ observing that $N \models T$ is η-closed. ⊣

We now give extender characterizations of large cardinals which usually are defined by elementary embeddings of V. For the purpose of this article the subsequent theorems could also be understood as definitions of those cardinals.

THEOREM 3.5. *The following are equivalent:*
(a) κ *is a* measurable cardinal.
(b) *There exists an extender* $E\colon V_\kappa \prec T$ *on* V.
(c) *There exists a* κ-*closed extender* $E\colon V_\kappa \prec T$ *on* V.

PROOF.
(a) → (b). κ is measurable if and only if there exists an elementary embedding $\pi\colon V \to M$ with M transitive and critical point κ. The extender E induced by π satisfies (b).

(b) → (c). Let $E\colon V_\kappa \prec T$ be an extender on V. We can assume that $\operatorname{dom}(E) = H_{\kappa^+}$: $E\colon H_{\kappa^+} \to H$ elementarily. Let $Z = \{\, E(f)(\kappa) \mid f \in H_{\kappa^+} \,\}$. Since there are sufficiently many Skolem functions among the $f \in H_{\kappa^+}$,

$$\operatorname{rng}(E) \subseteq Z \prec H.$$

Let $\sigma\colon (\bar H,\in) \cong (Z,\in)$ be the Mostowski isomorphism with $\bar H$ transitive. Define an extender $\bar E\colon H_{\kappa^+} \to \bar H$ by $\bar E = \sigma^{-1} \circ E$; $\bar E\colon V_\kappa \prec \bar T$ with $\bar T = \bar E(V_\kappa) \neq V\kappa$. We show that $\bar E$ satisfies (c), i.e., that $\bar T$ is κ-closed. Since $\bar T$ is κ-closed inside $\bar H$, it suffices to see that $\bar H$ or the isomorphic structure Z are κ-closed.

Let $s = (\, E(f_i)(\kappa) \mid i < \kappa \,) \in {}^{\kappa}Z$. Define

$$g\colon \kappa \to H_{\kappa^+} \quad \text{by} \quad g(\gamma) = (\, f_i(\gamma) \mid i < \gamma \,).$$

$g(\gamma)$ is a γ-sequence. Since E is elementary, $E(g)(\kappa)$ is a κ-sequence. Let $i < \kappa$.

$$H_{\kappa^+} \models \forall\gamma\ (\gamma > i \Longrightarrow g(\gamma)(i) = f_i(\gamma)),$$

and as E is elementary,

$$H \models \forall\gamma\ (\gamma > i \Longrightarrow (E(g)(\gamma))(i) = E(f_i)(\gamma)).$$

For $\gamma = \kappa$,

$$(E(g)(\kappa))(i) = E(f_i)(\kappa).$$

Hence $s = E(g)(\kappa) \in Z$.

(c) → (a). Let E be a κ-closed extender satisfying (c). By Theorem 2.2 and Lemma 3.3 one can define an elementary map $\pi\colon V \to N$, N transitive, with critical point κ. Hence κ is measurable. ⊣

THEOREM 3.6. *The following are equivalent*:

(a) κ *is a* strong cardinal.

(b) *For all* $x \in V$ *there exists an extender* $E: V_\kappa \prec T$ *on* V *such that* $x \in T$.

(c) *For all* $x \in V$ *there exists a* κ*-closed extender* $E: V_\kappa \prec T$ *on* V *such that* $x \in T$.

PROOF.

(a) $\to$ (b). κ is strong if and only if for all $x \in V$ there exists an elementary embedding $\pi: V \to M$ with M transitive, $\mathrm{crit}(\pi) = \kappa$ and $x \in (V_{\pi(\kappa)})^M$. The extenders induced by the embeddings π for varying x satisfy (b).

(b) $\to$ (c). Let $x \in V$ be given. Take some λ such that $x \in V_\lambda$ and V_λ is κ-closed. By (b), take an extender $E: V_\kappa \prec T$ on V such that $x \in V_\lambda \in T$. We continue as in the proof of Theorem 3.5. Assume that $E: H_{\kappa^+} \to H$ elementarily. Define Z by:

$$\mathrm{rng}(E) \subseteq Z = \{E(f)(a) \mid f \in H_{\kappa^+}, a \in V_\lambda\} \prec H.$$

Let $\sigma: (\bar{H}, \in) \cong (Z, \in)$, $\bar{H}$ transitive, $\sigma \restriction V_\lambda = \mathrm{id}$. Define $\bar{E}: H_{\kappa^+} \to \bar{H}$ by $\bar{E} = \sigma^{-1} \circ E$; $\bar{E}: V_\kappa \prec \bar{T}$ with $\bar{T} = \bar{E}(V_\kappa)$ is an extender on V with $x \in V_\lambda \subseteq \bar{T}$. An easy generalisation of the argument in Theorem 3.5 shows that $\bar{T}$ is κ-closed.

(c) $\to$ (a). Let $x \in V$. Let $E: V_\kappa \prec T$ be a κ-closed extender satisfying (c) for x. By Theorem 2.2 and Lemma 3.3, the elementary map $\pi_E: V \to \mathrm{Ext}(V, E)$ extends E and $x \in T \subseteq \mathrm{Ext}(V, E)$. Hence κ is strong. ⊣

THEOREM 3.7. *For a class* $A \subseteq V$ *the following are equivalent*:

(a) κ *is* strong in A.

(b) *For all* $\lambda \in \mathrm{On}$ *there exists an extender* $E: V_\kappa \prec T$ *on* V *such that* $V_\lambda \subseteq T$ *and* $E(A \cap V_\kappa) \cap V_\lambda = A \cap V_\lambda$.

(c) *For all* $\lambda \in \mathrm{On}$ *there exists a* κ*-closed extender* $E: V_\kappa \prec T$ *on* V *such that* $V_\lambda \subseteq T$ *and* $E(A \cap V_\kappa) \cap V_\lambda = A \cap V_\lambda$.

PROOF.

(a) $\to$ (b). κ is strong in A if and only if for all $\lambda \in \mathrm{On}$ there is an elementary map $\pi: (V, A) \to (M, A')$ with M transitive, $\mathrm{crit}(\pi) = \kappa$, $V_\lambda \subseteq V_{\pi(\kappa)} \cap M$ and $A' \cap V_\lambda = A \cap V_\lambda$. Then

$$\pi(A \cap V_\kappa) \cap V_\lambda = A' \cap (V_{\pi(\kappa)} \cap M) \cap V_\lambda = A' \cap V_\lambda = A \cap V_\lambda,$$

and the extender induced by π satisfies (b) for λ.

(b) $\to$ (c) can be shown like the corresponding step in Theorem 3.6.

(c) $\to$ (a). Let $\lambda \in \mathrm{On}$ and let E be an extender satisfying (c) for λ. Let

$$\pi: V \to_E \mathrm{Ext}(V, E)$$

with transitive extension $\mathrm{Ext}(V, E)$. The construction of the extension may be applied to the predicate A and one obtains a class A' such that

$$\pi: (V, \in, A) \to (\mathrm{Ext}(V, E), A')$$

is elementary. Then

$$\begin{aligned} A' \cap V_\lambda &= (A' \cap V_{\pi(\kappa)} \cap \mathrm{Ext}(V, E)) \cap V_\lambda \\ &= \pi(A \cap V_\kappa) \cap V_\lambda \\ &= E(A \cap V_\kappa) \cap V_\lambda \\ &= A \cap V_\lambda, \end{aligned}$$

and so κ is strong in A. ⊣

To characterize Woodin cardinals we define:

DEFINITION 3.8. κ is *strong in A up to δ* if $(V_\delta, A) \models$ "κ is strong in A".

THEOREM 3.9. *For a cardinal δ the following are equivalent*:
(a) *δ is a* Woodin cardinal.
(b) *For all $A \subseteq V_\delta$ there exists a $\kappa < \delta$ which is strong in A up to δ.*
(c) $\forall A \subseteq V_\delta\ \exists \kappa < \delta\ \forall \lambda < \delta\ \exists E \in V_\delta\ \exists T \in V_\delta$:

$$(E : V_\kappa \prec T \text{ is a } \kappa\text{-closed extender on } V \\ \wedge V_\lambda \subseteq T \wedge E(A \cap V_\kappa) \cap V_\lambda = A \cap V_\lambda).$$

PROOF. The equivalence of (a) and (c) is in essence proved in [10, Lemma 4.2]. The equivalence of (b) and (c) follows from Theorem 3.7. ⊣

Clauses 3.9 (b) and (c) are the characterisations of Woodin cardinals to be used later on. We conclude this section with some results on wellfounded extensions.

LEMMA 3.10. *Let M be a transitive model of set theory which is extendable by the extender $E : S \prec T$ with critical point κ and extension $\pi_E : M \to_E \mathrm{Ext}(M, E)$. Then*:
(a) $\forall \alpha \in \mathrm{On} \cap M\ \pi_E(\alpha) < \max(\bar{\bar{\alpha}}^{\bar{\bar{\kappa}}}, \bar{\bar{T}})^+$.
(b) *If τ is a cardinal $> \bar{\bar{T}}$ such that $\forall \alpha < \tau\ \bar{\bar{\alpha}}^{\bar{\bar{\kappa}}} < \tau$ then $\pi_E''\tau \subseteq \tau$.*
(c) *If τ satisfies the assumptions of* (b) *and* $\mathrm{cof}(\tau) > \omega$ *then there is a closed unbounded subset $C \subseteq \tau$ such that $\forall \gamma \in C\ \pi_E''\gamma \subseteq \gamma$.*
(d) *If $\gamma \in \mathrm{On} \cap M$, $\pi_E''\gamma \subseteq \gamma$ and* $\mathrm{cof}^M(\gamma) > \kappa$ *then $\pi_E(\gamma) = \gamma$.*
(e) *If τ satisfies the assumptions of* (b) *and* $\mathrm{cof}(\tau) > \kappa^+$ *then there is a κ^+−closed unbounded subset $D \subseteq \tau$ such that $\pi_E \restriction D = \mathrm{id} \restriction D$.*
(f) *The hypotheses of* (b), (c), *and* (e) *are satisfied for successor cardinals $\tau = \mu^+$ where μ is a strong limit cardinal of cofinality $> \kappa$.*

PROOF. Let $N = \mathrm{Ext}(M, E)$.
(a) Let $\alpha \in \mathrm{On} \cap M$. Every $[f', a] < \pi_E(\alpha)$ is equal to some $[f, a]$ with $f : S \to \alpha$. So

$$\pi_E(\alpha) = \{ [f, a] \mid f : S \to \alpha,\ a \in T \},$$

and

$$\overline{\overline{\pi_E(\alpha)}} \leq \mathrm{card}({}^S\alpha) \cdot \bar{\bar{T}}.$$

Hence $\pi_E(\alpha) < \max(\bar{\bar{\alpha}}^{\bar{\bar{\kappa}}}, \bar{\bar{T}})^+$.
(b) Property (a) yields: $\alpha < \tau \to \pi_E(\alpha) < \tau$.
(c) Follows directly from (b).

(d) Clearly $\pi_E(\gamma) \geq \gamma$. For the converse assume that $[f,a] < \pi_E(\gamma)$. As above, assume that $f\colon S \to \gamma$, $f \in M$. $\bar{\bar{S}}^M = \kappa < \operatorname{cof}^M(\gamma)$ and so there is $\alpha < \gamma$ such that $f\colon S \to \alpha$. Then $[f,a] < \pi_E(\alpha) < \gamma$ by assumption. Hence $\pi_E(\gamma) \leq \gamma$.

(e) Take $C \subseteq \tau$ as in (c) and let $D = \{\gamma \in C \mid \operatorname{cof}(\gamma) > \kappa\}$. Then $\pi_E \restriction D = \mathrm{id}$ by (d).

(f) We only have to check $\forall \alpha < \tau\ \bar{\bar{\alpha}}^{\bar{\bar{\kappa}}} < \tau$ as in (b). Since $\tau = \mu^+$ this comes down to seeing that $\mu^{\bar{\bar{\kappa}}} = \mu < \tau$:

$$\begin{aligned} \mu^{\bar{\bar{\kappa}}} &= \sum_{\lambda<\mu} \lambda^{\bar{\bar{\kappa}}} && \text{since } \operatorname{cof}(\mu) > \kappa, \\ &\leq \sum_{\lambda<\mu} \mu && \text{since } \mu \text{ is strong limit,} \\ &= \mu. \end{aligned}$$

⊣

Lemma 3.11. *Let $E\colon S \prec T$ be an extender with critical point κ and let the* ZFC*-model M be extendable by E with extension map $\pi = \pi_E$. Let $\gamma > \kappa$ be regular in M. Then* $\operatorname{Ext}(H_\gamma^M, E)$ *is welldefined and transitive and*

$$\operatorname{Ext}(H_\gamma^M, E) = H_{\pi(\gamma)}^{\operatorname{Ext}(M,E)}.$$

Proof. H_γ^M is a model of ZFC$^-$ and SP so that $\operatorname{Ext}(H_\gamma^M, E)$ is defined.

$$\operatorname{Ext}(H_\gamma^M, E) = \{[f,a]_0 \mid f \in H_\gamma^M,\ a \in T\}$$

where $[\,]_0$ denotes the collapsed equivalence classes for the extension of H_γ^M by E. It is easy to check that

$$\imath\colon [f,a]_0 \longmapsto \pi(f)(a)$$

defines an isomorphism

$$\imath\colon \operatorname{Ext}(H_\gamma^M, E) \cong H_{\pi(\gamma)}^{\operatorname{Ext}(M,E)}.$$

Then both sides are transitive and hence equal. ⊣

§4. **Trees of models.** In the next section we shall show the determinacy of sets of reals that can be represented by certain embedding normal forms, which are tree-like systems of models of set theory connected by elementary embeddings. Such normal forms will be obtained from other trees of models called iteration trees. Presently we consider properties which apply to embedding normal forms and iteration trees alike.

Definition 4.1. $T = (T, \leq_T)$ is an *ω-tree* if $\leq_T$ is a non-strict partial order on $T \neq \emptyset$ and if for all $t \in T$ the set $\{s \in T \mid s \leq_T t\}$ is linearly ordered by $\leq_T$ and is finite. We write $s <_T t$ if $s \leq_T t$ and $s \neq t$.

$b \subseteq T$ is a *branch* through T if b is a $\subseteq$-maximal subset of T which is linearly ordered by $\leq_T$. Let $[T]$ denote the set of all branches through T.

Definition 4.2. Let $T = (T, \leq_T)$ be an ω-tree. A system $\mathfrak{T} = (M_s)_{s \in T}, (\pi_{st})_{s \leq_T t}$ is called a *tree of models* over T provided:

(a) every M_s is a transitive model of ZFC$^-$ and the Skolem principle SP;

(b) $s \leq_T t \Longrightarrow \pi_{st}\colon M_s \to M_t$ is elementary;

(c) $r \leq_T s \leq_T t \Longrightarrow \pi_{rt} = \pi_{st} \circ \pi_{rs}$.
$\mathfrak{T}$ is *η-closed,* if every M_s in $\mathfrak{T}$ is η-closed.
The *critical point* of $\mathfrak{T}$ is

$$\operatorname{crit}(\mathfrak{T}) = \min\{ \operatorname{crit}(\pi_{st}) \mid s \leq_T t \}.$$

If $b \in [T]$ let

$$M_b,\ (\pi_{sb})_{s \in b} = \operatorname{dir\,lim}(M_s)_{s \in b},\ (\pi_{st})_{s \leq_T t \in b}$$

be the direct limit of the subsystem along the branch b. We require that the wellfounded part of M_b is transitive. If M_b is wellfounded b is called a *wellfounded branch* of $\mathfrak{T}$; otherwise b is *illfounded.*

The most important ω-tree is the tree $T = ({}^{<\omega}\omega, \subseteq)$ of finite sequences of natural numbers, partially ordered by inclusion. A branch through T corresponds canonically to a function from ω to ω and we may identify the set of real numbers with the set of branches through T: $\mathbb{R} = [({}^{<\omega}\omega, \subseteq)]$. We can now define the central notion for our presentation of the determinacy proofs:

Definition 4.3. Let $\mathfrak{T} = (M_s), (\pi_{st})$ be a tree of models over $T = ({}^{<\omega}\omega, \subseteq)$. Let $A \subseteq \mathbb{R}$. Then $\mathfrak{T}$ is an *embedding normal form* (ENF) for A with *base model* M_0 if

$$\forall b \in \mathbb{R}\ (b \in A \longleftrightarrow M_b \text{ is transitive}).$$

It will be important to work with trees of models where one can locally see some information about descending sequences in illfounded branches. The information is given by "witnesses":

Definition 4.4. Let $\mathfrak{T} = (M_s), (\pi_{st})$ be a tree of models over $T = (T, \leq_T)$. A system $(w_s)_{s \in T}$ is called a *system of witnesses* for $\mathfrak{T}$ if:
(a) $\forall s \in T$: $w_s \colon [T] \cap M_s \to \mathrm{On} \wedge w_s \in M_s$;
(b) $\forall s <_T t \in b \in [T] \cap M_t$ (b is illfounded $\Longrightarrow (\pi_{st}(w_s))(b) > w_t(b)$).

Condition (b) expresses that for an illfounded branch b of the form $s_0 <_T s_1 <_T s_2 <_T \ldots$ through T the ordinals $w_{s_0}(b), w_{s_1}(b), w_{s_2}(b), \ldots$ give rise to an infinitely descending $<$-chain in the limit model M_b:

$$M_b \models \pi_{s_0 b}(w_{s_0}(b)) > \pi_{s_1 b}(w_{s_1}(b)) > \cdots.$$

Lemma 4.5. *Let $\mathfrak{T} = (M_s), (\pi_{st})$ be a tree of* ZFC*-models over $T = (T, \leq_T)$. Assume that for every $s \leq_T t$: $[T] \in M_s$, $\pi_{st} \restriction [T] = id$ and M_s is* $\operatorname{card}([T])$*-closed. Then $\mathfrak{T}$ possesses a system of witnesses.*

Proof. Set $B = \{ b \in [T] \mid b \text{ is illfounded}\}$.

(1) For $b \in B$ there is a sequence $(\gamma_s^b \mid s \in b)$ of ordinals such that

$$s <_T t \in b \Longrightarrow \pi_{st}(\gamma_s^b) > \gamma_t^b.$$

Proof. Let $b = \{ s_n \mid n < \omega \} \in B$ with $s_0 <_T s_1 <_T s_2 <_T \cdots$.
Since b is illfounded there is an infinite sequence

$$n(0) < n(1) < \cdots < \omega$$

and ordinals

$$\xi_i \in M_{s_{n(i)}} \quad \text{for } i < \omega$$

so that for $i < j < \omega$:

$$\pi_{s_{n(i)} s_{n(j)}}(\xi_i) > \xi_j.$$

We may assume that $n(0) = 0$. Define for $n(i) \leq n < n(i+1)$:

$$(*) \qquad \gamma^b_{s_n} = \omega \cdot \pi_{s_{n(i)} s_n}(\xi_i) + (n(i+1) - n).$$

The sequence $(\gamma^b_{s_n})$ satisfies the claim since we have a descent in at least one of the two summands in $(*)$. ⊣ (1)

Now define for $s \in T$ functions $w_s : [T] \to \mathrm{On}$,

$$w_s(b) = \begin{cases} \gamma^b_s, & \text{if } s \in b \in B; \\ 0, & \text{else.} \end{cases}$$

$w_s \in M_s$ since $[T] \in M_s$ and M_s is card($[T]$)-closed. If $s <_T t \in b$, b illfounded:

$$\begin{aligned} (\pi_{st}(w_s))(b) &= \pi_{st}(w_s(b)), \quad \text{since } \pi_{st} \restriction [T] = \mathrm{id}, \\ &= \pi_{st}(\gamma^b_s) \\ &> \gamma^b_t = w_t(b). \end{aligned}$$

⊣

If *every* infinite branch through $\mathfrak{T}$ is illfounded, one can improve the above lemma so that the illfoundedness is witnessed by single ordinals instead of ordinal-valued functions.

LEMMA 4.6. *Let $\mathfrak{T} = (M_s), (\pi_{st})$ be a tree of models over $T = (T, \leq_T)$ which satisfies the assumptions of Lemma 4.5. Assume further that every infinite branch through $\mathfrak{T}$ is illfounded. Then there is a system $(\, \mu_s \mid s \in T\,)$ of ordinals such that for $s <_T t$: $\pi_{st}(\mu_s) > \mu_t$. In this case we say that $(\, \mu_s \mid s \in T\,)$ witnesses that $\mathfrak{T}$ is* continously illfounded.

PROOF. There is a system $(w_s)_{s \in T}$ of witnesses for $\mathfrak{T}$ which satisfies:

(1) $\forall s <_T t \in b \in [T]$: $w_t(b) < \pi_{st}(w_s)(b)$.

The system of witnesses given by 4.5 fulfills (1) for all infinite b; this can be modified easily to also encompass all finite $b \in [T]$. We can also assume:

(2) $\forall s \in T\ \forall b \in [T]\ (s \notin b \to w_s(b) = 0)$.

Define, in V, a strict partial order $<^*$ on $T \times {}^{[T]}\mathrm{On}$ by:

$$\begin{aligned} (t, g) <^* (s, f) := t >_T s \wedge \forall b \in [T]\ g(b) \leq f(b) \\ \wedge \forall b \in [T]\ (t \in b \to g(b) < f(b)). \end{aligned}$$

(3) $<^*$ is strongly wellfounded.

PROOF. The second clause in the definition of $<^*$ ensures that the class of $<^*$-predecessors of (s, f) is a set. Assume that for $n < \omega$: $(t_{n+1}, f_{n+1}) <^* (t_n, f_n)$. There is a unique branch $b \in [T]$ such that $\{\, t_n \mid n < \omega\,\} \subseteq b$. Now the third clause in the definition of $<^*$ yields that for $n < \omega$: $f_{n+1}(b) < f_n(b)$. Contradiction. ⊣ (3)

For $s \in T$ let $\mu_s =$ the $<^*$-rank of (s, w_s). The definition is absolute for every M_t in $\mathfrak{T}$ since $\mathfrak{T}$ is card$([T])$-closed. The system of μ_s satisfies the lemma: Let $s <_T t$. By (1) and (2):

(4) $(t, w_t) <^* (s, \pi_{st}(w_s))$.

Hence:

$$\begin{aligned} \pi_{st}(\mu_s) &= \pi_{st}(<^*\text{-rank of } (s, w_s)) \\ &= \text{the } <^*\text{-rank of } (s, \pi_{st}(w_s)) \\ &> \text{the } <^*\text{-rank of } (t, w_t), \quad \text{by (4)}, \\ &= \mu_t. \end{aligned}$$

⊣

§5. Determinacy and embedding normal forms. We consider games played on trees of finite sequences. Let $T \subseteq {}^{<\omega}V$ be closed under the formation of initial segments, $T \neq \emptyset$. Then $T = (T, \subseteq)$ is an ω-tree under the inclusion ordering. The elements of T are the *positions* of the game, the empty sequence $\emptyset$ is the *initial position*. A *play* on T is a branch $b \in [T]$; one often identifies the branch b with its union $\bigcup b$ which is a sequence of length $\leq \omega$. The *game* $G(T, A)$ on T is defined by a *winning set* $A \subseteq [T]$: I *wins the play* b in $G(T, A)$ if $b \in A$, otherwise II *wins the play* b.

The motivating idea is that two "players" I and II produce a play $b \triangleq (a_n \mid n < l)$, $l \leq \omega$, in T as follows: I plays a_0, II plays a_1, I plays a_2, etc. such that $(a_n \mid n < k) \in T$ for each k. Schematically:

I	a_0		a_2		...	
II		a_1		a_3		....

The play continues until a branch b through T is completed. I's aim is to steer that branch into the winning set A.

A *strategy* on T is a partial function $\sigma\colon T \to V$ so that $\forall t \in \operatorname{dom}(\sigma)\ t^\frown\sigma(t) \in T$. A play $b \triangleq (a_n \mid n < l)$ on T is *played by* I *according to the strategy* σ if

$$\forall i\ (2i < l \Longrightarrow a_{2i} = \sigma(a_0, a_1, \ldots, a_{2i-1}));$$

b is *played by* II *according to the strategy* σ if

$$\forall i\ (2i + 1 < l \Longrightarrow a_{2i+1} = \sigma(a_0, a_1, \ldots, a_{2i})).$$

σ is a *winning strategy* for I (respectively II) in $G(T, A)$ if I (respectively II) wins every play b in $G(T, A)$ which is played by I (respectively II) according to σ. We say that $G(T, A)$, or just A, is *determined* if I or II possesses a winning strategy in $G(T, A)$.

One is interested in topological or other conditions which imply the determinacy of a set A. There is a natural topology on $[T]$ which is generated by the basis sets $\{b \in [T] \mid t \in b\}$ for all $t \in T$. Gale and Stewart [5] have shown that $A \subseteq [T]$ is determined in case A is open or closed.

Descriptive set theory is particularly interested in games played on the tree $T = ({}^{<\omega}\omega, \subseteq)$. Plays on T are real numbers $b \in [T] = \mathbb{R}$. A set $A \subseteq \mathbb{R}^l$ is $\mathbf{\Pi}^1_n$ with $n \geq 1$ if A is of the form:

$$\forall \vec{x} \in \mathbb{R}^l\ (\vec{x} \in A \longleftrightarrow \forall z_n \in \mathbb{R}\ \exists z_{n-1} \in \mathbb{R}\ \ldots\ Qz_1 \in \mathbb{R}\ (\vec{x}, z_1, \ldots, z_n) \in B),$$

where $B \subseteq \mathbb{R}^{n+l}$ is open/closed if n is odd/even; the set B can be coded by a single real number p which is called a *defining parameter* for A. A set $C \subseteq \mathbb{R}^l$ is $\mathbf{\Sigma}^1_n$ if $\mathbb{R}^l \setminus C$ is $\mathbf{\Pi}^1_n$. $A \subseteq \mathbb{R}$ is *projective* if A is $\mathbf{\Pi}^1_n$ for some n. $\mathbf{\Pi}^1_n$*-determinacy* is the statement that all $\mathbf{\Pi}^1_n$-sets $A \subseteq \mathbb{R}^l$ are determined. *Projective determinacy* (PD) states that all projective sets $A \subseteq \mathbb{R}^l$ are determined. The *axiom of determinacy* (AD) requires that *all* sets of reals are determined. We shall use some basic properties of projective sets, in particular the absoluteness of Π^1_1-relations and normal forms for $\mathbf{\Pi}^1_1$-sets (see [7] or [13]).

Sets of reals and large cardinals can be linked using embedding normal forms. We shall see that an embedding normal form with witnesses for a set $A \subseteq \mathbb{R}$ implies the determinacy of A.

DEFINITION 5.1. Let $A \subseteq \mathbb{R}$. An *embedding normal form with witnesses* (ENFW) for A is a system $\mathfrak{T} = (M_s), (\pi_{st}), (w_s)$ where $(M_s), (\pi_{st})$ is an embedding normal form for A with witnesses (w_s).

Working with ENFWs is equivalent to working with projections of homogeneous trees:

THEOREM 5.2. *A set $A \subseteq \mathbb{R}$ has an* ENFW*s with base model V if and only if A is the projection of a homogeneous tree.*

A homogeneous tree yields an ENFW consisting of ultrapowers of V by the homogeneity measures. Conversely, given an ENFW, use the witnesses as generators for the required homogeneity measures. This equivalence is the key observation of [14] but is already implicitely proved in [10]. In the context of ENFWs the basic determinacy result takes the following form:

THEOREM 5.3. *Let $A \subseteq \mathbb{R}$ have an ENFW $(M_s), (\pi_{st}), (w_s)$ with base model M_0. Assume at least one of*

(a) *$A \in M_0$ and $\mathbb{R} \in M_0$, or*

(b) *A is $\mathbf{\Pi}^1_1$ with a defining parameter in M_0.*

Then A is determined.

PROOF. We reduce the game $G(A)$ to a game $\tilde{G}$ on an auxiliary tree with a closed winning set; the definition takes place inside the base model M_0:

$\tilde{G}$:						
	I	a_0, f_0		a_2, f_2		$\dots$
	II		a_1		a_3	$\dots,$

with $a_i \in \omega$, $f_{2i}\colon \mathbb{R} \to \theta$ where $\theta \in \mathrm{On}$ is chosen sufficiently large, e.g., $\theta = \sup \mathrm{rge}(w_0) + 1$. Player I wins the play $(a_0, f_0, a_1, a_2, f_2, a_3, \dots)$ if and only if the following rule $(\mathscr{R})$ is satisfied:

$$(\mathscr{R}) \qquad \forall n < \omega\, \forall z \in \mathbb{R} \setminus A\, ((a_0, \dots, a_{2n+2}) \in z \to f_{2n+2}(z) < f_{2n}(z)).$$

Note that in case (b) of the assumptions, $A^{M_0} = A \cap M_0$ by Π^1_1-absoluteness, as M_0 contains a defining parameter for A. So $(\mathscr{R})$ and the definition of $\tilde{G}$ make sense inside M_0. If a play in $\tilde{G}$ violates $(\mathscr{R})$ this already takes place on a finite initial segment of the play. The "losing set" for I in $\tilde{G}$ is thus open, hence $\tilde{G}$ is a closed game which is determined by the Gale-Stewart result.

Let $\tilde{\sigma} \in M_0$ be a winning strategy for I or II in $\tilde{G}$ inside the model M_0.

CASE 1. $M_0 \models \tilde{\sigma}$ is a winning strategyy for I in $\tilde{G}$.

Let σ be the strategy derived from $\tilde{\sigma}$ by "hiding" the auxiliary moves $f_0, f_2, \dots$:

$$\sigma(\emptyset) = a_0 \quad \text{where } \tilde{\sigma}(\emptyset) = (a_0, f_0);$$
$$\sigma(a_0, a_1) = a_2 \quad \text{where } \tilde{\sigma}(\emptyset) = (a_0, f_0) \text{ and } \tilde{\sigma}(a_0, f_0, a_1) = (a_2, f_2);$$
$$\sigma(a_0, a_1, a_2, a_3) = a_4 \quad \text{where } \tilde{\sigma}(\emptyset) = (a_0, f_0) \text{ and } \tilde{\sigma}(a_0, f_0, a_1) = (a_2, f_2) \text{ and } \tilde{\sigma}(a_0, f_0, a_1, a_2, f_2, a_3) = (a_4, f_4);$$

etc.

Obviously $\sigma \in M_0$.

CLAIM 1. *σ is a winning strategy for* I *in $G(A)$ (in V!).*

PROOF. Assume not. Then

(1) $V \models$ there is a play $(a_0, a_1, \dots)$ played by I according to σ so that

$$(a_0, a_1, \dots) \notin A.$$

(2) $M_0 \models$ there is a play $(a_0, a_1, \dots)$ played by I according to σ so that

$$(a_0, a_1, \dots) \notin A.$$

PROOF. Clear in case (a) when $\mathbb{R} \in M_0$ and $A \in M_0$.

In case (b) the statement "there is a play ..." is $\mathbf{\Sigma}^1_1$ in the parameter $\sigma \in M_0$ and some defining parameter $p \in M_0$ for the $\mathbf{\Pi}^1_1$-set A. Then (2) follows from (1) by Π^1_1-absoluteness. ⊣ (2)

Let $x = (a_0, a_1, \dots) \in M_0$ satisfy (2). By the definition of σ there is a play

I	a_0, f_0		a_2, f_2		...
II		a_1		a_3	...

in $\tilde{G}$ in which I follows the winning strategy σ. Since $x \notin A$, rule $(\mathscr{R})$ implies:

$$f_0(x) > f_2(x) > f_4(x) > \cdots,$$

contradiction ⊣ (Claim 1)

CASE 2. $M_0 \models \tilde{\sigma}$ is a winning strategy for II in $\tilde{G}$.

To use $\tilde{\sigma}$ in the original game $G(A)$ player II has to "simulate" moves f_0, f_2, ... for I. To do this, II uses the witnesses w_t of the ENFW for A. These are "descending" along the ENF and provide arbitrarily long sequences of functions satisfying rule $(\mathscr{R})$. Define a strategy σ for II in $G(A)$ by:

$$\sigma(a_0) = \pi_{\emptyset, a_0}(\tilde{\sigma})(a_0, w_{a_0}),$$
$$\sigma(a_0, a_1, a_2) = \pi_{\emptyset, a_0 a_1 a_2}(\tilde{\sigma})(a_0, \pi_{a_0, a_0 a_1 a_2}(w_{a_0}), a_1, a_2, w_{a_0 a_1 a_2})$$
$$\vdots$$
$$\sigma(s) = \pi_{\emptyset, s}(\tilde{\sigma})(s, \pi_{s\restriction 1, s}(w_{s\restriction 1}), \pi_{s\restriction 3, s}(w_{s\restriction 3}), \dots, w_s), \quad \text{for } |s| \text{ odd.}$$

Note that in defining $\sigma(s)$ the strategy $\tilde{\sigma}$ and the witnesses employed all are mapped up to the model M_s of the tree of models where all these images "live together".

CLAIM 2. *σ is a winning strategy for* II *in $G(A)$ (in V!).*

PROOF. Let $x = (a_0, a_1, \dots) \in \mathbb{R}$ be a play in $G(A)$ in which II plays according to σ but assume that $x \in A$. By the normal form property, the direct limit

$$M_x,\ (\pi_{sx})_{s\in x} = \text{dir lim}(M_s)_{s\in x},\ (\pi_{st})_{s\subseteq t\in x}$$

is a transitive $\in$-model. We apply the maps π_{sx} to the defining equations of σ where we set $\tilde{\sigma}^x = \pi_{\emptyset x}(\tilde{\sigma})$ and $w_s^x = \pi_{sx}(w_s)$ for $s \in x$:

$$\begin{aligned} a_1 &= \tilde{\sigma}^x(a_0, w_{a_0}^x) \\ a_3 &= \tilde{\sigma}^x(a_0, w_{a_0}^x, a_1, a_2, w_{a_0a_1a_2}^x) \\ &\vdots \\ a_{2n+1} &= \tilde{\sigma}^x(a_0, w_{a_0}^x, a_1, \dots, w_{a_0a_1\dots a_{2n}}^x). \end{aligned}$$

This amounts to a play

I	$a_0, w_{a_0}^x$		$a_2, w_{a_0a_1a_2}^x$		$\dots$
II		a_1		a_3	$\dots$

in $\pi_{\emptyset x}(\tilde{G})$ in which II plays according to the strategy $\tilde{\sigma}^x$. The play follows the rule $(\mathscr{R})$ for reals in M_x:

$$\begin{aligned} \text{if } n < \omega,\ z \in (\mathbb{R} \cap M_x) \setminus A \text{ and } (a_0, \dots, a_{2n+2}) \in z: \\ w_{x\restriction 2n+3}^x(z) &= \pi_{x\restriction 2n+3,x}(w_{x\restriction 2n+3}(z)) \\ &< \pi_{x\restriction 2n+3,x}(\pi_{x\restriction 2n+1,x\restriction 2n+3}(w_{x\restriction 2n+1})(z)), \\ &\qquad \text{since the } w_s \text{ are witnesses,} \\ &= w_{x\restriction 2n+1}^x(z). \end{aligned}$$

In general, this play according to $\tilde{\sigma}^x$ will not be an element of M_x but we can find an analogous play in M_x by an absoluteness argument. Consider, in M_x, the set of all positions in $\pi_{\emptyset x}(\tilde{G})$ which are obtained by II playing according to $\tilde{\sigma}^x$ and which satisfy the rule $(\mathscr{R})$ for all functions already played. This is a tree in M_x for which the above play a_0, $w_{a_0}^x$, a_1, a_2, $w_{a_0a_1a_2}^x, \dots$ yields an infinite branch in V. Since M_x is a transitive inner model, M_x also contains an infinite branch through the same tree by the absoluteness of wellfoundedness. So in M_x there is a play in which II plays according to $\tilde{\sigma}^x$ and in which $(\mathscr{R})$ is satisfied. That play is won by I and so

$$M_x \models \tilde{\sigma}^x \text{ is } \textit{not} \text{ a winning strategy for II in } \pi_{\emptyset x}(\tilde{G}).$$

Since $\pi_{\emptyset x}$ is elementary,

$$M_0 \models \tilde{\sigma} \text{ is } \textit{not} \text{ a winning stategy for II in } \tilde{G},$$

contradicting the assumption of Case 2. ⊣

§6. Normal forms for Π_1^1-sets. We are going to obtain embedding normal forms with witnesses for Π_1^1-sets from iterated ultrapowers and from Silver indiscernibles ("sharps"). We start from an ordinary normal form which will be lifted into the realm of large cardinals by an Ehrenfeucht-Mostowski technique.

THEOREM 6.1. *Let $A \subseteq \mathbb{R}$ be a $\mathbf{\Pi}^1_1$-set. Then there is a system $(|s|)_{s\in T}$, $(e_{st})_{s\leq_T t}$ over the tree $(T, \leq_T) = ({}^{<\omega}\omega, \subseteq)$ which is a normal form for A in the following sense:*
(a) $s \leq_T t \Longrightarrow e_{st}\colon |s| \to |t|$ *is orderpreserving;*
(b) $r \leq_T s \leq_T t \Longrightarrow e_{rt} = e_{st} \circ e_{rs}$;
(c) $\forall x \in \mathbb{R}\ (x \in A \longleftrightarrow (|s|, <)_{s\in x}, (e_{st})_{s\leq_T t\in x}$ *has a wellfounded direct limit).*
Such a system can be constructed recursively from any defining parameter for A.

REMARK. Clauses (a) and (b) express that the system is a tree of natural numbers connected by orderpreserving maps, in analogy to the trees of models introduced in 4.2, (c) corresponds to the crucial property for embedding normal forms (Definition 4.3).

PROOF. It is essentially shown in [13, Lemma 6G.6] that A has a representation of the following form: there is an assignment $s \mapsto <_s$ for $s \in {}^{<\omega}\omega$ such that:
(1) $<_s$ linearly orders $|s|$;
(2) $s \leq_T t \in {}^{<\omega}\omega \to <_s \subseteq <_t$;
(3) $\forall x \in \mathbb{R}\ (x \in A \longleftrightarrow <_x := \bigcup_{s\in x} <_s$ is a wellordering of ω).
For $s \in T$ let

$$h_s\colon (|s|, <) \cong (|s|, <_s)$$

be $<$-$<_s$-orderpreserving. For $s \leq_T t \in T$ define

$$e_{st} = h_t^{-1} \circ h_s.$$

By (2), e_{st} is orderpreserving and (a) holds. Clause (b) follows directly from the definition of the e_{st}. For (c), consider $x \in \mathbb{R}$. The system

$$(|s|, <_s)_{s\in x},\ (\mathrm{id} \restriction |s|)_{s\leq_T t\in x}$$

is via $(h_s^{-1})_{s\in x}$ isomorphic to

$$(|s|, <)_{s\in x},\ (e_{st})_{s\leq_T t\in x}.$$

Property (3) implies:

$$x \in A \longleftrightarrow (|s|, <)_{s\in x}, (e_{st})_{s\leq_T t\in x} \text{ has a wellfounded direct limit.}$$

Inspection of the proof in [13] shows that a system $(<_s)_{s\in T}$ as above can be found recursively from any defining parameter for A. By definition, the system $(e_{st})_{s\leq_T t\in T}$ is explicitly recursive in $(<_s)_{s\in T}$. ⊣

Let us now recall some key facts about *iterated ultrapowers*. These could be constructed as iterated extensions but it is easier here to keep to the standard presentation as in [6].

From a normal ultrafilter U on a measurable cardinal κ one defines the following linear system of ZFC-models.

$$N_0 = V, \quad \pi_{00} = \mathrm{id}, \quad \kappa_0 = \kappa, \quad U_0 = U;$$

$$N_{\alpha+1} = \mathrm{Ult}(N_\alpha, U_\alpha) \text{ is the ultrapower of } N_\alpha \text{ by } U_\alpha,$$

$$\pi_{\alpha,\alpha+1}\colon N_\alpha \to_{U_\alpha} N_{\alpha+1} \text{ is the natural embedding into the ultrapower,}$$

$$\pi_{\alpha+1,\alpha+1} = \mathrm{id}, \quad \pi_{\gamma,\alpha+1} = \pi_{\alpha,\alpha+1} \circ \pi_{\gamma,\alpha} \quad \text{for } \gamma < \alpha,$$

$$\kappa_{\alpha+1} = \pi_{0,\alpha+1}(\kappa_0), \quad U_{\alpha+1} = \pi_{0,\alpha+1}(U_0);$$

for limit ordinals λ let N_λ, $(\pi_{\alpha\lambda})_{\alpha\le\lambda}$ be the transitive direct limit of $(N_\alpha)_{\alpha<\lambda}$, $(\pi_{\alpha\beta})_{\alpha\le\beta<\lambda}$, $\kappa_\lambda = \pi_{0,\lambda}(\kappa_0)$, $U_\lambda = \pi_{0,\lambda}(U_0)$.

The following two statements express that N_α is the Ehrenfeucht-Mostowski model for the (class-sized) theory of $(V, \in)$ with constant symbols for every set $x \in V$; that model is generated by the wellorder α.

LEMMA 6.2. *The set* $\{ \kappa_i \mid i < \alpha \}$ *is a set of order-indiscernibles for* N_α *relative to parameters from* $\operatorname{rng}(\pi_{0\alpha})$.

LEMMA 6.3.

$$N_\alpha = \{ \pi_{0\alpha}(f)(\kappa_{i_1}, \dots, \kappa_{i_n}) \mid n \in \omega,\ f : \kappa^n \to V,\ i_1 < \cdots < i_n < \alpha \}.$$

These facts yield lifting properties for orderpreserving maps.

LEMMA 6.4. *Let* $e : \alpha \to \beta$ *be strictly orderpreserving,* $\alpha \le \beta \in \mathrm{On}$. *Then there is a canonical map*

$$e^* : N_\alpha \to N_\beta$$

defined by:

$$e^*(\pi_{0\alpha}(f)(\kappa_{i_1}, \dots, \kappa_{i_n})) = \pi_{0\beta}(f)(\kappa_{e(i_1)}, \dots, \kappa_{e(i_n)}),$$

for all $n < \omega$, $f : \kappa^n \to V$, $i_1 < \cdots < i_n < \alpha$.

LEMMA 6.5. *If* $(e_{mn})_{m\le n<\omega}$ *is a commutative system of orderpreserving maps* $e_{mn} : m \to n$, *then* $(e^*_{mn})_{m\le n<\omega}$ *commutes. Moreover, the system* $(m)_{m<\omega}$, $(e_{mn})_{m\le n<\omega}$ *has a wellfounded direct limit if and only if the system* $(N_m)_{m<\omega}$, $(e^*_{mn})_{m\le n<\omega}$ *has a wellfounded direct limit.*

PROOF. Commutativity is trivial. For the other statement observe that the system (m), (e_{mn}) is orderpreservingly embedded into (N_m), (e^*_{mn}) by the maps $m \to N_m$, $i \mapsto \kappa_i$. So if (m), (e^*_{mn}) has an illfounded direct limit so has (N_m), (e^*_{mn}). On the other hand let (m), (e_{mn}) have a wellfounded direct limit, say

$$\alpha,\ (e_m)_{m<\omega} = \operatorname{dir\,lim}(m),\ (e_{mn}),$$

where α is an ordinal. It is straightforward to check that N_α, $(e^*_m)_{m<\omega}$ is the transitive direct limit of (N_m), (e^*_{mn}). ⊣

THEOREM 6.6. *Assume there is a measurable cardinal* κ. *Then every* $\mathbf{\Pi}^1_1$*-set possesses an embedding normal form with witnesses with base model* V *and critical point* $\ge \kappa$.

PROOF. Let $A \subseteq \mathbb{R}$ be $\mathbf{\Pi}^1_1$ and let $(|s|)_{s\in T}$, $(e_{st})_{s\le_T t}$ be the normal form for A given by Theorem 6.1. Let $(N_\alpha)_{\alpha\in\mathrm{On}}$, $(\pi_{\alpha\beta})_{\alpha\le\beta\in\mathrm{On}}$ be the iterated ultrapowers of V by a measure on κ. Then define

$$\mathfrak{T} = (N_{|s|})_{s\in T},\ (e^*_{st})_{s\le_T t}.$$

For $x \in \mathbb{R}$,

$$x \in A \longleftrightarrow (|s|, <)_{s\in x},\ (e_{st})_{s\le_T t\in x} \text{ has a wellfounded direct limit} \quad \text{(Theorem 6.1 (c))}$$
$$\longleftrightarrow (N_{|s|})_{s\in x},\ (e^*_{st})_{s\le_T t\in x} \text{ has a wellfounded direct limit} \quad \text{(Lemma 6.5).}$$

Hence $\mathfrak{T}$ is an ENF for A with base model $N_0 = V$ and critical point $\ge \kappa$. $\mathfrak{T}$ is built from finite iterates of V and each of these is κ-closed; this is a standard fact, see also Theorem 3.5 (c). By Lemma 4.5, $\mathfrak{T}$ has a system of witnesses. ⊣

An immediate corallary using Theorem 5.3 is the classic result of Martin [9]:

THEOREM 6.7. *If there is a measurable cardinal then* $\mathbf{\Pi}^1_1$*-determinacy holds.*

The usual strengthening from measurable cardinals to "sharps" can also be carried out for embedding normal forms. This will also be used for a strong form of the Martin-Steel result.

Let $w = (w_0, <_0)$ consist of a transitive set w_0 wellordered by $<_0$. We want to define the notion "$w^\sharp$ exists". Let $N_0 = L(w)$ be the smallest inner model containing w as an element. $L(w)$ satisfies AC since w is a wellorder. Assume now that

$$I = \{ \kappa_i \mid i \in \mathrm{On} \} \subseteq \mathrm{On}$$

is a class of *Silver-indiscernibles* for $L(w)$, i.e.:

(a) $i < j \to \kappa_i < \kappa_j$;

(b) I is a class of order-indiscernibles for the structure $(L(w), (z \mid z \in \mathrm{TC}(w)))$: if $\varphi(\vec{u}, \vec{v})$ is an $\in$-formula, $\vec{z} \in \mathrm{TC}(w)$, $\vec{\kappa}$, $\vec{\lambda} \in I$ strictly increasing sequences of appropriate length then

$$L(w) \models \varphi(\vec{z}, \vec{\kappa}) \longleftrightarrow L(w) \models \varphi(\vec{z}, \vec{\lambda}).$$

(c) I generates the structure $(L(w), (z \mid z \in \mathrm{TC}(w)))$: there is a ZF-term $t(v_0, v_1)$ such that

$$L(w) = \{ t^{L(w)}(\vec{z}, \vec{\kappa}) \mid \vec{z} \in \mathrm{TC}(w), \vec{\kappa} \in I \}.$$

We describe two cases of particular interest to us:

1. $w_0 = \mathrm{TC}(\{a\})$ for some real $a \in \mathbb{R}$ and $<_0$ a natural wellorder of w_0. Then $L(w) = L(a)$ and we paraphrase properties (a)–(c) as "$a^\sharp$ exists".

2. $w = (V_\delta, <_0)$ for some "big" ordinal δ. We then abbreviate (a)–(c) as "$V_\delta^\sharp$ exists", although correctly speaking this depends on the choice of $<_0$.

In general, (a)–(c) are described as "$w^\sharp$ exists". Note that usually one normalizes the indiscernible class by some minimality condition which is called "remarkability"; this is not necessary here. We can use the Silver-indiscernibles to define an "iteration" of $L(w)$ which behaves much like iterated ultrapowers: For $\alpha \in \mathrm{On}$ let $N_\alpha = L(w)$; define

$$\pi_{0\alpha} \colon N_0 \to N_\alpha$$

by:

$$t^{L(w)}(\vec{z}, \kappa_{i_1}, \ldots, \kappa_{i_n}) \longmapsto t^{L(w)}(\vec{z}, \kappa_{\alpha+i_1}, \ldots \kappa_{\alpha+i_n})$$

for $\vec{z} \in \mathrm{TC}(w)$ and $i_1 < \cdots < i_n \in \mathrm{On}$. Conditions (b) and (c) imply that Lemmas 6.2 and 6.3 transfer verbatim to the new situation:

LEMMA 6.8. *For each* $\alpha \in \mathrm{On}$:

(a) *The set* $\{ \kappa_i \mid i < \alpha \}$ *is a set of order-indiscernibles for* N_α *relative to parameters from* $\mathrm{rng}(\pi_{0\alpha})$.

(b) $N_\alpha = \{ \pi_{0\alpha}(f)(\kappa_{i_1}, \ldots, \kappa_{i_n}) \mid n \in \omega, f \colon \kappa^n \to V, i_1 < \cdots < i_n < \alpha \}$.

We can then define the liftings $e \mapsto e^*$ with the properties described in Lemmas 6.4 and 6.5 as before.

THEOREM 6.9. *Let $A \subseteq \mathbb{R}$ be a $\mathbf{\Pi}^1_1$-set in a defining parameter $a \in \mathbb{R}$. Assume that $w^\sharp$ exists where $w = (w_0, <_0)$ and $a \in w_0$. Then A possesses an embedding normal form with witnesses with base model $L(w)$ and critical point $> \mathrm{rk}(w)$.*

PROOF. Let $\mathfrak{N} = (|s|)_{s \in T}, (e_{st})_{s \leq_T t}$ be a normal form for A as in Theorem 6.1, where $\mathfrak{N}$ is recursive in a. Hence $\mathfrak{N} \in L(w)$. As in the proof of Theorem 6.6, $\mathfrak{N}$ lifts to an embedding normal form

$$\mathfrak{T} = (N_{|s|})_{s \in T},\ (e^*_{st})_{s \leq_T t}$$

for A with base model $N_0 = L(w)$. Since every ordinal $\leq \mathrm{rk}(w)$ is definable from constants in $L(w)$, the critical point of $\mathfrak{T}$ is $> \mathrm{rk}(w)$. It remains to find a system of witnesses for $\mathfrak{T}$.

Work inside the model $L(w)$. We construct a kind of witnesses for the system $\mathfrak{N}$. If $x \in \mathbb{R} \setminus A$, the corresponding branch through $\mathfrak{N}$ is illfounded and we can choose a sequence $(i^x_n \mid n < \omega)$ such that:

(1) $i^x_n \in n$, for $0 < n < \omega$;

(2) $e_{x \restriction m, x \restriction n}(i^x_m) \geq i^x_n$, for $0 < m \leq n < \omega$;

(3) $e_{x \restriction n, x \restriction n+1}(i^x_n) > i^x_{n+1}$, for infinitely many $n < \omega$.

Define a further sequence $(k^x_n \mid n < \omega)$:

$$k^x_n = \text{the smallest } k \text{ such that } e_{x \restriction n+k,\, x \restriction n+k+1}(i^x_{n+k}) > i^x_{n+k+1}.$$

By (3), there is always some "strict" descent for the (i^x_n) or the (k^x_n):

(4) $\forall x \in \mathbb{R} \setminus A\ \forall 0 < m < n < \omega$: $e_{x \restriction m, x \restriction n}(i^x_m) > i^x_n$ or $k^x_m > k^x_n$.

Now define $(w_s)_{s \in T}$ in V by:

$$w_0(x) = \begin{cases} \kappa_0, & \text{if } x \in \mathbb{R} \setminus A; \\ 0, & \text{else.} \end{cases}$$

$$w_s(x) = \begin{cases} \kappa_{i^x_n} + k^x_n, & \text{if } s = x \restriction n \neq \emptyset \text{ and } x \in \mathbb{R} \setminus A; \\ 0, & \text{else.} \end{cases}$$

(5) $w_s \in N_{|s|} = L(w)$, since the definition of w_s refers to $\mathfrak{N} \in L(w)$ and the *finite* set $\{\kappa_0, \dots, \kappa_{|s|}\} \in L(w)$ and can be carried out in $L(w)$.

(6) $(w_s)_{s \in T}$ is a system of witnesses for $\mathfrak{T}$.

PROOF. Let $s <_T t \in x \in (\mathbb{R} \cap L(w)) \setminus A$.
If $0 < m = |s| < n = |t|$:

$$\begin{aligned} e^*_{st}(w_s)(x) &= e^*_{st}(w_s(x)) = e^*_{st}(\kappa_{i^x_m} + k^x_m) \\ &= \kappa_{e_{st}(i^x_m)} + k^x_m \\ &> \kappa_{i^x_n} + k^x_n, \quad \text{by (4)}, \\ &= w_t(x). \end{aligned}$$

If $0 = m = |s| < n = |t|$:

$$\begin{aligned} e^*_{st}(w_s)(x) &= \pi_{0n}(w_0(x)) = \pi_{0n}(\kappa_0) = \kappa_n \\ &> \kappa_{i^x_n} + k^x_n = w_t(x). \end{aligned}$$

So we get the stronger theorem of Martin's:

THEOREM 6.10.
(a) *Let* $A \subseteq \mathbb{R}$ *be a* $\mathbf{\Pi}^1_1$*-set in a defining parameter* $a \in \mathbb{R}$, *and assume that* $a^\sharp$ *exists. Then* A *is determined.*
(b) *If* $\forall a \in \mathbb{R}\ a^\sharp$ *exists then* $\mathbf{\Pi}^1_1$*-determinacy holds.*

Let us briefly discuss the necessity of some witness property for the determinacy proofs. We get ENFs for any set of reals from $0^\sharp$, hence in general ENFs without witnesses are not strong enough to prove determinacy.

LEMMA 6.11. *Assume that* $0^\sharp$ *exists. Then every set* $A \subseteq \mathbb{R}$ *has an embedding normal form with base model* L.

PROOF. $L = L(w)$ with $w = (\emptyset, \emptyset)$. $0^\sharp$ yields an "iteration" $(N_\alpha)_{\alpha \in \mathrm{On}}, (\pi_{\alpha\beta})_{\alpha \leq \beta}$ as described in Lemma 6.8. $N_\alpha = L$ for every $\alpha \in \mathrm{On}$. Let $\langle x_r \mid r < \delta \rangle$ be an enumeration of $\mathbb{R}$ where δ is some infinite cardinal. For $s \leq_T t \in {}^{<\omega}\omega$ define

$$e_{st} : \delta \rightarrow \delta$$

by

$$e_{st}(\omega \cdot r + k) = \begin{cases} \omega \cdot r + k + 1, & \text{if } t \in x_r \text{ and } x_r \notin A; \\ \omega \cdot r + k, & \text{else;} \end{cases}$$

where we assume $r < \delta$ and $k < \omega$. Then

$$(L)_{s \in T}, \ (e^*_{st})_{s \leq_T t}$$

is an ENF for A. The details are left to the reader. ⊣

§7. Iteration trees and Steel's lemma. The determinacy results of the preceding section rest on the construction of embedding normal forms from measures and sharps. Consistency strength considerations imply that we cannot prove $\mathbf{\Pi}^1_2$-determinacy from a measurable cardinal, and so one cannot build good ENFs for arbitrary $\mathbf{\Pi}^1_2$-sets from ordinary iterated ultrapowers. In the proof of the Martin-Steel-theorem more complicated iteration mechanisms which allow to code more information into the iterates are employed.

DEFINITION 7.1. A system $\mathfrak{I} = (M_i)_{i<l}, (i^*, E_i)_{i+1<l}$ is called an *iteration tree* if:
(a) $l \leq \omega$; l is the *length* of the tree $\mathfrak{I}$; $\mathfrak{I}$ is *finite* if $l < \omega$ and *infinite* otherwise;
(b) each M_i is a transitive model of ZFC;
(c) $E_i : S_i \prec T_i$ is an extender on M_i; $E_i \in M_i$;
(d) $i^* \leq i$;
(e) $\mathscr{P}(S_i) \cap M_{i^*} = \mathscr{P}(S_i) \cap M_i \in T_{i^*}$;
(f) $M_{i+1} = \mathrm{Ext}(M_{i^*}, E_i)$;
(g) $T_i \subseteq T_{i+1}$.
$\mathfrak{I}$ is an η*-closed* iteration tree if each M_i and each E_i in $\mathfrak{I}$ is η-closed.

REMARK. Our iteration trees are more usually called *iteration trees of length* $\leq \omega$.

We imagine the iteration tree $\mathfrak{I}$ as a recursive construction in l stages. At stage i, where $i+1 < l$, an extender E_i is chosen in M_i. Then a stage $i^* \leq i$ is chosen for the application of the extender. The tree of models generated can attain a complicated branching structure. To form $\mathrm{Ext}(M_{i^*}, E_i)$ sufficient agreement between M_i and M_{i^*} is required. This is expressed in condition (e). Putting $M_{i+1} = \mathrm{Ext}(M_{i^*}, E_i)$ continues the construction. The agreement between the models M_i is controlled by the targets T_i of the extenders. T_i is a subset of M_i, of $\mathrm{Ext}(M_i, E_i)$, and of $M_{i+1} = \mathrm{Ext}(M_{i^*}, E_i)$. By the growth condition (g) this implies $T_i \subseteq M_j$ for all further $j > i$. Condition (e) says that when we go back to the model M_{i^*} at stage i, the necessary agreement between M_i and M_{i^*} is already in the guaranteed agreement set T_{i^*}.

An iteration tree is also a tree of models: Let $I = (l, \leq_I)$ be the tree order on l generated as the transitive reflexive closure of all pairs $(i^*, i+1)$. Set

$$\pi_{i^*,i+1} = \pi_{M_{i^*},E_i} : M_{i^*} \to_{E_i} M_{i+1}$$

and let

$$\tilde{\mathfrak{I}} = (M_i)_{i \in I},\ (\pi_{ij})_{i \leq_I j}$$

be the tree of models generated from the $\pi_{i^*,i+1}$ by compositions along the $\leq_I$-ordering. For a branch b through $I = (l, \leq_I)$ let

$$M_b,\ (\pi_{ib})_{i \in b} = \operatorname{dir\,lim}(M_i)_{i \in b},\ (\pi_{ij})_{i \leq_I j \in b}$$

be the direct limit along the branch with the wellfounded part of M_b being transitive.

Later we shall piece together ENFs from branches of iteration trees. The crucial device for controlling the wellfoundedness of branches is the following result of Martin and Steel of which we present a simple but sufficient instance. The argument was suggested by a more general proof in [11].

THEOREM 7.2. *Let $\mathfrak{I}$ be an infinite $2^{\aleph_0}$-closed iteration tree. Then $\mathfrak{I}$ possesses at least one infinite branch $b \subseteq \omega$ such that M_b is transitive.*

PROOF. Assume that $\mathfrak{I} = (M_i)_{i<\omega}, (i^*, E_i)_{i<\omega}$ is a counterexample. We use the notations introduced in this section so far. Let $\eta = 2^{\aleph_0}$. By Lemma 4.6, the tree $\tilde{\mathfrak{I}} = (M_i)_{i\in\omega}, (\pi_{ij})_{i\leq_I j}$ is continuously illfounded with a system $(\mu_i)_{i\in\omega}$ of ordinals satisfying $\pi_{ij}(\mu_i) > \mu_j$ whenever $i <_I j$. By Lemma 3.10 there is a strong limit cardinal $\gamma = \beth_\gamma$ which is a fixed point of all the embeddings π_{ij} and such that $\gamma > \mathrm{rk}(E_i)$ for all $i < \omega$. For $i < \omega$ let $\gamma_i = (\beth^+_{\gamma+\mu_i\cdot 2})^{M_i}$. γ_i is a successor cardinal inside M_i. Let $M'_i = (H_{\gamma_i})^{M_i}$. The following properties of the system $(M'_i, i^*, E_i)_{i<\omega}$ correspond to the conditions in Definition 7.1 (b)–(g):

(1) M'_i is an η-closed transitive model of ZFC^- and the Skolem principle SP;

(2) $E_i : S_i \prec T_i$ is an extender on M'_i, $E_i \in M'_i$;

(3) $i^* \leq i$;

(4) $\mathscr{P}(S_i) \cap M'_{i^*} = \mathscr{P}(S_i) \cap M'_i \in T_{i^*}$;

(5) $M'_{i+1} \in \mathrm{Ext}(M'_{i^*}, E_i)$;

(6) $T_i \subseteq T_{i+1}$, $T_i \in M'_{i+1}$.

PROOF OF (5).

$$\begin{aligned}\pi_{i^*,i+1}(\gamma_{i^*}) &= \pi_{i^*,i+1}((\beth^+_{\gamma+\mu_{i^*}\cdot 2})^{M_{i^*}}) \\ &= (\beth^+_{\gamma+\pi_{i^*,i+1}(\mu_{i^*})\cdot 2})^{M_{i+1}} \\ &> (\beth_{\gamma+\pi_{i^*,i+1}(\mu_{i^*})\cdot 2})^{M_{i+1}} \\ &\geq (\exp_2(\exp_2(\beth_{\gamma+\mu_{i+1}\cdot 2})))^{M_{i+1}}, \quad \text{since } \pi_{i^*,i+1}(\mu_{i^*}) > \mu_{i+1}, \\ &\geq (\exp_2(\beth^+_{\gamma+\mu_{i+1}\cdot 2}))^{M_{i+1}} \\ &= (\exp_2(\gamma_{i+1}))^{M_{i+1}} \\ &\geq \operatorname{card}(H_{\gamma_{i+1}})^{M_{i+1}}.\end{aligned}$$

Therefore

$$\begin{aligned}M'_{i+1} = (H_{\gamma_{i+1}})^{M_{i+1}} &\in (H_{\pi_{i^*,i+1}(\gamma_{i^*})})^{M_{i+1}} \\ &= \operatorname{Ext}((H_{\gamma_{i^*}})^{M_{i^*}}, E_i), \quad \text{by Lemma 3.11,} \\ &= \operatorname{Ext}(M'_{i^*}, E_i). \qquad \dashv (5)\end{aligned}$$

By a downward Löwenheim-Skolem argument the situation (1)–(6) is reflected down to the hereditarily countable sets. Let H be a transitive model of sufficiently many axioms of ZFC and let $(M'_i, i^*, E_i)_{i<\omega} \in H$. Let $X \prec H$ be countable such that $(M'_i, i^*, E_i)_{i<\omega} \in X$. Let $\sigma\colon \bar{H} \cong X \prec H$, $\bar{H}$ transitive and let $\sigma((\bar{M}_i, i^*, \bar{E}_i)_{i<\omega}) = (M'_i, i^*, E_i)_{i<\omega}$, $\sigma(\bar{\eta}) = \eta$. Properties (1)–(6) imply:

(7) $\bar{M}_i$ is a countable transitive model of $\mathrm{ZFC}^- + \mathrm{SP}$;

(8) $\bar{M}_i \models \bar{E}_i\colon \bar{S}_i \prec \bar{T}_i$ is an $\bar{\eta}$-closed extender on V;

(9) $i^* \leq i$;

(10) $\mathscr{P}(\bar{S}_i) \cap \bar{M}_{i^*} = \mathscr{P}(\bar{S}_i) \cap \bar{M}_i \in \bar{T}_{i^*}$;

(11) $\bar{M}_{i+1} \in \operatorname{Ext}(\bar{M}_{i^*}, \bar{E}_i)$;

(12) $\operatorname{Ext}(\bar{M}_{i^*}, \bar{E}_i) \models \bar{M}_{i+1}$ is $\bar{\eta}$-closed;

(13) $\bar{T}_i \subseteq \bar{T}_{i+1}$, $\bar{T}_i \in \bar{M}_{i+1}$;

(14) $\sigma_0\colon \bar{M}_0 \to M'_0$ is elementary, where $\sigma_0 = \sigma \restriction \bar{M}_0$;

(15) M'_0 is η-closed.

Now we lift the countable system $(\bar{M}_i, i^*, \bar{E}_i)_{i<\omega}$ up into the uncountable again so that the "descent" in (11) is transformed into an infinite descending $\in$-chain (19) which establishes the desired contradiction. We shall construct a system $(\tilde{M}_i, \sigma_i)_{i<\omega}$ by recursion satisfying:

(16) $\tilde{M}_i$ is a transitive η-closed model of $\mathrm{ZFC}^- + \mathrm{SP}$;

(17) $\sigma_i\colon \bar{M}_i \to \tilde{M}_i$ is elementary;

(18) $i \leq j \Longrightarrow \sigma_i \restriction \bar{T}_i = \sigma_j \restriction \bar{T}_i$;

(19) $\tilde{M}_i \in \tilde{M}_{i-1}$ for $i \geq 1$.

For $i = 0$ let $\tilde{M}_0 = M_0'$ and σ_0 as described in (14). Then (16)–(19) are trivially satisfied up to $i = 0$.

Assume the system $(\tilde{M}_j, \sigma_j)_{j\le i}$ has been constructed satisfying (16)–(19) and we have to define $\tilde{M}_{i+1}$ and σ_{i+1}. Let $(\tilde{E}_i, \tilde{S}_i, \tilde{T}_i) = \sigma_i(\bar{E}_i, \bar{S}_i, \bar{T}_i)$. By (17),

$$\tilde{M}_i \models \tilde{E}_i\colon \tilde{S}_i \prec \tilde{T}_i \text{ is an } \eta\text{-closed extender on } V.$$

Since $\tilde{M}_i$ is η-closed (16), the universe V satisfies

$$\tilde{E}_i\colon \tilde{S}_i \prec \tilde{T}_i \text{ is an } \eta\text{-closed extender on } \tilde{M}_i.$$

(20) $\mathscr{P}(\tilde{S}_i) \cap \tilde{M}_i = \mathscr{P}(\tilde{S}_i) \cap \tilde{M}_{i^*}$.

Proof.

$$\begin{aligned}
\mathscr{P}(\tilde{S}_i) \cap \tilde{M}_i &= \sigma_i(\mathscr{P}(\bar{S}_i) \cap \bar{M}_i), && \text{by (10), (17),}\\
&= \sigma_i(\mathscr{P}(\bar{S}_i) \cap \bar{M}_{i^*}), && \text{by (10),}\\
&= \sigma_{i^*}(\mathscr{P}(\bar{S}_i) \cap \bar{M}_{i^*}), && \text{by (10), (18),}\\
&= \mathscr{P}(\tilde{S}_i) \cap \tilde{M}_{i^*}, && \text{by (10), (18).}
\end{aligned} \qquad \dashv (20)$$

So $\tilde{E}_i\colon \tilde{S}_i \prec \tilde{T}_i$ is also an η-closed extender on $\tilde{M}_{i^*}$. Let

$$\tilde{\pi}\colon \tilde{M}_{i^*} \to_{\tilde{E}_i} \operatorname{Ext}(\tilde{M}_{i^*}, \tilde{E}_i).$$

By Theorem 3.3, $\operatorname{Ext}(\tilde{M}_{i^*}, \tilde{E}_i)$ is transitive and η-closed. Let

$$\bar{\pi}\colon \bar{M}_{i^*} \to_{\bar{E}_i} \operatorname{Ext}(\bar{M}_{i^*}, \bar{E}_i)$$

be the corresponding map for the countable structures.

(21) There is an elementary embedding $\sigma\colon \operatorname{Ext}(\bar{M}_{i^*}, \bar{E}_i) \to \operatorname{Ext}(\tilde{M}_{i^*}, \tilde{E}_i)$ defined by $\bar{\pi}(f)(a) \longmapsto \tilde{\pi}(\sigma_{i^*}(f))(\sigma_i(a))$.

Proof. Let $\varphi(v_1, \ldots, v_n)$ be an $\in$-formula and $\bar{\pi}(f_k)(a_k) \in \operatorname{Ext}(\bar{M}_{i^*}, \bar{E}_i)$, $f_k\colon \bar{S}_i \to \bar{M}_{i^*}$, $f_k \in \bar{M}_{i^*}$, $a_k \in \bar{T}_i$ for $k = 1, \ldots, n$. Then

$$\operatorname{Ext}(\bar{M}_{i^*}, \bar{E}_i) \models \varphi(\bar{\pi}(f_1)(a_1), \ldots, \bar{\pi}(f_n)(a_n))$$

if and only if

$$(a_1, \ldots, a_n) \in \bar{E}_i\{\, (u_1, \ldots, u_n) \in \bar{S}_i^n \mid \bar{M}_{i^*} \models \varphi(f_1(u_1), \ldots, f_n(u_n)) \,\},$$

by the Łoś-property of Lemma 2.5, if and only if

$$\begin{aligned}
&\sigma_i(a_1, \ldots, a_n)\\
&\quad \in \tilde{E}_i \sigma_i\{\, (u_1, \ldots, u_n) \in \bar{S}_i^n \mid \bar{M}_{i^*} \models \varphi(f_1(u_1), \ldots, f_n(u_n)) \,\}\\
&\qquad = \tilde{E}_i \sigma_{i^*}\{\, (u_1, \ldots, u_n) \in \bar{S}_i^n \mid \bar{M}_{i^*} \models \varphi(f_1(u_1), \ldots, f_n(u_n)) \,\},
\end{aligned}$$

by (10), (18),

$$= \tilde{E}_i\{\, (u_1, \ldots, u_n) \in \tilde{S}_i^n \mid \tilde{M}_{i^*} \models \varphi(\sigma_{i^*}(f_1)(u_1), \ldots, \sigma_{i^*}(f_n)(u_n)) \,\}$$

if and only if

$$\operatorname{Ext}(\tilde{M}_{i^*}, \tilde{E}_i) \models \varphi(\tilde{\pi}(\sigma_{i^*}(f_1))(\sigma_i(a_1)), \ldots, \tilde{\pi}(\sigma_{i^*}(f_n))(\sigma_i(a_n))). \qquad \dashv (21)$$

By (11) we can apply σ to $\bar{M}_{i+1}$ and then by (12), $\text{Ext}(\tilde{M}_{i^*}, \tilde{E}_i) \models \sigma(\bar{M}_{i+1})$ is η-closed. Since $\text{Ext}(\tilde{M}_{i^*}, \tilde{E}_i)$ is η-closed, V satisfies:

(22) $\sigma(\bar{M}_{i+1})$ is η-closed.

(23) $\sigma\restriction\bar{M}_{i+1}: \bar{M}_{i+1} \to \sigma(\bar{M}_{i+1})$ is elementary.

(24) $\sigma\restriction\bar{M}_{i+1} \in \sigma(\bar{M}_{i+1})$, since $\sigma\restriction\bar{M}_{i+1}$ is a map with hereditarily countable domain and $\sigma(\bar{M}_{i+1})$ is η-closed.

(25) $\tilde{T}_i \subseteq \sigma(\bar{M}_{i+1})$.

PROOF.

$$\begin{aligned} \tilde{T}_i = \tilde{\pi}(\tilde{S}_i) = \tilde{\pi}(\sigma_i(\bar{S}_i)) = \tilde{\pi}(\sigma_{i^*}(\bar{S}_i)), &\quad \text{by (10) and (18)},\\ = \sigma(\bar{\pi}(\bar{S}_i)) = \sigma(\bar{T}_i) \subseteq \sigma(\bar{M}_{i+1}), &\quad \text{by (13)}. \end{aligned}$$

⊣ (25)

Inside $\text{Ext}(\tilde{M}_{i^*}, \tilde{E}_i)$ let Y be an η-closed elementary substructure of $\sigma(\bar{M}_{i+1})$ such that $Y \supseteq \tilde{T}_i \cup \{\sigma\restriction\bar{M}_{i+1}\}$ and such that Y is of minimal size. Y exists since $\sigma(\bar{M}_{i+1})$ itself is η-closed inside $\text{Ext}(\tilde{M}_{i^*}, \tilde{E}_i)$. Since $\tilde{T}_i$ is η-closed in $\text{Ext}(\tilde{M}_{i^*}, \tilde{E}_i)$ and Y has the minimal possible size:

(26) There is a bijection $\tilde{T}_i \longleftrightarrow Y$ in $\text{Ext}(\tilde{M}_{i^*}, \tilde{E}_i)$.

Let $\rho: Y \cong \tilde{M}_{i+1}$, $\tilde{M}_{i+1}$ transitive, be the Mostowski collapse of Y and set

$$\sigma_{i+1} = \rho \circ \sigma : \bar{M}_{i+1} \to \tilde{M}_{i+1}.$$

We have to check (16)–(19). (16) and (17) are immediate. For (18) it suffices to show $\sigma_i\restriction\bar{T}_i = \sigma_{i+1}\restriction\bar{T}_i$: if $a \in \bar{T}_i$,

$$\begin{aligned} \sigma_i(a) &= \sigma(a), &&\text{by the definition of } \sigma,\\ &= \rho(\sigma(a)), &&\text{since } \sigma(a) \in \tilde{T}_i \subseteq Y \text{ and } \tilde{T}_i \text{ is transitive},\\ &= \sigma_{i+1}(a). \end{aligned}$$

Finally, (26) implies that there is some $Z \subseteq \tilde{T}_i$, $Z \in \text{Ext}(\tilde{M}_{i^*}, \tilde{E}_i)$ which codes the isomorphism type of Y and hence codes $\tilde{M}_{i+1}$.

(27) $Z \in \tilde{M}_i$.

PROOF. $Z = \tilde{\pi}(f)(a)$ for some $f: \tilde{S}_i \to \tilde{M}_{i^*}$, $f \in \tilde{M}_{i^*}$, $a \in \tilde{T}_i$. Since $Z \subseteq \tilde{T}_i = \tilde{\pi}(\tilde{S}_i)$ we may assume that $f: \tilde{S}_i \to \mathscr{P}(\tilde{S}_i)$. Then f can be coded by a subset of $\tilde{S}_i$ and since $\mathscr{P}(\tilde{S}_i) \cap \tilde{M}_{i^*} = \mathscr{P}(\tilde{S}_i) \cap \tilde{M}_i$ (20) we get $f \in \text{dom}(\tilde{E}_i)$. Then $Z = \tilde{\pi}(f)(a) = \tilde{E}_i(f)(a) \in \tilde{M}_i$ since $\tilde{E}_i \in \tilde{M}_i$. ⊣ (27)

In $\tilde{M}_i$ we can decode Z and obtain $\tilde{M}_{i+1} \in \tilde{M}_i$.

This concludes the recursive definition of the system $(\tilde{M}_i)_{i<\omega}$ and (19) contradicts the initial assumption. ⊣

§8. Growing alternating trees. The Martin-Steel-theorem will be proved by constructing embedding normal forms with witnesses for projective sets. The branches through those ENFs will be the main branches through certain *alternating trees.*

The wellfoundedness of the main branches will be controlled by injecting information from given witnesses into the side branches of the alternating trees. We shall construct alternating trees by recursion and the present section describes a method by which a finite alternating tree may be end-extended.

Infinite alternating trees look like the "sum" of one linear main branch and a copy of the tree ${}^{<\omega}\omega$. We introduce a partial order $\leq_I$ on ω with the corresponding order-type. Let $h\colon \omega \longleftrightarrow {}^{<\omega}\omega$ be a recursive bijection satisfying $h(k) \subseteq h(l) \Longrightarrow k \leq l$; thus initial segments are enumerated first. Then define

$$i \leq_I j \longleftrightarrow \exists i', j' \ (i = 2i' \wedge j = 2j' \wedge i' \leq j') \\ \vee \exists i', j' \ (i = 2i' \dot{-} 1 \wedge j = 2j' \dot{-} 1 \wedge h(i') \subseteq h(j')),$$

where $m \dot{-} n = \max\{m - n, 0\}$. $\{0, 2, 4, \dots\}$ is called the *main branch* of $I = (\omega, \leq_I)$. For $i \in \omega$ let i^* be the immediate $<_I$-predecessor of $i + 1$. An iteration tree is called an *alternating tree* if its i^*-function is equal to a proper or improper initial segment of the function i^* just defined.

We now describe a method for endextending an alternating tree of length $2n + 1$ to an alternating tree of length $2n + 3$. Let us first introduce some notation for describing the agreement between models of set theory. For a class X and $\alpha \in \mathrm{On}$ let $X \restriction \alpha = X \cap V_\alpha$. If M is a transitive $\in$-model, $\gamma \leq \mathrm{On} \cap M$, $\vec{y} \in M \restriction \gamma$, and $\kappa \leq \gamma$ let $\mathrm{Th}(M \restriction \gamma, \vec{y}; \kappa)$ be the first order theory of the structure

$$(M \restriction \gamma, \in, \vec{y}, (a \mid a \in M \restriction \kappa))$$

where the members of the finite tuple $\vec{y}$ and every $a \in M \restriction \kappa$ are taken as constants. We assume some natural Gödelization of the language so that for λ, κ limit ordinals, $\lambda \leq \kappa \leq \gamma$:

(1) $\mathrm{Th}(M \restriction \gamma, \vec{y}; \lambda) \subseteq M \restriction \lambda$ and $\mathrm{Th}(M \restriction \gamma, \vec{y}; \lambda) = \mathrm{Th}(M \restriction \gamma, \vec{y}; \kappa) \restriction \lambda$.

We shall argue in the presence of a fixed Woodin cardinal δ. We only consider alternating trees with base model V which are formed by extenders from V_δ. Let $\mathfrak{F}$ be the class of sets which are fixed points in all those trees. Lemma 3.10 shows that $\mathfrak{F}$ is a proper class containing lots of big ordinals. Also δ which is strongly inaccessible is an element of $\mathfrak{F}$.

All objects to be determined in the subsequent construction as well as in the next section can be found in some sufficiently high V_θ. By a simple pigeonhole argument there are $c_0, c_1, c_2 \in \mathfrak{F}$, $\theta < c_0 < c_1 < c_2$ so that:

(2) $\mathrm{Th}(V \restriction c_2, c_0; \theta + 1) = \mathrm{Th}(V \restriction c_2, c_1; \theta + 1)$.

Let us remark already here that c_0, c_1, c_2 are not really needed when certain things are chosen in the construction. We rather refer to theories definable from c_0 or c_1 but which are themselves rather small objects.

Now let an η-closed alternating tree

$$\mathfrak{T} = (M_i)_{i \leq 2n}, \ (i^*, E_i)_{i < 2n}$$

of length $2n + 1$ be given with base model $M_0 = V$ and $\forall i < 2n \ E_i \in V_\delta$. Let

$$\tilde{\mathfrak{T}} = (M_i)_{i \leq 2n}, \ (\pi_{ij})_{i \leq_I j \leq 2n}$$

be the finite tree of models associated with $\mathfrak{T}$. Assume that $\aleph_1 \leq \eta < \delta$.

Let $(2n)^* = 2m \dot{-} 1$ be the immediate $<_I$-predecessor of $2n+1$. We end-extend $\mathfrak{T}$ in two stages:

I. Extend $M_{2m \dot{-} 1}$ by an extender $E_{2n} \in M_{2n} \upharpoonright \delta$ to obtain M_{2n+1}.

II. Extend M_{2n} by an extender $E_{2n+1} \in M_{2n+1} \upharpoonright \delta$ to obtain M_{2n+2}.

In our later applications we have to realize certain 1st-order properties of M_{2n} in the model M_{2n+1} and we formulate sufficient conditions for this. The resulting end-extension will also satisfy appropriate versions of these conditions so that a recursive continuation is possible. These conditions, for the particular $m \leq n$, are as follows:

There are $\kappa_{2m}, \gamma_{2m}, \vec{y}, \vec{y}^*$ satisfying (3)–(7):

(3) $\eta < \kappa_{2m} < \delta, \delta < \gamma_{2m} < \theta, \vec{y} \in M_{2m} \upharpoonright \gamma_{2m}, \vec{y} \in \mathfrak{F}, \vec{y}^* \in M_{2m \dot{-} 1} \upharpoonright c_0 + 1$;

(4) $M_{2m} \models \kappa_{2m}$ is strong in $\mathrm{Th}(M_{2m} \upharpoonright \gamma_{2m} + 1, \delta, \vec{y}; \delta)$ up to δ;

(5) $M_{2m} \upharpoonright \kappa_{2m} + 1 = M_{2m \dot{-} 1} \upharpoonright \kappa_{2m} + 1$;

(6) $\pi_{2m,2n} \upharpoonright \kappa_{2m} + 1 = \mathrm{id}$ and $M_{2m} \upharpoonright \kappa_{2m} + 1 = M_{2n} \upharpoonright \kappa_{2m} + 1$;

(7) $\mathrm{Th}(M_{2m} \upharpoonright \gamma_{2m} + 1, \delta, \vec{y}; \kappa_{2m}) = \mathrm{Th}(M_{2m \dot{-} 1} \upharpoonright c_0 + 1, \delta, \vec{y}^*; \kappa_{2m})$.

By (4), there are strong extenders in M_{2m} with critical point κ_{2m}. By (5) and (6), these can be mapped up to M_{2n} and applied to $M_{2m \dot{-} 1}$. Moreover we want to incorporate first order properties of a further parameter into the extension. Let this parameter be

$$z \in M_{2n}, \quad \text{with } \mathrm{rk}(z) \leq \mathrm{rk}(y_i) \text{ for all } y_i \text{ in } \vec{y}, \text{ and } z \in \mathfrak{F}.$$

Let us now begin the construction by applying $\pi_{2m,2n}$ to (4), (5), and (7); observe that most parameters are fixed by $\pi_{2m,2n}$:

(8) $M_{2n} \models \kappa_{2m}$ is strong in $\mathrm{Th}(M_{2n} \upharpoonright \pi_{2m,2n}(\gamma_{2m}) + 1, \delta, \vec{y}; \delta)$ up to δ;

(9) $M_{2n} \upharpoonright \kappa_{2m} + 1 = M_{2m \dot{-} 1} \upharpoonright \kappa_{2m} + 1$, by (5), (6);

(10) $\mathrm{Th}(M_{2n} \upharpoonright \pi_{2m,2n}(\gamma_{2m}) + 1, \delta, \vec{y}; \kappa_{2m}) = \mathrm{Th}(M_{2m \dot{-} 1} \upharpoonright c_0 + 1, \delta, \vec{y}^*; \kappa_{2m})$.

(11) $M_{2n} \models \delta$ is a Woodin cardinal, since δ is Woodin in V and $\pi_{0,2n}(\delta) = \delta$.

We apply the Woodinness of δ also to first order properties of the new parameter z: there is κ_{2n+1}, $\kappa_{2n} < \kappa_{2n+1} < \delta$ such that

(12) $M_{2n} \models \kappa_{2n+1}$ is strong in $\mathrm{Th}(M_{2n} \upharpoonright \pi_{2m,2n}(\gamma_{2m}), \delta, \vec{y}, z; \delta)$ up to δ.

We choose an extender which injects the strongness of κ_{2n+1} into its extensions: by (8), take an η-closed extender $E_{2n} \in M_{2n} \upharpoonright \delta$, $E_{2n} \colon S_{2n} \prec T_{2n}$, $\mathrm{crit}(E_{2n}) = \kappa_{2m}$, $T_{2n} \supseteq M_{2n} \upharpoonright \kappa_{2n+1} + \omega$ with the following “strength”:

(13) $$E_{2n}(\mathrm{Th}(M_{2n} \upharpoonright \pi_{2m,2n}(\gamma_{2m}) + 1, \delta, \vec{y}; \kappa_{2m})) \upharpoonright \kappa_{2n+1} + \omega = \mathrm{Th}(M_{2n} \upharpoonright \pi_{2m,2n}(\gamma_{2m}) + 1, \delta, \vec{y}; \kappa_{2n+1} + \omega).$$

Let $\pi_{2m \dot{-} 1,2n+1} = \pi_{E_{2n}} \colon M_{2m \dot{-} 1} \to_{E_{2n}} M_{2n+1} = \mathrm{Ext}(M_{2m \dot{-} 1}, E_{2n})$.

(14) $$\mathrm{Th}(M_{2n} \upharpoonright \pi_{2m,2n}(\gamma_{2m}) + 1, \delta, \vec{y}; \kappa_{2n+1} + \omega) = \mathrm{Th}(M_{2n+1} \upharpoonright c_0 + 1, \delta, \pi_{2m \dot{-} 1,2n+1}(\vec{y}^*); \kappa_{2n+1} + \omega).$$

PROOF.

$$\begin{aligned}
&\mathrm{Th}(M_{2n}\restriction\pi_{2m,2n}(\gamma_{2m})+1,\delta,\vec{y};\kappa_{2n+1}+\omega)\\
&\quad=\pi_{2m\dot{-}1,2n+1}(\mathrm{Th}(M_{2n}\restriction\pi_{2m,2n}(\gamma_{2m})+1,\delta,\vec{y};\kappa_{2m}))\restriction\kappa_{2n+1}+\omega, && \text{by (13)},\\
&\quad=\pi_{2m\dot{-}1,2n+1}(\mathrm{Th}(M_{2m\dot{-}1}\restriction c_0+1,\delta,\vec{y}^{\,*};\kappa_{2m}))\restriction\kappa_{2n+1}+\omega, && \text{by (10)},\\
&\quad=\mathrm{Th}(M_{2n+1}\restriction c_0+1,\delta,\pi_{2m\dot{-}1,2n+1}(\vec{y}^{\,*});\kappa_{2n+1}+\omega). && \dashv (14)
\end{aligned}$$

The type-equality (14) allows us to transport properties of z over to M_{2n+1}. Let

$$\tau=\mathrm{Th}(M_{2n}\restriction\pi_{2m,2n}(\gamma_{2m}),\delta,\vec{y},z;\kappa_{2n+1}).$$

$\tau\in M_{2n}\restriction\kappa_{2n+1}+\omega$, hence it is a constant of the structure on the left hand side of (14). If we use $\dot{x}$ as a canonical name for a constant x the left hand side of (14) contains the statement

$$\begin{aligned}
&\exists u\,\exists v\ (u \text{ is the largest ordinal} \wedge \dot{\tau}=\mathrm{Th}(V\restriction u,\dot{\delta},\dot{\vec{y}},v;\dot{\kappa}_{2n+1})\\
&\qquad\qquad \wedge\ \dot{\kappa}_{2n+1} \text{ is strong in } \mathrm{Th}(V\restriction u,\dot{\delta},\dot{\vec{y}},v;\dot{\delta}) \text{ up to } \dot{\delta}).
\end{aligned}$$

By (14), the same statement holds in the structure on the right hand side of the equality. The largest ordinal of $M_{2n+1}\restriction c_0+1$ is c_0. As a witness for the quantifier $\exists v$ we get a $z^*\in M_{2n+1}\restriction c_0$ so that (15) and (16) hold:

(15) $\mathrm{Th}(M_{2n}\restriction\pi_{2m,2n}(\gamma_{2m}),\delta,\vec{y},z;\kappa_{2n+1})=\tau$
$=\mathrm{Th}(M_{2n+1}\restriction c_0,\delta,\pi_{2m\dot{-}1,2n+1}(\vec{y}^{\,*}),z^*;\kappa_{2n+1});$

(16) $M_{2n+1}\models\kappa_{2n+1}$ is strong in $\mathrm{Th}(M_{2n+1}\restriction c_0,\delta,\pi_{2m\dot{-}1,2n+1}(\vec{y}^{\,*}),z^*;\delta)$ up to δ.

Since we intend a recursive construction which continues for ω stages we have to get back to properties similiar to the initial assumptions. In particular we have to "top up" c_0 to prevent a descending sequence of ordinals. By the indiscernibility property (2) we may substitute c_1 for c_0 in (15) and (16):

(17) $\mathrm{Th}(M_{2n}\restriction\pi_{2m,2n}(\gamma_{2m}),\delta,\vec{y},z;\kappa_{2n+1})$
$=\mathrm{Th}(M_{2n+1}\restriction c_1,\delta,\pi_{2m\dot{-}1,2n+1}(\vec{y}^{\,*}),z^*;\kappa_{2n+1});$

(18) $M_{2n+1}\models\kappa_{2n+1}$ is strong in $\mathrm{Th}(M_{2n+1}\restriction c_1,\delta,\pi_{2m\dot{-}1,2n+1}(\vec{y}^{\,*}),z^*;\delta)$ up to δ.

Since $T_{2n}\supseteq M_{2n}\restriction\kappa_{2n+1}+\omega$ we have

(19) $M_{2n+1}\restriction\kappa_{2n+1}+1=M_{2n}\restriction\kappa_{2n+1}+1.$

The situation (17)–(19) is similiar to (8)–(10) and we continue in a parallel way. Choose κ_{2n+2}, $\kappa_{2n+1}<\kappa_{2n+2}<\delta$ so that

(20) $M_{2n+1}\models\kappa_{2n+2}$ is strong in $\mathrm{Th}(M_{2n+1}\restriction c_0+1,\delta,\pi_{2m\dot{-}1,2n+1}(\vec{y}^{\,*}),z^*;\delta)$ up to δ.

By (18), choose an η-closed extender $E_{2n+1}\in M_{2n+1}\restriction\delta$ on M_{2n+1}, $E_{2n+1}: S_{2n+1}\prec T_{2n+1}$ so that $\mathrm{crit}(E_{2n+1})=\kappa_{2n+1}$, $T_{2n+1}\supseteq T_{2n}$, $T_{2n+1}\supseteq M_{2n+1}\restriction\kappa_{2n+2}+\omega$ with the following "strength":

$$(21)\ E_{2n+1}(\mathrm{Th}(M_{2n+1}\restriction c_1,\delta,\pi_{2m\dot{-}1,2n+1}(\vec{y}^*),z^*;\kappa_{2n+1}))\restriction\kappa_{2n+2}+\omega = \mathrm{Th}(M_{2n+1}\restriction c_1,\delta,\pi_{2m\dot{-}1,2n+1}(\vec{y}^*),z^*;\kappa_{2n+2}+\omega).$$

Let $\pi_{2n,2n+2}=\pi_{E_{2n+1}}: M_{2n}\to_{E_{2n+1}} M_{2n+2}=\mathrm{Ext}(M_{2n},E_{2n+1})$.

$$\begin{aligned}(22)\ \mathrm{Th}(M_{2n+1}\restriction c_1,\delta,\pi_{2m\dot{-}1,2n+1}(\vec{y}^*),z^*;\kappa_{2n+2}+\omega)\\ =\pi_{2n,2n+2}(\mathrm{Th}(M_{2n+1}\restriction c_1,\delta,\pi_{2m\dot{-}1,2n+1}(\vec{y}^*),z^*;\kappa_{2n+1}))\restriction\kappa_{2n+2}+\omega\\ =\pi_{2n,2n+2}(\mathrm{Th}(M_{2n}\restriction\pi_{2m,2n}(\gamma_{2m}),\delta,\vec{y},z;\kappa_{2n+1}))\restriction\kappa_{2n+2}+\omega\\ =\mathrm{Th}(M_{2n+2}\restriction\pi_{2m,2n+2}(\gamma_{2m}),\delta,\vec{y},z;\kappa_{2n+2}+\omega);\end{aligned}$$

the first equality follows by the definition of $\pi_{2n,2n+2}\supseteq E_{2n+1}$ and (21), the second from (17), and the third by the elementarity of $\pi_{2n,2n+2}$, observing that several parameters are fixed points of the iteration tree. Let

$$\tau'=\mathrm{Th}(M_{2n+1}\restriction c_0+1,\delta,\pi_{2m\dot{-}1,2n+1}(\vec{y}^*),z^*;\kappa_{2n+2}).$$

The left hand side of (22) contains the statement

$$\exists u\ (u \text{ is a successor ordinal}\wedge\dot{\tau}'=\mathrm{Th}(V\restriction u,\dot{\delta},\dot{\vec{y}},\dot{z};\dot{\kappa}_{2n+2})\\ \wedge\ \dot{\kappa}_{2n+2}\text{ is strong in }\mathrm{Th}(V\restriction u,\dot{\delta},\dot{\vec{y}},\dot{z};\dot{\delta})\text{ up to }\dot{\delta}).$$

By (22), the same statement holds in the structure on the right hand side of the equality. Hence there is some γ_{2n+2} corresponding to "$u-1$" with

$$(23)\ \gamma_{2n+2}<\pi_{2m,2n+2}(\gamma_{2m})$$

such that (24) and (25) hold:

$$(24)\ M_{2n+2}\models\kappa_{2n+2}\text{ is strong in }\mathrm{Th}(M_{2n+2}\restriction\gamma_{2n+2}+1,\delta,\vec{y},z;\delta)\text{ up to }\delta;$$

$$(25)\ \mathrm{Th}(M_{2n+2}\restriction\gamma_{2n+2}+1,\delta,\vec{y},z;\kappa_{2n+2}) =\mathrm{Th}(M_{2n+1}\restriction c_0+1,\delta,\pi_{2m\dot{-}1,2n+1}(\vec{y}^*),z^*;\kappa_{2n+2}).$$

$$(26)\ M_{2n+2}\restriction\kappa_{2n+2}+1=M_{2n+1}\restriction\kappa_{2n+2}+1,$$

because $T_{2n+1}\supseteq M_{2n+1}\restriction\kappa_{2n+2}+\omega$, and

$$(27)\ \pi_{2n,2n+2}\restriction\kappa_{2n}+1=\mathrm{id},$$

because $\kappa_{2n+1}>\kappa_{2n}$.

This concludes the construction of the alternating tree of length $2n+3$. Our argument basically is a twofold application of the "One-Step-Lemma" of [10]. Properties (24)–(27) are in close analogy to the initial assumptions (4)–(7); extending M_{2n+1} later in the construction can be done just like we have extended $M_{2m\dot{-}1}$ right now.

REMARKS.

1. The construction would yield an illfounded main branch due to (23). This will be mollified in the next section where the construction steps are carried out inside varying models.

2. We chose objects κ_{2n+1}, E_{2n}, z^*, κ_{2n+2}, E_{2n+1}, γ_{2n+2} in the course of the construction. One easily checks that the conditions for choosing these objects refer to c_0 or c_1 only via theories of the form $\mathrm{Th}(M_{\dots}\restriction c_0,\dots;\delta)$ or $\mathrm{Th}(M_{\dots}\restriction c_1,\dots;\delta)$ which are elements of V_θ. Since V_θ is nicely closed all choices can be done within V_θ. If we also assume a fixed wellorder $<_\theta$ of V_θ we may stipulate that all choices are made $<_\theta$-minimal.

§9. The Martin-Steel-theorem. We shall prove the determinacy of projective sets by constructing embedding normal forms with witnesses. We proceed by induction on the complexity of sets in the projective hierarchy. For this we have to discuss higher dimensional embedding normal forms since projective sets are formed by complementations and by projections of simpler but higher dimensional sets.

Let $T = ({}^{<\omega}\omega, \subseteq)$ be the usual tree of finite sequences of natural numbers. For $1 \leq l < \omega$, the product tree T^l is defined by

$$T^l = \{ (s_1, \dots, s_l) \in T \times \dots \times T \mid |s_1| = \dots = |s_l| \},$$
$$(s_1, \dots, s_l) \leq (s_1', \dots, s_l') \quad \text{if and only if} \quad s_1 \subseteq s_1' \wedge \dots \wedge s_l \subseteq s_l'.$$

We usually write $s_1 \dots s_l$ for $(s_1, \dots, s_l)$. Naturally $[T^l] \cong [T]^l = \mathbb{R}^l$. On the other hand, T^l is ω-branching and of height ω, hence T^l is canonically isomorphic to T. This gives rise to a canonical homeomorphism

$$\zeta : \mathbb{R} = [T] \cong [T^l] \cong [T]^l = \mathbb{R}^l.$$

Obviously $A \subseteq \mathbb{R}$ is open or $\mathbf{\Pi}^1_n$ if and only if $\zeta''A \subseteq \mathbb{R}^l$ is open or $\mathbf{\Pi}^1_n$, respectively

Definitions 4.3 and 4.4 are easily generalized to embedding normal forms (with witnesses) for sets $A \subseteq \mathbb{R}^l$, so that $A \subseteq \mathbb{R}$ has an embedding normal form (with witnesses) if and only if $\zeta''A \subseteq \mathbb{R}^l$ has an embedding normal form (with witnesses). We are now able to formulate the crucial theorem for the inductive proof of the Martin-Steel-theorem:

THEOREM 9.1. *Let $A \subseteq \mathbb{R} \times \mathbb{R}$ have an* ENFW

$$\mathfrak{N} = (N_{st})_{st \in T^2}, \ (\sigma_{st,s't'})_{st \leq s't'}, \ (w_{st})_{st \in T^2},$$

with $V_\delta \subseteq N_{00}$ and critical point $> \delta$. Let δ be a Woodin cardinal and $\eta < \delta$. Then

$$\neg \mathrm{p} A = \{ x \in \mathbb{R} \mid \neg \exists y \in \mathbb{R}\ (x, y) \in A \}$$

has an η-closed ENFW *with base model V and critical point $> \eta$.*

Before proving this theorem let us deduce the Martin-Steel result:

THEOREM 9.2. *Let $\delta_n < \dots < \delta_1$ be Woodin cardinals, $n \geq 1$, and assume that $V^\sharp_{\delta_1}$ exists. Let $\eta < \delta_n$. Then every $\mathbf{\Pi}^1_{n+1}$-set has an η-closed* ENFW *with base model V and critical point $> \eta$.*

PROOF. By induction on $n \geq 1$. Let $n = 1$ and let $B \subseteq \mathbb{R}$ be a $\mathbf{\Pi}^1_2$-set. Then there is a $\mathbf{\Pi}^1_1$-set $A \subseteq \mathbb{R}^2$ such that

$$x \in B \longleftrightarrow \forall y\ \neg(x, y) \in A \longleftrightarrow \neg \exists y\ (x, y) \in A,$$

i.e., $B = \neg \mathrm{p} A$. By Theorem 6.9, A has an ENFW which satisfies the assumptions of Theorem 9.1 with $\delta = \delta_1$. By Theorem 9.1, B has an η-closed ENFW with base model V and critical point $> \eta$.

Now let $n = m + 1$, $m \geq 1$, and assume the theorem holds for m. Let $B \subseteq \mathbb{R}$ be $\mathbf{\Pi}^1_{n+1}$. As above, $B = \neg \mathrm{p} A$ for some $\mathbf{\Pi}^1_n$-set $A \subseteq \mathbb{R} \times \mathbb{R}$. Let us apply the inductive assumption to the Woodin cardinals $\delta_m < \dots < \delta_1$ and the set A with $\eta = \delta_n < \delta_m$: A has an ENFW with base model V and critical point $> \eta = \delta_n$. Then the hypothesis of Theorem 9.1 with $\delta = \delta_n$ is satisfied and yields an η-closed ENFW with base model V and critical point $> \eta$ for $B = \neg \mathrm{p} A$. ⊣

With Theorem 5.3 we arrive at the Martin-Steel result:

THEOREM 9.3.

(a) *Let* $\delta_n < \cdots < \delta_1$ *be Woodin cardinals,* $n \geq 1$, *and assume that* $V^\sharp_{\delta_1}$ *exists. Then* $\mathbf{\Pi}^1_{n+1}$*-determinacy holds.*

(b) *If there are infinitely many Woodin cardinals, projective determinacy* (*PD*) *holds.*

PROOF OF THEOREM 9.1. We are going to build an ENFW

$$\mathfrak{M} = (M_s)_{s\in T},\ (\pi_{st})_{s\leq_T t},$$

for the set $\neg pA$. So we want that for $x \in \mathbb{R}$: $x \in \neg pA$ if and only if the direct limit $M_x, (\pi_{sx})_{s\in x}$ of the branch $(M_s)_{s\in x}, (\pi_{st})_{s\leq_T t\in x}$ through $\mathfrak{M}$ is wellfounded. To control the wellfoundedness of $M_x, (\pi_{sx})_{s\in x}$ we make $(M_s)_{s\in x}, (\pi_{st})_{s\leq_T t\in x}$ the main branch of some alternating tree $\mathfrak{T}^x$. Let us give a brief motivation for this procedure: If $x \in \neg pA$ then $\forall y\ (x, y) \notin A$ and any branch

$$(N_{st})_{s\in x},\ (\sigma_{st,s't'})_{st\leq s't',s'\in x}$$

through the "x-section" of the given ENFW $\mathfrak{N}$ for A is illfounded. This is witnessed by the witnesses $(w_{st})_{s\in x}$. In the subsequent construction, properties of these witnesses are reflected into the odd part of the alternating tree $\mathfrak{T}^x$ so that any branch through the odd part is illfounded. By Steel's Lemma 7.2, the main branch of $\mathfrak{T}^x$ which is the only other branch through $\mathfrak{T}^x$ must be wellfounded, which establishes part of the ENF-property.

Several technical problems have to be dealt with in the construction:

1. The main branches of $\mathfrak{T}^x$ and $\mathfrak{T}^{x'}$ have to agree as long as x and x' agree. This is achieved by defining an increasing system of finite alternating trees $\mathfrak{T}^s$ for $s \in {}^{<\omega}\omega$ so that $\mathfrak{T}^x$ is the "union" of all $\mathfrak{T}^s$ with $s \in x$.

2. To refer to relevant properties of a witness w_{st} we have to work in the model N_{st} where w_{st} is "living". So the construction process is spread out over the given system $\mathfrak{N}$.

3. When we have to choose objects in the course of the construction we always take the least possible choice according to some wellordering. So we assume that (sufficiently long initial segments of) the structures N_{st} are equipped with a wellorder $<_{st}$ so that the embeddings $\sigma_{st,s't'}$ respect the wellorders.

4. All finite iteration trees $\mathfrak{T}^s$ will be determined by extenders which are elements of V_δ. Although these extenders are not moved by the maps in the given ENFW $\mathfrak{N}$ the models of the tree $\mathfrak{T}^s$ will depend on whether we work in V or in N_{st}. Therefore we work with certain *terms* $\dot{M}^s_i$ for the models of $\mathfrak{T}^s$. These terms are abstraction terms of the language of set theory with an added relation symbol $\dot{<}$; the terms may use parameters which are fixed points of the System $\mathfrak{N}$. Such terms can be evaluated in every model N_{st} where $\dot{<}$ is interpreted by $<_{st}$. We introduce similiar terms $\dot{\pi}^s_{ij}$, $\dot{w}^s_i$, and $\dot{\gamma}^s_i$ for the maps in $\mathfrak{T}^s$, the "reflections" of the witnesses, and for some "descending ordinals", respectively.

5. We assume that every strong limit cardinal of sufficiently high cofinality is a fixed point for all the embeddings $\sigma_{st,s't'}$ of the system $\mathfrak{N}$. If necessary, the given system can be modified by the formation of elementary substructures and their transitivisations to obtain the fixed point property. We don't want to go into any details since with respect to the Martin-Steel theorem the ENFs constructed in Theorems 6.6, 6.9 and the present proof all satisfy the fixed point property.

6. Fixed points are also convenient in our considerations of iteration trees. We shall construct iteration trees from extenders in V_δ and in Section 8 a class $\mathfrak{F}$ of fixed points for all such iteration trees was defined. Again, $\mathfrak{F}$ will vary between various N_{st} and we let $\dot{\mathfrak{F}}$ be a canonical term for the fixed point class. If ν is a strong limit cardinal of sufficient high cofinality, $N_{st} \models \nu \in \dot{\mathfrak{F}}$ for all $st \in T^2$.

7. Now choose θ, c_0, c_1, c_2 strong limit cardinals of sufficiently high cofinality, so that property (2) of Section 8 holds:

$$\mathrm{Th}(V \upharpoonright c_2, c_0; \theta + 1) = \mathrm{Th}(V \upharpoonright c_2, c_1; \theta + 1).$$

8. We may also assume that for $st \in T^2$: $N_{st} \models w_{st} \in \dot{\mathfrak{F}}$ because otherwise we could replace w_{st} by $w'_{st} \in N_{st}$,

$w'_{st}(x, y) =$ the $w_{st}(x, y)$th element of $\dot{\mathfrak{F}}$, computed in N_{st}.

9. As a last preparation we assume that the parameter η of the theorem is $\geq 2^{\aleph_0}$ so that the resulting ENF will be sufficiently closed for the automatic existence of witnesses (see Lemma 4.5).

Let us now begin the actual construction. We determine for every $s \in {}^{<\omega}\omega$ terms for a finite alternating tree

$$\dot{\mathfrak{T}}^s = (\dot{M}_i^s)_{i \leq 2|s|},\ (i^*, E_i^s)_{i<2|s|}$$

of length $2|s| + 1$ with embeddings

$$(\dot{\pi}_{ij}^s)_{i \leq_I j \leq 2|s|}.$$

Moreover we determine ordinals κ_i^s for $i \leq 2|s|$ and terms $\dot{w}^s_{2m \dot{-} 1}$ and $\dot{\gamma}^s_{2m}$ for $m \leq |s|$.

For any $s \in {}^{<\omega}\omega$ the following properties will hold:

(1) $\dot{\mathfrak{T}}^s$ is the canonical term for an iteration tree constructed from $\dot{M}_0^s = \{x \mid x = x\}$ as base model with extenders $E_i^s \in V_\delta$.

For $m \leq n = |s|$ we require analogues of properties (4)–(7) of Section 8. So for $\bar{t} = h(2m \dot{-} 1)$, $\bar{s} = s \upharpoonright |\bar{t}|$ postulate conditions (2)–(5):

(2) $N_{\bar{s}\bar{t}} \models$ "$\dot{M}^s_{2m} \models \kappa^s_{2m}$ is strong in $\mathrm{Th}(\dot{M}^s_{2m} \upharpoonright \dot{\gamma}^s_{2m} + 1, \delta, (\sigma_{\bar{s}\bar{t} \upharpoonright i, \bar{s}\bar{t}}(w_{\bar{s}\bar{t} \upharpoonright i}) \mid i \leq |\bar{t}|); \delta)$ up to δ";

(3) $N_{\bar{s}\bar{t}} \models \dot{M}^s_{2m} \upharpoonright \kappa^s_{2m} + 1 = \dot{M}^s_{2m \dot{-} 1} \upharpoonright \kappa^s_{2m} + 1$;

(4) $N_{\bar{s}\bar{t}} \models \dot{\pi}^s_{2m,2n} \upharpoonright \kappa^s_{2m} + 1 = \mathrm{id}$;

(5) $N_{\bar{s}\bar{t}} \models \mathrm{Th}(\dot{M}^s_{2m} \upharpoonright \dot{\gamma}^s_{2m} + 1, \delta, (\sigma_{\bar{s}\bar{t} \upharpoonright i, \bar{s}\bar{t}}(w_{\bar{s}\bar{t} \upharpoonright i}) \mid i \leq |\bar{t}|); \kappa^s_{2m})$
$$= \mathrm{Th}(\dot{M}^s_{2m \dot{-} 1} \upharpoonright c_0 + 1, \delta, (\dot{\pi}^s_{i,2m \dot{-} 1}(\dot{w}^s_i) \mid i \leq_I 2m \dot{-} 1); \kappa^s_{2m}).$$

These conditions correspond to the assumptions of Section 8 with

$$\vec{y} = (\sigma_{\bar{s}\bar{t} \upharpoonright i, \bar{s}\bar{t}}(w_{\bar{s}\bar{t} \upharpoonright i}) \mid i \leq |\bar{t}|), \quad \vec{y}^* = (\dot{\pi}^s_{i,2m \dot{-} 1}(\dot{w}^s_i) \mid i \leq_I 2m \dot{-} 1).$$

Also the $\dot{\gamma}$-terms satisfy a certain descent-property along the main branch:

(6) If $2k \dot{-} 1 <_I 2l \dot{-} 1 \leq 2n \dot{-} 1$ then $N_{\bar{s}\bar{t}} \models \dot{\gamma}^s_{2l} < \dot{\pi}^s_{2k,2l}(\dot{\gamma}^s_{2k})$.

The construction of these terms proceeds by recursion on $s \in {}^{<\omega}\omega$:

Let $s = \emptyset$. Set $\dot{M}_0^\emptyset = \{ x \mid x = x \}$, the universal term. Let $\dot{\gamma}_0^\emptyset = c_0$. Because $V_\delta \subseteq N_{00}$, δ is a Woodin cardinal in N_{00}. Choose $\kappa_0^\emptyset < \delta$ so that

$$N_{00} \models \kappa_0^\emptyset \text{ is strong in } \mathrm{Th}(V \upharpoonright c_0 + 1, \delta, w_{00}; \delta) \text{ up to } \delta.$$

Let $\dot{w}_0^\emptyset$ be the canonical term (involving the symbol $\dot{<}$) so that in N_{00}:

$$N_{00} \models \mathrm{Th}(V \upharpoonright c_0 + 1, \delta, \dot{w}_0^\emptyset; \delta) = \mathrm{Th}(V \upharpoonright c_0 + 1, \delta, w_{00}; \delta).$$

Let $\dot{\pi}_{00}^\emptyset$ be the canonical term for the identity function.

It is straightforward to check (1)–(6) for these choices of terms and parameters.

Now let $s \neq \emptyset$, $|s| = n + 1$, and assume that

$$\dot{\mathfrak{T}}^{s \upharpoonright n} = (\dot{M}_i^{s \upharpoonright n})_{i \leq 2n}, \ (i^*, E_i^{s \upharpoonright n})_{i < 2n}$$

with embeddings $(\dot{\pi}_{ij}^{s \upharpoonright n})_{i \leq_I j \leq 2n}$ is constructed satisfying (1)–(6). $\dot{\mathfrak{T}}^s$ will be an endextension of $\dot{\mathfrak{T}}^{s \upharpoonright n}$ by two more structures $\dot{M}_{2n+1}^s$ and $\dot{M}_{2n+2}^s$. Let $2m \dot{-} 1 = (2n)^*$ be the immediate $<_I$-predecessor of $2n + 1$. Let $\bar{t} = h(2m \dot{-} 1)$, $\bar{s} = s \upharpoonright |\bar{t}|$ and $\tilde{t} = h(2n + 1)$, $\tilde{s} = s \upharpoonright |\tilde{t}|$. We want $\dot{M}_{2n+1}^s$ to be an extension of $\dot{M}_{2m \dot{-} 1}^{s \upharpoonright n}$ which imitates some aspects of the embedding $\sigma_{\bar{s}\bar{t},\tilde{s}\tilde{t}} \colon N_{\bar{s}\bar{t}} \to N_{\tilde{s}\tilde{t}}$ as regards the witness $w_{\tilde{s}\tilde{t}}$. The subsequent construction will thus take place in $N_{\tilde{s}\tilde{t}}$, the natural habitat for $w_{\tilde{s}\tilde{t}}$. To simplify our notation let us omit the superscripts $s \upharpoonright n$ and s in this construction step. Properties (2)–(6) hold in $N_{\bar{s}\bar{t}}$ by our recursive assumption. Let us first apply the elementary map $\sigma_{\bar{s}\bar{t},\tilde{s}\tilde{t}}$ to (2)–(6). Then *inside* $N_{\tilde{s}\tilde{t}}$ we note:

(7) $\dot{M}_{2m} \models \kappa_{2m}$ is strong in $\mathrm{Th}(\dot{M}_{2m} \upharpoonright \dot{\gamma}_{2m} + 1, \delta, (\sigma_{\bar{s}\bar{t} \upharpoonright i, \tilde{s}\tilde{t}}(w_{\bar{s}\bar{t} \upharpoonright i}) \mid i \leq |\bar{t}|); \delta)$ up to δ.

(8) $\dot{M}_{2m} \upharpoonright \kappa_{2m} + 1 = \dot{M}_{2m \dot{-} 1} \upharpoonright \kappa_{2m} + 1$.

(9) $\dot{\pi}_{2m,2n} \upharpoonright \kappa_{2m} + 1 = \mathrm{id}$.

(10) $\mathrm{Th}(\dot{M}_{2m} \upharpoonright \dot{\gamma}_{2m} + 1, \delta, (\sigma_{\bar{s}\bar{t} \upharpoonright i, \tilde{s}\tilde{t}}(w_{\bar{s}\bar{t} \upharpoonright i}) \mid i \leq |\bar{t}|); \kappa_{2m})$
$= \mathrm{Th}(\dot{M}_{2m \dot{-} 1} \upharpoonright c_0 + 1, \delta, (\dot{\pi}_{i,2m \dot{-} 1}(\dot{w}_i) \mid i \leq_I 2m \dot{-} 1); \kappa_{2m})$.

(11) If $2k \dot{-} 1 <_I 2l \dot{-} 1 \leq 2n \dot{-} 1$ then $\dot{\gamma}_{2l} < \dot{\pi}_{2k,2l}(\dot{\gamma}_{2k})$.

Now (7)–(10) correspond exactly to properties (4)–(7) in Section 8 with

$$\vec{y} = (\sigma_{\bar{s}\bar{t} \upharpoonright i, \tilde{s}\tilde{t}}(w_{\bar{s}\bar{t} \upharpoonright i}) \mid i \leq |\bar{t}|) \quad \text{and} \quad \vec{y}^* = (\dot{\pi}_{i,2m \dot{-} 1}(\dot{w}_i) \mid i \leq_I 2m \dot{-} 1).$$

We apply the construction of the previous section *inside* $N_{\tilde{s}\tilde{t}}$ with $z = w_{\tilde{s}\tilde{t}}$. This yields objects

$$\kappa_{2n+1}, \quad E_{2n}, \quad \dot{z}^* = \dot{w}_{2n+1}, \quad \kappa_{2n+2}, \quad E_{2n+1}, \quad \dot{\gamma}_{2n+2}$$

belonging to an endextension of $\dot{\mathfrak{T}}^{s \upharpoonright n}$ by two more structures. E_{2n} and E_{2n+1} are η-closed extenders with critical points $> \eta$. We then define

$$\dot{\mathfrak{T}}^s = (\dot{M}_i^s)_{i \leq 2n+2}, \ (i^*, E_i^s)_{i < 2n+2}$$

by

$$\dot{M}_i^s = \dot{M}_i^{s\restriction n} \text{ for } i \leq 2n,$$
$$\dot{M}_{2n+1}^s \text{ is the canonical term for } \mathrm{Ext}(\dot{M}_{2m \dot{-} 1}^s, E_{2n}),$$
$$\dot{M}_{2n+2}^s \text{ is the canonical term for } \mathrm{Ext}(\dot{M}_{2n}^s, E_{2n+1}),$$
$$\kappa_i^s = \kappa_i^{s\restriction n} \text{ for } i \leq 2n,$$
$$\kappa_{2n+1}^s = \kappa_{2n+1},$$
$$\kappa_{2n+2}^s = \kappa_{2n+2},$$

and we proceed analogously for the $\dot{w}^s$ and $\dot{\gamma}^s$.

We have to show that (1)–(6) hold for the extended alternating tree. For $m \leq n$ this is given by the recursive assumption and we only have to consider the case $m = n + 1$. But then the properties follow from (24), (26), (27), (25) and (23) of Section 8. This concludes the recursive construction of the alternating trees $\dot{\mathfrak{T}}^s$.

Now define (a term for) a tree

$$\dot{\mathfrak{M}} = (\dot{M}_s)_{s\in T}, \ (\dot{\pi}_{st})_{s\leq_T t}$$

of models over $T = {}^{<\omega}\omega$ by:

$$\dot{M}_s = \dot{M}_{2|s|}^s, \quad \dot{\pi}_{st} = \dot{\pi}_{2|s|,2|t|}^t.$$

The term $\dot{\mathfrak{M}}$ essentially only involves the universal term $\dot{M}_\emptyset = \{ x \mid x = x \}$ and parameters which are extenders $\in V_\delta$. So $\dot{\mathfrak{M}}$ may be evaluated in V and in every N_{st} of the ENFW $\mathfrak{N}$. We first show that $\dot{\mathfrak{M}}$ is an ENF for $\neg\mathrm{p}A$ inside the base model $N_{\emptyset\emptyset}$ of $\mathfrak{N}$. This will later transfer to V.

(12) Let $x \notin \neg\mathrm{p}A$. Then $N_{\emptyset\emptyset} \models$ "$\dot{M}_x$ is illfounded", where $\dot{M}_x$ is the canonical term for the limit model along the branch x.

Proof. $x \in \mathrm{p}A$ and there is a $y \in \mathbb{R}$ such that $(x, y) \in A$. Since $\mathfrak{N}$ is an ENF for A the limit N_{xy}, $(\sigma_{xy\restriction n,xy})$ along the branch xy is transitive. We apply the maps $\sigma_{xy\restriction n,xy}$ to (11) and obtain:

$$N_{xy} \models \text{If } h(2k \dot{-} 1) <_T h(2l \dot{-} 1) \in y \text{ then } \dot{\gamma}_{2l} < \dot{\pi}_{2k,2l}(\dot{\gamma}_{2k}).$$

So these $\dot{\gamma}_{2l}$ form an infinite descending $\in$-chain in $\dot{M}_x$ as evaluated in N_{xy}. As N_{xy} is a transitive $\in$-model, the absoluteness of illfoundedness yields that $\dot{M}_x$ is illfounded inside N_{xy}. Since $\sigma_{\emptyset\emptyset,xy}$ is elementary,

$$N_{\emptyset\emptyset} \models \text{"}\dot{M}_x \text{ is illfounded"}. \qquad \dashv (12)$$

(13) Let $x \in \neg\mathrm{p}A$. Then $N_{\emptyset\emptyset} \models$ "$\dot{M}_x$ is wellfounded".

Proof. Proof] For all $y \in \mathbb{R}$, $(x, y) \notin A$ and the limit N_{xy}, $(\sigma_{xy\restriction n,xy})$ is illfounded. This is witnessed by the original witnesses $w_{xy\restriction n}$: if $\bar{s}\bar{t}\restriction i <_{T^2} \bar{s}\bar{t}$ and $\bar{s}\bar{t} \in xy$ then

$$\sigma_{\bar{s}\bar{t}\restriction i,\bar{s}\bar{t}}(w_{\bar{s}\bar{t}\restriction i}(x, y)) > w_{\bar{s}\bar{t}}(x, y).$$

This fact is expressed on the lefthand side of equation (5) when $\bar{s}\bar{t} \in xy$. By the equality, if $j <_I 2m \dot{-} 1$, where $\bar{s} = h(2m \dot{-} 1)$, then

$$N_{\bar{s}\bar{t}} \models \dot{\pi}_{j,2m \dot{-} 1}^{\bar{s}}(\dot{w}_j^{\bar{s}}(x, y)) > \dot{w}_{2m \dot{-} 1}^{\bar{s}}(x, y).$$

The terms can be pulled back to $N_{\emptyset\emptyset}$:

$$(*) \qquad N_{\emptyset\emptyset} \models \dot\pi^{\bar s}_{j,2m\dot-1}(\dot w^{\bar s}_j(x,y)) > \dot w^{\bar s}_{2m\dot-1}(x,y).$$

Let $\dot{\mathfrak{T}}^x$ be a canonical term for the unique alternating tree of height ω which end-extends all the $\dot{\mathfrak{T}}^{\bar s}$ for $\bar s \in x$. Since property $(*)$ holds for every $\bar s \in x$ and $y \in \mathbb{R}$:

$N_{\emptyset\emptyset} \models$ "each branch through $\dot{\mathfrak{T}}^x$

which is not the main branch through $\dot{\mathfrak{T}}^x$ is illfounded".

Since $N_{\emptyset\emptyset}$ satisfies Steel's lemma 7.2,

$$N_{\emptyset\emptyset} \models \text{"the main branch through } \dot{\mathfrak{T}}^x \text{ is wellfounded"}.$$

Now the main branch of $\dot{\mathfrak{T}}^x$ consists of the even models $\dot M^{\bar s}_{2|\bar s|}$ for $\bar s \in x$ and this is exactly the branch through $\dot{\mathfrak{M}}$ indexed by x. Hence

$$N_{\emptyset\emptyset} \models \text{"}\dot M_x \text{ is wellfounded"}. \qquad \dashv (13)$$

We transfer (12) and (13) from $N_{\emptyset\emptyset}$ to the universe V by showing:

(14) For $x \in \mathbb{R}$, $N_{\emptyset\emptyset} \models$ "$\dot M_x$ is transitive" if and only if $V \models$ "$\dot M_x$ is transitive".

Proof. The term $\dot M_x$ is defined from the sequence of extenders in the above recursive construction and the real x. $\dot M_x$ is illfounded if and only if there is a system of functions representing, in the various extensions, an infinite descending sequence of ordinals. To check whether the functions represent such a descent is definable using only bounded quantifiers. So there is a Σ_1-formula $\varphi(x)$ in parameters from V_δ so that in ZFC:

$$\dot M_x \text{ is illfounded} \longleftrightarrow \varphi(x).$$

A straightforward transitivisation argument shows that Σ_1-formulae in parameters from V_δ are absolute for V_δ, i.e.,

$$\varphi(x) \longleftrightarrow V_\delta \models \varphi(x).$$

Together we obtain:

$$\begin{aligned} N_{\emptyset\emptyset} \models \dot M_x \text{ is transitive} &\longleftrightarrow N_{\emptyset\emptyset} \models \neg\varphi(x) \\ &\longleftrightarrow N_{\emptyset\emptyset} \models \text{"}V_\delta \models \neg\varphi(x)\text{"} \\ &\longleftrightarrow V_\delta \models \neg\varphi(x), \quad \text{since } V_\delta^{N_{\emptyset\emptyset}} = V_\delta, \\ &\longleftrightarrow \neg\varphi(x) \\ &\longleftrightarrow \dot M_x \text{ is transitive}. \qquad \dashv (14) \end{aligned}$$

Now let the system $\mathfrak{M} = (M_s), (\pi_{st})$ be the interpretation of $\dot{\mathfrak{M}}$ in V: $M_s = \dot M_s^V$, $\pi_{st} = \dot\pi^V_{st}$. By (12), (13), and (14), $\mathfrak{M}$ is an ENF for $\neg pA$. Its base model is V and all extenders used in defining the extension-maps π_{st} are η-closed with critical points $> \eta$. Therefore $\mathfrak{M}$ is η-closed with critical point $> \eta$. Since $\eta \geq 2^{\aleph_0}$ $\mathfrak{M}$ has a system of witnesses by Lemma 4.5, which concludes the proof of Theorem 9.1. $\dashv$

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