A MINIMAL PRIKRY-TYPE FORCING FOR SINGULARIZING A MEASURABLE CARDINAL

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Abstract. Recently, Gitik, Kanovei and the first author proved that for a classical Prikry forcing extension the family of the intermediate models can be parametrized by $\mathscr{P}(\omega)$/finite. By modifying the standard Prikry tree forcing we define a Prikry-type forcing which also singularizes a measurable cardinal but which is minimal, i.e. there are *no* intermediate models properly between the ground model and the generic extension. The proof relies on combining the rigidity of the tree structure with indiscernibility arguments resulting from the normality of the associated measures.

1. Introduction

The classical Prikry forcing first appeared in 1970 in Prikry's dissertation [Pri70]. It positively answered the following question of Silver and Solovay:

Is there a forcing preserving all cardinals while some cofinality changes?

In fact, the singularization of regular cardinals by some forcing is necessarily connected with Prikry forcing. In such an extension there must be a Prikry generic filter over an inner model with a measurable cardinal by the covering theorem of Dodd and Jensen (see [DJ82]).
Prikry forcing is equivalent to a Prikry tree forcing where conditions are trees with trunks, where the splitting sets above the trunk are always large with respect to the chosen normal measure. Many variants of Prikry forcings are known, see [Git10].

Let us give some examples for the analogous situation when a forcing adds a subset of ω instead of κ. It is easily seen that Cohen forcing adds a perfect set of mutually generic reals. On the other hand several forcings adding reals are minimal. Here a generic extension is called minimal if it has only trivial intermediate models. Furthermore, a forcing is said to be minimal if every generic extension by it is minimal. The first known forcing with this property was Sacks forcing introduced in [Sac71]. Also plain Laver forcing is minimal, see [Gra80]. Mathias forcing, the analog of the classical Prikry forcing for ω, is not minimal, as the subsequence of even digits generates a proper intermediate model. This holds as well for plain Mathias forcing as for Mathias forcing with an ultrafilter associated. In contrast to plain Laver forcing, the version with a Ramsey ultrafilter associated is not minimal, because it is equivalent to Mathias forcing with the same ultrafilter, see [JS89].

Classical Prikry forcing is not minimal. The main result of Gitik, Kanovei and the first author in [GKK10] reads:

Theorem. *Let $V[G]$ be a generic extension by classical Prikry forcing for some normal measure on a measurable cardinal κ. Then every intermediate model is a Prikry extension by this forcing and is generated by some subsequence of the associated Prikry sequence. Moreover, the intermediate models of V and $V[G]$ ordered by inclusion are isomorphic to* $\mathscr{P}(\omega)/\text{finite}$ *ordered by almost inclusion.*

Other Prikry-type forcings also have many intermediate models. Gitik showed in [Git10] that for a 2^μ-supercompact cardinal κ and a normal measure on $\mathscr{P}_\kappa(2^\mu)$ every $<\kappa$-distributive forcing of size μ is a subforcing of the associated supercompact Prikry forcing. Thus all results so far have shown that generic extensions by Prikry-type forcings have many intermediate models.

In contrast, this paper provides a minimal Prikry-type forcing preserving all cardinals while singularizing a measurable cardinal from the ground model. Inspired by the classical Prikry tree forcing, we introduce the partial order $\mathbb{P}_{\mathcal{U}}$, where $\mathcal{U}$ is a sequence of κ-complete nonprincipal ultrafilters over κ. The conditions of $\mathbb{P}_{\mathcal{U}}$ are $\mathcal{U}$-trees whose splitting sets are large with respect to certain ultrafilters in $\mathcal{U}$. In Section 3 we prove a Ramsey theorem for such trees and a Prikry lemma for $\mathbb{P}_{\mathcal{U}}$, which justifies calling it a Prikry-type forcing. Thereafter, in Section 4, we are going to investigate the intermediate models of generic extensions by $\mathbb{P}_{\mathcal{U}}$ if $\mathcal{U}$ is sequence of pairwise distinct normal measures. The minimality of $\mathbb{P}_{\mathcal{U}}$ is a direct consequence of:

Theorem. *Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$ where $\mathcal{U}$ is κ-sequence of pairwise distinct normal measures on κ. Then for every $X \in V[G]$ either X was already in the ground model or X generates the whole generic extension, i.e., $V[X] = V[G]$.*

Since the proof heavily uses the normality of the associated measures, we discuss the situation in the more general setting without the requirement of normality in Section 5. This may be helpful to gain information about generic extensions by the classical Prikry tree forcing.

The results of this paper have grown out of the diploma project of the second author under supervision of the first. In the diploma thesis [Räs10] only a part of the previous theorem was proved. For the remaining part the correspondence with the third author was of indispensible importance.

2. Setting

The notation follows common conventions. We will typically think of $u, v \in [\kappa]^{<\omega}$ as strictly increasing sequences of ordinals in κ. By $u \trianglelefteq v$ we mean that u is an initial segment of v. Moreover, we will use the typical operations on sequences, namely concatenation denoted by the symbol $^\frown$ and the restriction of the domain to some subset of ω denoted by $\restriction$. In addition $-$ corresponds to the operation $\setminus$, i.e., thinking of sequences remove the one range from the other and enumerate the result increasingly. By a tree we understand a non-empty subset of $[\kappa]^{<\omega}$ which is closed under initial segments. If T is a tree and $k \in \omega$, then we denote by $\mathrm{Lev}_k(T)$ the k-th level of T, which consists of all elements of T of length k.

2.1. **Normal measures.**

For our construction we shall fix a measurable cardinal κ, and we shall assume that there is a sequence $\mathcal{U} = \langle U_\alpha : \alpha < \kappa \rangle$ of pairwise distinct normal measures on κ. This assumption has the consistency strength of ZFC+"there exists a measurable cardinal" by the following result of Kunen and Paris:

Theorem 2.1 (Kunen-Paris forcing, [KP71])**.** *Assume that V is a model of* GCH *and let κ be a measurable cardinal and $\lambda > \kappa^+$ a regular cardinal. Then in some generic extension with the same cardinals and cofinalities there are λ many pairwise distinct normal measures on κ.*

In our minimality proof we will use a family $\langle A_\alpha : \alpha < \kappa \rangle$ of pairwise disjoint subsets of κ with $A_\alpha \in U_\alpha$. In the case of normal measures such a sequence always exists:

Lemma 2.2. *Let κ be a measurable cardinal carrying κ-many pairwise distinct normal measures $\langle U_\alpha : \alpha < \kappa \rangle$. Then there is a family $\langle A_\alpha : \alpha < \kappa \rangle$ of pairwise disjoint subsets of κ with $A_\alpha \in U_\alpha$.*

Proof. For ordinals $\alpha, \beta \in \kappa$, $\alpha < \beta$ pick $X_{\alpha,\beta} \subseteq \kappa$ such that $X_{\alpha,\beta} \in U_\alpha \setminus U_\beta$. Moreover define $X_{\beta,\alpha} := \kappa \setminus X_{\alpha,\beta}$ and $X_{\alpha,\alpha} := \kappa$. Let

$$A_\alpha := \Big(\mathop{\triangle}_{\beta<\kappa} X_{\alpha,\beta} \Big) \setminus (\alpha + 1).$$

By the normality we have $A_\alpha \in U_\alpha$. Assume that there is $\xi \in A_\alpha \cap A_\beta$. Then $\xi > \max(\alpha, \beta)$ and therefore $\xi \in X_{\alpha,\beta} \cap X_{\beta,\alpha}$ – a contradiction. ∎

2.2. **Intermediate models of generic extensions.**

Every intermediate model of ZFC of a generic extension $V[G]$ is generated by a single set, namely some $\mathbb{Q}$-generic filter H for some forcing $\mathbb{Q}$ over V, see [Jec06, pp 247-248]. We may further assume $H \subseteq \lambda$ for some cardinal λ. Hence, we restrict our attention to all sets of ordinals in $V[G]$ in the minimality proof. The smallest inner model N of ZFC with $V \subseteq N \subseteq V[G]$ and $H \in N$ for a set of ordinals $H \in V[G]$ can be defined without reference to forcing as $\bigcup_{z \subseteq \mathrm{Ord},\, z \in V} L[z, H]$ (such a model need not exist for arbitrary sets in $V[G]$). It is easy to check that this is a model of ZFC for any $H \in V[G]$. We say X is V-constructibly equivalent to Y, in short $X \equiv_V Y$, if $V[X] = V[Y]$.

3. Tree Prikry Forcing for Sequences of Ultrafilters

We now define and study a Prikry-type forcing $\mathbb{P}_{\mathcal{U}}$ where $\mathcal{U} = \langle U_\alpha : \alpha < \kappa \rangle$ is a sequence of ultrafilters and the conditions are trees whose branching sets are controlled by $\mathcal{U}$. This generalizes the classical situation where all branching sets are controlled by a single ultrafilter. Nevertheless the forcing satisfies several properties of classical Prikry forcing, in particular the Prikry lemma. In the next section we shall let $\mathcal{U}$ be a sequence of pairwise distinct normal measures and obtain the desired minimality result.

For the rest of this section let $\mathcal{U} = \langle U_\alpha : \alpha < \kappa \rangle$ denote some fixed sequence of κ-complete nonprincipal ultrafilters over the measurable cardinal κ.

3.1. **$\mathcal{U}$-trees and the partial order $\mathbb{P}_{\mathcal{U}}$.**

The conditions in the forcing are trees of the following type.

Definition 3.1. A set $T \subseteq [\kappa]^{<\omega}$ is called $\mathcal{U}$-tree with trunk t if

(T1) $\langle T, \trianglelefteq \rangle$ is a tree.
(T2) $t \in T$ and for all $u \in T$ we have $u \trianglelefteq t$ or $t \trianglelefteq u$.
(T3) For all $u \in T$ if $t \trianglelefteq u$ then

$$\mathrm{Suc}_T(u) := \{ \xi < \kappa : u^\frown\langle \xi \rangle \in T \} \in U_{\max(u)}{}^*.$$

Note that each such tree has a unique trunk.

*Because $u = \varnothing$ is possible for correctness one should use "sup" instead of "max" but this seems to be less intuitive.

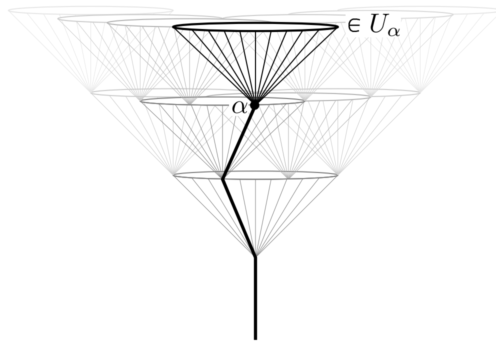

FIGURE 1. An image of a $\mathcal{U}$-tree.

Definition 3.2. Let $\mathbb{P}_{\mathcal{U}} := \{ \langle t, T \rangle : T$ is a $\mathcal{U}$-tree with trunk $t \}$.
Furthermore for $\langle s, S \rangle, \langle t, T \rangle \in \mathbb{P}_{\mathcal{U}}$ define

$$\langle s, S \rangle \leqslant \langle t, T \rangle \quad :\Longleftrightarrow \quad S \subseteq T$$
$$\langle s, S \rangle \leqslant^* \langle t, T \rangle \quad :\Longleftrightarrow \quad S \subseteq T \text{ and } s = t$$

In the latter case we call $\langle s, S \rangle$ a direct extension of $\langle t, T \rangle$.

Note that $\langle s, S \rangle \leqslant \langle t, T \rangle$ implies $s \trianglerighteq t$.

The following lemma introduces several possibilities to alter a $\mathcal{U}$-tree to obtain a new one, namely to restrict it, to remove an initial part or to attach the $\mathcal{U}$-tree on top of some finite strictly increasing sequence of ordinals in κ.

Lemma 3.3 (and **Definition**). *Let T be a $\mathcal{U}$-tree with trunk t.*

(1) *If $u \in T$, $u \trianglerighteq t$ then $T \upharpoonright u := \{ v \in T : u \trianglelefteq v \vee v \trianglelefteq u \}$ is a $\mathcal{U}$-tree with trunk u and $\langle t, T \rangle \geqslant \langle u, T \upharpoonright u \rangle$. The case $u \trianglelefteq t$ is not of interest since then $T \upharpoonright u = T$.*
(2) *For $u \in T$ let $T_{\trianglerighteq u} := \{ v \in [\kappa]^{<\omega} : u^\frown v \in T \}$. This is a $\mathcal{U}$-tree with trunk $t - u$ if $\mathrm{Suc}_T(u) \in U_0$ or $t - u \neq \varnothing$.*
(3) *If $u \in [\kappa]^{<\omega}$ and $\max(u) = \max(t)$ then $u \oplus T_{\trianglerighteq t} := \{ u^\frown v : t^\frown v \in T \}$ is a $\mathcal{U}$-tree with trunk u.*

Proof. In every case all properties are evident from the respective definition. ∎

What is essentially needed in (3) are the properties $\mathrm{Suc}_T(t) \in U_{\max(v)}$ and $\max(v) < \min(\mathrm{Suc}_T(t))$.

Lemma 3.4. *Let $\langle T_\xi : \xi < \lambda \rangle$, $\lambda < \kappa$, be a sequence of $\mathcal{U}$-trees with the same trunk t. Then $\bigcap_{\xi < \lambda} T_\xi$ is again a $\mathcal{U}$-tree with trunk t.*

Proof. This is obvious as U_α is κ-complete for every $\alpha < \kappa$. ∎

Let us now characterize compatibility.

Lemma 3.5. *Let $\langle s,S\rangle$, $\langle t,T\rangle \in \mathbb{P}_{\mathcal{U}}$. Then $\langle s,S\rangle \| \langle t,T\rangle$ iff ($s \in T$ and $t \in S$). In particular $\langle s,S\rangle \| \langle t,T\rangle$ implies $s \trianglelefteq t$ or $t \trianglelefteq s$.*

Proof. Clearly $\langle r,R\rangle \leqslant \langle s,S\rangle, \langle t,T\rangle$ implies $s,t \trianglelefteq r$ and hence $s,t \in R \subseteq S,T$.
If $s \in T$, $t \in S$ and $s \trianglelefteq t$, then $\langle t, (S \upharpoonright t) \cap (T \upharpoonright s)\rangle \in \mathbb{P}_{\mathcal{U}}$ is a common extension of $\langle s,S\rangle$ and $\langle t,T\rangle$. ∎

3.2. **Ramsey properties of $\mathcal{U}$-trees.**

In this subsection we are going to prove that for every coloring of some $\mathcal{U}$-tree one can find a sub-$\mathcal{U}$-tree which is homogeneous in the sense that all elements on the same level have the same color. This is a version of the Rowbottom-Theorem for colorings of $[\kappa]^{<\omega}$. We will use this to prove a slightly more involved property of this sort, namely that every graph on a $\mathcal{U}$-tree can be restricted to a sub-$\mathcal{U}$-tree such that whether two nodes are connected only depends on the order configuration of the ordinals in the two nodes.

Lemma 3.6. *Let T be a $\mathcal{U}$-tree and $c : T \to \lambda$ with $\lambda < \kappa$. Then there is a $\mathcal{U}$-tree $\bar{T} \subseteq T$ with the same trunk homogeneous for c, i.e. every two elements of $\bar{T}$ on the same level have the same color.*

Proof. We show by induction on n that for every coloring of a $\mathcal{U}$-tree T with trunk t with less than κ many colors there is a $\mathcal{U}$-tree $T_n \subseteq T$ with the same trunk such that the coloring is constant on all Levels of T_n up to $|t|+n$. By letting $\bar{T} := \bigcap_{n<\omega} T_n$ where for all n we know that c is constant on $\mathrm{Lev}_k(T_n)$ for all $k \leqslant |t|+n$ we obtain a $\mathcal{U}$-tree homogeneous for c.
The assertion is obvious for $n = 0$. Thus let $c : T \to \lambda$ with $\lambda < \kappa$ be a coloring of a $\mathcal{U}$-tree T. Consider for every $\xi \in \mathrm{Suc}_T(t)$ the coloring $c\upharpoonright (T \upharpoonright t^\frown\langle\xi\rangle)$. By the induction hypothesis there are $\mathcal{U}$-trees $S_\xi \subseteq T \upharpoonright t^\frown\langle\xi\rangle$ such that c is constant on $\mathrm{Lev}_k(S_\xi)$ for all $k \leqslant |t|+n+1$. Further, $S := \bigcup_{\xi \in \mathrm{Suc}_T(t)} S_\xi$ is a $\mathcal{U}$-tree and for every $\xi \in \mathrm{Suc}_S(t) = \mathrm{Suc}_T(t)$ and all $u,v \in S \upharpoonright t^\frown\langle\xi\rangle = S_\xi$ with $|u|-|t| = |v|-|t| \leqslant n+1$ we obtain $c(u) = c(v)$. Denote this value of c by $\gamma_{\xi,|u|-|t|}$. Then by the κ-completeness of $U_{\max(t)}$ there is $H \subseteq \mathrm{Suc}_S(t)$ in $U_{\max(t)}$ such that all $\xi \in H$ have the same sequence $\langle \gamma_{\xi,k} : k \leqslant n+1\rangle$ and hence c is constant up to Level $|t|+n+1$ on $T_{n+1} := \bigcup_{\xi \in H} S_\xi$. ∎

Now we establish partition results for colorings of T^2. The colors of a pair $\langle u,v\rangle \in T^2$ may depend on the type of u,v, i.e., the way in which the sequences u and v are interlaced. We shall prove that on a sub-$\mathcal{U}$-tree the color of each pair only depends on its type.

Definition 3.7. Let u, v be finite strictly increasing sequences of ordinals. Then $\operatorname{type}(u,v) \in 3^{|\operatorname{ran}(u)\cup\operatorname{ran}(v)|}$ denotes the order configuration of u and v, i.e., enumerate $u \cup v$ strictly increasing as $\{\, \xi_i : i < n \,\}$ and define

$$\operatorname{type}(u,v)(i) = \begin{cases} 0 & \text{if } \xi_i \in u \setminus v \\ 1 & \text{if } \xi_i \in v \setminus u \\ 2 & \text{if } \xi_i \in u \cap v \end{cases}$$

For example $\langle 0,2,0,0,1\rangle$ tells us that u has four and v has two elements, that the first element of v is the same as the second element of u, and that the second element of v is bigger than all elements of u. This is depicted in Figure 2.

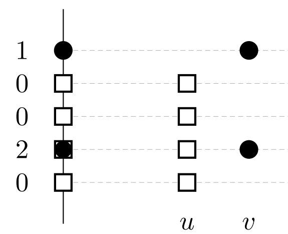

FIGURE 2. The type $\langle 0,2,0,0,1\rangle$.

Theorem 3.8. *Let T be a $\mathcal{U}$-tree and $c : T^2 \to \lambda$ for some $\lambda < \kappa$ where all the U_α are normal measures on κ. Then there is a $\mathcal{U}$-tree $\bar{T} \subseteq T$ with the same trunk such that for all $u, v \in \bar{T}$ the value of c only depends on the type of u, v.*

This is an immediate consequence of the following.

Lemma 3.9. *Let S, T be $\mathcal{U}$-trees and $c : S \times T \to \lambda$ for some $\lambda < \kappa$ where all the U_α are normal measures on κ. Then there are $\mathcal{U}$-trees $\bar{S} \subseteq S$ and $\bar{T} \subseteq T$ with the same trunks, respectively, such that for all $u \in \bar{S}$, $v \in \bar{T}$ the value of c only depends on the type of u, v.*

Proof. For, we conclude by induction on $\langle m, n\rangle$:

> Let S, T be $\mathcal{U}$-trees with the trunks s and t, respectively. Further let $c : S \times T \to \lambda$ for some $\lambda < \kappa$. Then there are $\mathcal{U}$-trees $S_m \subseteq S$ with trunk s and $T_n \subseteq T$ with trunk t such that for all $u \in S_m$, $v \in T_n$ with $|u| - |s| \leqslant m$, $|v| - |t| \leqslant n$ the value of c only depends on the type of u, v.

We prove this by induction along the order on $\omega \times \omega$ defined by $\langle k, l\rangle < \langle m, n\rangle$ iff $k \leqslant m$ and $l \leqslant n$, and $k < m$ or $l < n$.
Since the cases where $m = 0$ or $n = 0$ follow from the previous lemma, let us assume $m, n \neq 0$. We may first apply the induction hypothesis and therefore assume that c

on $S \times T$ behaves as required for u, v with $\langle |u| - |s|, |v| - |t| \rangle < \langle m, n \rangle$. Now we will successively thin out both trees in three ways to cover the different arrangements of the largest elements of u and v.

To deal with the case $\max(u) = \max(v)$, we define another coloring c' on $S \times T$ by $c'(u, w) = c(u, w^\frown\langle\xi\rangle)$ if $\max(u) = \xi$ and $w^\frown\langle\xi\rangle \in T$, and $c'(u, w) = 0$ otherwise. By the induction hypothesis we can thin out S, T so that $c'(u, w)$ depends only on $\operatorname{type}(u, w)$ for u, w with $\langle |u| - |s|, |w| - |t| \rangle < \langle m, n \rangle$. Then $c(u, v)$ is constant for all u, v with $|u| - |s| \leqslant m$, $|v| - |t| \leqslant n$ and the same type $\mathbb{t}$ if the last entry of $\mathbb{t}$ is 2 since then $c(u, v) = c'(u, v - \langle \max(v) \rangle)$.

Now we handle the case $\max(u) > \max(v)$. For each pair $\langle w, v \rangle \in S \times T$ let

$$X_{w,v} := \{ \xi \in \operatorname{Suc}_S(w) \,:\, c(w^\frown\langle\xi\rangle, v) = \gamma_{w,v} \}$$

where $\gamma_{w,v} < \lambda$ is chosen so that $X_{w,v} \in U_{max(w)}$. Let us define a coloring c'' on $S \times T$ by $c''(w, v) = \gamma_{w,v}$. We can assume that S, T are thinned out so that $c''(w, v)$ depends only on $\operatorname{type}(w, v)$ for w, v with $\langle |w| - |s|, |v| - |t| \rangle < \langle m, n \rangle$ by the induction hypothesis. Let $X_w = \triangle_{v \in T} X_{w,v}$ for $w \in S$, i.e., $\xi \in X_w$ iff $\xi \in X_{w,v}$ for all $v \in T$ with $\max(v) < \xi$. Then $X_w \in U_{\max(w)}$ by normality. We restrict $\operatorname{Suc}_S(w)$ to X_w for each $w \in S$ with $|w| - |s| = m - 1$. To see that $c(u, v)$ is constant for all u, v with $|u| - |s| \leqslant m$, $|v| - |t| \leqslant n$ and the same type $\mathbb{t}$ if the last entry of $\mathbb{t}$ is 0, note that in this case $\max(u) \in X_{u - \langle \max(u) \rangle, v}$ and hence $c(u, v) = c''(u - \langle \max(u) \rangle, v)$.

The procedure for the remaining case $\max(u) < \max(v)$ is similar. ∎

It is possible to generalize Theorem 3.8 to arbitrary colorings of n-products of $\mathcal{U}$-trees.

3.3. Forcing with $\mathbb{P}_{\mathcal{U}}$.

Now we investigate whether several properties of Prikry forcings also apply to $\mathbb{P}_{\mathcal{U}}$. For example we discuss a Prikry lemma to show the preservation of all cofinalities but κ's and the reconstruction of the generic filter from the Prikry sequence.

Lemma 3.10. *Let G be generic on $\mathbb{P}_{\mathcal{U}}$, then*

$$f_G := \bigcup \{ t \,:\, \exists T \, \langle t, T \rangle \in G \}$$

is an ω-sequence cofinal in κ. Hence in $V[G]$ we have $\operatorname{cf}(\kappa) = \aleph_0$. We call f_G a Prikry sequence for $\mathcal{U}$.

Proof. For every $\alpha < \kappa$ the set

$$D_\alpha := \{ \langle t, T \rangle \in \mathbb{P}_{\mathcal{U}} \,:\, \alpha < \max(t) \}$$

is dense and hence we know that f_G is cofinal in κ. By Lemma 3.5 all trunks of elements of G are totally ordered by $\trianglelefteq$. Accordingly, f_G has to be an ω-sequence. ∎

Lemma 3.11. *$\mathbb{P}_{\mathcal{U}}$ satisfies the κ^+-cc. Thus it preserves cofinalities and cardinals greater than κ.*

Further, $\langle \mathbb{P}_{\mathcal{U}}, \leqslant^* \rangle$ is κ-closed.

Proof. The first claim is an immediate conclusion from the fact that there are only κ many possible trunks and Lemma 3.4. The second statement is a direct consequence of the same lemma. ∎

For the next result we convey the proof of the Prikry lemma for the classical Prikry forcing but use Lemma 3.6 as the analog of Rowbottom's Theorem. It is also possible to copy the proof given in [Git10] for the classical Prikry tree forcing.

Lemma 3.12 (Prikry lemma)**.** *Let $\langle t, T \rangle \in \mathbb{P}_{\mathcal{U}}$ and φ a statement in the forcing language. Then there is a direct extension $\langle s, S \rangle \in \mathbb{P}_{\mathcal{U}}$ of $\langle t, T \rangle$ deciding φ.*

Proof. Let $\langle t, T \rangle$ and φ as stated above. We apply Proposition 3.6 to the coloring on T defined by

$$u \mapsto \begin{cases} 0 & \text{if there exists a } \mathcal{U}\text{-tree } X \text{ with trunk } u \text{ such that } \langle u, X \rangle \Vdash \varphi \\ 1 & \text{otherwise} \end{cases}$$

and obtain a $\mathcal{U}$-tree $\bar{T} \subseteq T$ homogeneous for this coloring. Then $\langle t, \bar{T} \rangle$ decides φ. ∎

Corollary 3.13. *The forcing $\mathbb{P}_{\mathcal{U}}$ does not add bounded subsets of κ. In fact $V_\kappa = V_\kappa^{V[G]}$.*

Proof. This follows from the previous two lemmas by standard methods. ∎

Corollary 3.14. *The forcing $\mathbb{P}_{\mathcal{U}}$ preserves cofinalities of ordinals less than κ. Hence it also preserves all cardinals less or equal to κ.*

The following theorem sums up the preceding results.

Theorem 3.15. *Let G be a generic filter on $\mathbb{P}_{\mathcal{U}}$. Then in $V[G]$*

(1) *κ is singular with $\mathrm{cf}(\kappa) = \aleph_0$,*
(2) *all cardinals are preserved and also all cofinalities but κ's.*
Furthermore $V_\kappa = V_\kappa^{V[G]}$.

As for the classical Prikry forcing it is possible to reconstruct the generic filter from the Prikry sequence.

Lemma 3.16. *Let G be a generic filter on $\mathbb{P}_{\mathcal{U}}$ and $f := f_G$ the associated Prikry sequence. Then $G = G_f$ where*

$$G_f := \{ \langle t, T \rangle \in \mathbb{P}_{\mathcal{U}} \ : \ \forall n < \omega \ f \restriction n \in T \}.$$

Also the other direction works. If f is an ω-sequence of ordinals in κ and G_f is generic on $\mathbb{P}_{\mathcal{U}}$, then $f_{G_f} = f$.

Proof. It is enough to show $G \subseteq G_f$ and that every two elements in G_f are compatible. Let $\langle t, T \rangle \in G$. For every $n < \omega$ there is some $\langle s, S \rangle \in G$ with $f \restriction n \trianglelefteq s$ and we can assume $\langle s, S \rangle \leqslant \langle t, T \rangle$ which yields $f \restriction n \in S \subseteq T$ and hence $\langle t, T \rangle \in G_f$.
For the second requirement let $\langle s, S \rangle$, $\langle t, T \rangle \in G_f$. According to the definition of G_f we have $s = f \restriction |s|$, $t = f \restriction |t|$ and thus $s \in T$ and $t \in S$. Hence we finally obtain $\langle s, S \rangle \parallel \langle t, T \rangle$ by Lemma 3.5. The second assertion is easily seen as $\langle t, [\kappa]^{<\omega} \restriction t \rangle \in G_f$ for every $t \trianglelefteq f$. ∎

It would be nice to have a characterization of Prikry sequences for $\mathbb{P}_{\mathcal{U}}$ as there is of Prikry sequences for the classical Prikry forcing in form of the Mathias criterion (see [Mat73]). Although we do not know of such a characterization, we have the following proposition:

Proposition 3.17. *Let $f = f_G$ for some generic filter G on $\mathbb{P}_{\mathcal{U}}$ and let $\tilde{f}$ be an ω-sequence equal to f on all but finitely many natural numbers. Then $\tilde{f}$ also is a Prikry sequence for $\mathcal{U}$.*

Proof. We will show that $G_{\tilde{f}}$ is a generic filter on $\langle \mathbb{P}_{\mathcal{U}}, \leqslant \rangle$ because G_f is. From the proof of Lemma 3.16 we already know that every two conditions in $G_{\tilde{f}}$ are compatible. Further, from $\langle t, T \rangle \leqslant \langle s, S \rangle$ and $\langle t, T \rangle \in G_{\tilde{f}}$ it directly follows that $\langle s, S \rangle \in G_{\tilde{f}}$.
Suppose $f(i) = \tilde{f}(i)$ for all $i \geqslant m$ and let $t = f \restriction (m+1)$, $\tilde{t} = \tilde{f} \restriction (m+1)$, so $\max(t) = \max(\tilde{t})$. Let $p := \langle t, [\kappa]^{<\omega} \restriction t \rangle$ and $\tilde{p} := \langle \tilde{t}, [\kappa]^{<\omega} \restriction \tilde{t} \rangle$. Further, define

$$h : \{ q \in \mathbb{P}_{\mathcal{U}} \ : \ q \leqslant p \} \to \{ q \in \mathbb{P}_{\mathcal{U}} \ : \ q \leqslant \tilde{p} \}, \quad \langle t^\frown u, T \rangle \mapsto \langle \tilde{t}^\frown u, \tilde{t}^\frown u \oplus T_{\trianglerighteq t^\frown u} \rangle.$$

It is easy to see that h is an isomorphism. Since $p \in G_f = G$, we observe that G is $\mathbb{P}_{\mathcal{U}}$-generic below p. Thus, $h[G_f \cap \{ q \in \mathbb{P}_{\mathcal{U}} \ : \ q \leqslant p \}]$ is $\mathbb{P}_{\mathcal{U}}$-generic below $\tilde{p}$ and moreover $h[G_f \cap \{ q \in \mathbb{P}_{\mathcal{U}} \ : \ q \leqslant p \}] = G_{\tilde{f}} \cap \{ q \in \mathbb{P}_{\mathcal{U}} \ : \ q \leqslant \tilde{p} \}$ by the definition of $G_{\tilde{f}}$. ∎

4. Intermediate Models of Generic Extensions by $\mathbb{P}_{\mathcal{U}}$

In this section, we show that the Prikry tree forcing for a sequence of pairwise distinct normal measures is minimal. Let us consider the relation $\equiv_V$ introduced in the first section. The result can be stated as follows:

Theorem 4.1. *Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$ where $\mathcal{U}$ is sequence of pairwise distinct normal measures on κ. Then for every set $X \in V[G]$ of ordinals either $X \in V$ or $X \equiv_V f_G$.*

For the rest of this section we will refer to $\mathcal{U}$ as a sequence of pairwise distinct normal measures on κ and as we have seen in the first section in Lemma 2.2 with this comes a sequence $\langle A_\alpha : \alpha < \kappa\rangle$ of pairwise disjoint subsets of κ such that $A_\alpha \in U_\alpha$. Fix such a sequence $\langle A_\alpha : \alpha < \kappa\rangle$.

The proof is split into two parts where the first one only handles new subsets of κ and the second part uses this to obtain a general result for all sets of ordinals in the generic extension. The arguments are quite different and therefore we prove both results in separate subsections dedicated to the particular step in the proof.

4.1. **Subsets of κ in the generic extension.**

This subsection shows how to deal with $\mathscr{P}(\kappa)^{V[G]}$. More precisely, we are going to prove the following theorem.

Theorem 4.2. *Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$ where $\mathcal{U}$ is sequence of pairwise distinct normal measures on κ. Then for every $X \in \mathscr{P}(\kappa)^{V[G]}$ either $X \in V$ or $X \equiv_V f_G$.*

Before we start to prove the theorem we provide a helpful lemma.

Lemma 4.3. *Let T be a $\mathcal{U}$-tree. Then there is a $\mathcal{U}$-tree $\bar{T} \subseteq T$ with the same trunk such that for all $u, v \in T$ with $u(n) \neq v(n)$ for some $n < \min\{|u|, |v|\}$, we have $u(m) \neq v(m)$ for all $m \geqslant n$ below $\min\{|u|, |v|\}$.*

Proof. Let T be a $\mathcal{U}$-tree and shrink T to a $\mathcal{U}$-tree $\bar{T}$ in which all sets of successors have been restricted to the appropriate A_α as follows

$$\begin{aligned}
\bar{T}_0 &:= \{\, u \in T \ : \ u \trianglelefteq t \,\} \\
\bar{T}_{n+1} &:= \{\, u \in S \ : \ \exists\, \xi < \kappa \ \ \exists\, v \in \bar{T}_n \\
&\qquad\qquad (\, v \trianglerighteq s \ \wedge \ \xi \in \operatorname{Suc}_T(v) \cap A_{\max(v)} \ \wedge \ u = v^\frown\langle \xi\rangle \,)\,\} \\
\bar{T} &:= \bigcup_{n<\omega} \bar{T}_n.
\end{aligned}$$

Note that $\operatorname{Lev}_{|s|+n}(\bar{T}) = \bar{T}_n$ for all $n > 0$. Obviously $\bar{T}$ is as required. ∎

Since we have proved that the set of all $\mathcal{U}$-trees with the property from the lemma is dense, we also have the following proposition which introduces a proof idea for Proposition 4.2.

Proposition 4.4. *Let $f = f_G$ for some generic filter G on $\mathbb{P}_{\mathcal{U}}$ and let $d \in V[G]$ be a subsequence of f. Then either d is finite or $d \equiv_V f$.*

Proof. Let d be an infinite subsequence of some Prikry sequence f. Let h be the function mapping every $\xi < \kappa$ to the unique α with $\xi \in A_\alpha$ if $\xi \in \bigcup_{\alpha<\kappa} A_\alpha$ and to 0 otherwise. As we have seen in the previous lemma, the set

$$D := \{ \langle t, T \rangle \in \mathbb{P}_{\mathcal{U}} \ : \ \forall u \in T \ \ \mathrm{Suc}_T(u) \subseteq A_{\max(u)} \}$$

is dense in $\mathbb{P}_{\mathcal{U}}$. The density yields some condition $\langle t, T \rangle \in D \cap G$.

We will now reconstruct f from d. This construction works recursively although the idea is quite simple since from knowing $d(n+1) = f(n+1)$ we also know $f(n) = h(f(n+1))$. The latter is true because $f \upharpoonright m \in T$ for all m, hence it makes sense to consider $\mathrm{Suc}_T(f \upharpoonright (n+1))$ which is a subset of $A_{f(n)}$ for $n+1 \geqslant \mathrm{dom}(t)$ and therefore $h(\xi) = f(n)$ for all $\xi \in \mathrm{Suc}_T(f \upharpoonright (n+1))$, especially for $f(n+1)$. We define

$$f_0 := d$$
$$f_{n+1} := f_n \cup \{ (k, \alpha) \in \omega \times \kappa \ : \ k+1 \in \mathrm{dom}(f_n) \wedge h(f_n(k+1)) = \alpha \}$$

and conclude $f = t^\frown(\bigcup_{n<\omega} f_n \upharpoonright (\omega \setminus \mathrm{dom}(t)))$.

The inclusion from left to right follows by showing $f_n \upharpoonright (\omega \setminus \mathrm{dom}(t)) \subseteq f - t$ for all $n < \omega$. Clearly this is true for $f_0 = d$ and the argument for the induction step has been explained already.

Let us finally check $f \setminus t \subseteq \bigcup_{n<\omega} f_n$. Since d is infinite $\mathrm{dom}(d)$ is unbounded in ω and hence for every $k < \omega$ there is $l_k \in \mathrm{dom}(d)$ greater or equal to k. Now it is easy to prove $(k, f(k)) \in f_{l_k - k}$ by induction on $l_k - k$ for $k \geqslant \mathrm{dom}(t)$.

Since f war recursively defined from d, using $h \in V$, we have $f \in V[d]$. ∎

Now we prove Theorem 4.2.

Proof of the theorem. Let $\dot{X}$ be a name for some subset of κ and $\langle t, T \rangle \in \mathbb{P}_{\mathcal{U}}$. We will show that there is $p \leqslant \langle t, T \rangle$ such that

$$p \Vdash (\dot{X} \in V \vee \dot{X} \equiv_V \dot{f})$$

where $\dot{f}$ denotes the obvious name for the associated Prikry sequence.

By the Prikry lemma, we may assume that for every $u \in T$ the condition $\langle u, T \upharpoonright u \rangle$ already decides $\dot{X}$ up to $\max(u)$. For $u \in T$ define

$$X_u := \{ \xi < \max(u) \ : \ \langle u, T \upharpoonright u \rangle \Vdash \check{\xi} \in \dot{X} \}.$$

Moreover, consider the following coloring

$$c : T \times T \to 2, \quad \langle u, v \rangle \mapsto \begin{cases} 1 & \text{if } X_u \cap \max(v) = X_v \cap \max(u) \\ 0 & \text{otherwise} \end{cases}$$

By Theorem 3.8 there is $\bar{T} \subseteq T$ such that the values of c on $\bar{T} \times \bar{T}$ only depend on the type of the arguments.

Further, we may also assume by employing Lemma 4.3 that for all $u, v \in T$ with $u(n) \neq v(n)$ for some $n < \min\{|u|, |v|\}$, we have $u(m) \neq v(m)$ for all $m \geqslant n$ below $\min\{|u|, |v|\}$. Note that we have not changed the coloring by thinning out T.

Claim. For every $s \in \bar{T}$ and $n < \omega$ the coloring c is constant on the set

$$\{\langle s^\frown u, s^\frown v\rangle : u, v \in \mathrm{Lev}_n(\bar{T}_{\trianglerighteq s}) \text{ with } u(0) \neq v(0)\}.$$

Assume there are $u, v \in \mathrm{Lev}_n(\bar{T}_{\trianglerighteq s})$ with $u(0) \neq v(0)$ and $c(s^\frown u, s^\frown v) = 1$. We may further assume that $\max(u) < \max(v)$.

To begin with, we show that all $u', v' \in \mathrm{Lev}_n(\bar{T}_{\trianglerighteq s})$ with $\mathrm{type}(u', v') = \mathbb{t}_{alt}$ satisfy $c(s^\frown u', s^\frown v') = 1$, where $\mathbb{t}_{alt} := \langle 0, 1, \ldots, 0, 1\rangle$ denotes the alternating type of two sequences of length n. By our assumptions the type of u and v only consists of 0's and 1's and hence is of length $2n$. Furthermore, it ends with 1. We construct three sequences $w_0, w_1, w_2 \in \mathrm{Lev}_n(\bar{T}_{\trianglerighteq s})$ with $\mathrm{type}(w_0, w_1) = \mathrm{type}(w_2, w_1) = \mathrm{type}(u, v)$ and $\mathrm{type}(w_0, w_2) = \mathbb{t}_{alt}$.

To understand the procedure look at Figure 3. There $n = 3$ and the exemplary situation in which $\mathrm{type}(u, v) = \langle 1, 0, 0, 1, 0, 1\rangle$ is on the left side, where the squares stand for u and the dots for v. The right side shows what we are going to construct.

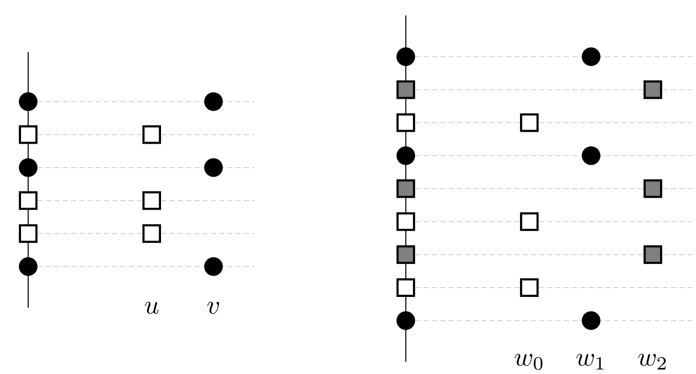

FIGURE 3. $\langle 1, 0, 0, 1, 0, 1\rangle \rightsquigarrow \mathbb{t}_{alt}$.

We proceed by recursion on $i < 2n$. Let $\ell(i) = |\{j < i : \mathrm{type}(u, v)(j) = 0\}|$ denote the number of 0's in $\mathrm{type}(u, v)$ up to component i. If $\mathrm{type}(u, v)(i) = 0$, then pick $\xi^0_{\ell(i)} > \max(\xi^1_{i-\ell(i)-1}, \xi^2_{\ell(i)-1})$ in $\mathrm{Suc}_{\bar{T}}(s^\frown \xi^0_0{}^\frown \cdots {}^\frown \xi^0_{\ell(i)-1})$ and $\xi^2_{\ell(i)} > \xi^0_{\ell(i)}$ in $\mathrm{Suc}_{\bar{T}}(s^\frown \xi^2_0{}^\frown \cdots {}^\frown \xi^2_{\ell(i)-1})$. If $\mathrm{type}(u, v)(i) = 1$, then pick $\xi^1_{i-\ell(i)} > \xi^2_{\ell(i)-1}$ in $\mathrm{Suc}_{\bar{T}}(s^\frown \xi^1_0{}^\frown \cdots {}^\frown \xi^1_{i-\ell(i)-1})$. Of course if $\ell(i) = 0$ or $i - \ell(i) = 0$, then there are less requirements to the choice of $\xi^0_{\ell(i)}, \xi^2_{\ell(i)}$ or $\xi^1_{i-\ell(i)}$.

Clearly $w_0 := s^\frown\xi_0^0{}^\frown\cdots^\frown\xi_{n-1}^0$, $w_1 := s^\frown\xi_0^1{}^\frown\cdots^\frown\xi_{n-1}^1$ and $w_2 := s^\frown\xi_0^2{}^\frown\cdots^\frown\xi_{n-1}^2$ are as required. Therefore $c(s^\frown w_0, s^\frown w_2) = 1$ because

$$\begin{aligned} X_{s^\frown w_0} &= X_{s^\frown w_1} \cap \max(w_0) \\ &= X_{s^\frown w_1} \cap \max(w_2) \cap \max(w_0) = X_{s^\frown w_2} \cap \max(w_0). \end{aligned}$$

Now we continue by showing $c(s^\frown u', s^\frown v') = 1$ for all $u', v' \in \mathrm{Lev}_n(\bar{T}_{\trianglerighteq s})$ with $\mathrm{type}(u', v') = \mathbb{t}_{suc}$, where $\mathbb{t}_{suc} := \langle 0, \ldots, 0, 1, \ldots, 1\rangle$ denotes the successive type of two sequences of length n. Similar to what we did so far, we construct sequences $w_0, w_1, \ldots, w_n \in \mathrm{Lev}_n(\bar{T}_{\trianglerighteq s})$ with $\mathrm{type}(w_j, w_{j+1}) = \mathbb{t}_{alt}$ for all $j < n$ and $\mathrm{type}(w_0, w_n) = \mathbb{t}_{suc}$.

Again we first look at the special case $n = 3$ and Figure 4 illustrates how we are going to proceed.

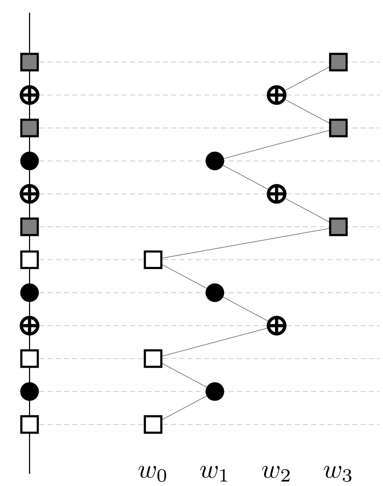

FIGURE 4. $\mathbb{t}_{alt} \rightsquigarrow \mathbb{t}_{suc}$ for $n = 3$.

We proceed by recursion on the Cantorian well-ordering of $\omega \times \omega$ restricted to $n \times (n+1)$. Pick $\xi_0^0 \in \mathrm{Suc}_{\bar{T}}(s)$. For $(i,j) \in n \times (n+1)$ with $(0,0) < (i,j)$, we distinguish two cases. If $i = 0$, then pick $\xi_0^j \in \mathrm{Suc}_{\bar{T}}(s)$ with $\xi_0^j > \xi_{j-1}^0$. If $i \neq 0$, then pick $\xi_i^j \in \mathrm{Suc}_{\bar{T}}(s^\frown\xi_0^j{}^\frown\cdots^\frown\xi_{i-1}^j)$ with $\xi_i^j > \xi_{i-1}^{j+1}$.
As before it is easy to see that $w_0, \ldots, w_n$ with $w_j := s^\frown\xi_0^j{}^\frown\cdots^\frown\xi_{n-1}^j$ are as required and hence the above argument shows $c(s^\frown w_0, s^\frown w_n) = 1$.

Finally, we easily obtain $c(s^\frown u', s^\frown v') = 1$ for all $u', v' \in \mathrm{Lev}_n(\bar{T}_{\trianglerighteq s})$. This is because for such u', v', there is $w \in \mathrm{Lev}_n(\bar{T}_{\trianglerighteq s})$ with $\mathrm{type}(u', w) = \mathrm{type}(v', w) = \mathbb{t}_{suc}$. For $n = 3$ and $\mathrm{type}(u', v') = \langle 1, 0, 0, 0, 1, 1\rangle$ this can be seen in Figure 5.

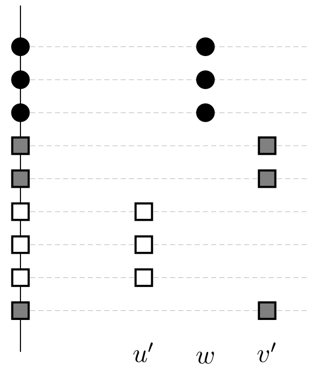

FIGURE 5. $\mathbb{t}_{suc} \rightsquigarrow \langle 1,0,0,0,1,1\rangle$.

Claim. The condition $\langle t, \bar{T}\rangle$ forces $\dot{X} \in V \vee \dot{X} \equiv_V \dot{f}$.

Let G be generic over $\mathbb{P}_{\mathcal{U}}$ with $\langle t, \bar{T}\rangle \in G$. We first prove that for all k

$$X_{f_G\restriction(k+1)} = \dot{X}^G \cap f_G(k).$$

Let $k < \omega$. Since $\langle f_G\restriction(k+1), \bar{T}\restriction(f_G\restriction(k+1))\rangle \in G$ holds, we obviously obtain $X_{f_G\restriction(k+1)} \subseteq \dot{X}^G \cap f_G(k)$. Now let $\alpha \in \dot{X}^G \cap f_G(k)$. Then there is a $\mathcal{U}$-tree S with trunk $v \trianglelefteq f_G$ of length greater than k such that $\langle v, S\rangle \Vdash \check{\alpha} \in \dot{X}$ and $\langle v, S\rangle \in G$. Since the conditions $\langle v, S\rangle$ and $p := \langle f_G\restriction(k+1), \bar{T}\restriction(f_G\restriction(k+1))\rangle$ are compatible and p decides $\dot{X}$ up to $f_G(k) > \alpha$, also $p \Vdash \check{\alpha} \in \dot{X}$. Thus $\alpha \in X_{f_G\restriction(k+1)}$.

An even easier observation is that $X_u = X_v \cap \max(u)$ for all $u \trianglelefteq v$ in $\bar{T}$. This holds because $\langle u, \bar{T}\restriction u\rangle$ decides $\dot{X}$ up to $\max(u)$ and will be useful in the end.

The rest of the proof describes a way to construct f_G from $\dot{X}^G$ if $\dot{X}^G \notin V$. Of course $t \trianglelefteq f_G$.

Now assume we already have constructed $s := f_G\restriction m$ for some $m < \omega$. Starting from s we now obtain $f_G\restriction(m+1)$. Assume there is $n > 0$ such that the only value of c on $\{\langle s^\frown u, s^\frown v\rangle : u, v \in \mathrm{Lev}_n(\bar{T}_{\trianglerighteq s})$ with $u(0) \neq v(0)\}$ is 0. Then $u := f_G\restriction(m+n) - s \in \mathrm{Lev}_n(\bar{T}_{\trianglerighteq s})$ satisfies

$$X_{s^\frown u} = X_{f_G\restriction(m+n)} = \dot{X}^G \cap f_G(m+n-1) = \dot{X}^G \cap \max(u)$$

and therefore $X_{s^\frown v} \neq \dot{X}^G \cap \max(v)$ holds for all $v \in \mathrm{Lev}_n(\bar{T}_{\trianglerighteq s})$ with $u(0) \neq v(0)$. It is clear that this uniquely determines $f_G(m)$.

By the previous lemma the other case is that for all $n > 0$ the only value of c on $\{\langle s^\frown u, s^\frown v\rangle : u, v \in \mathrm{Lev}_n(\bar{T}_{\trianglerighteq s})$ with $u(0) \neq v(0)\}$ is 1. In this case $\dot{X}^G$ equals $\bigcup_{\xi\in\mathrm{Suc}_{\bar{T}}(s)} X_{s^\frown\langle\xi\rangle} \in V$. In order to see this, we show $X_{s^\frown\langle\xi\rangle} = \dot{X}^G \cap \xi$ for all $\xi \in \mathrm{Suc}_{\bar{T}}(s)$. Let $\xi \in \mathrm{Suc}_{\bar{T}}(s)$. Then there is k such that $f_G(k) > \xi$ and by letting $n := k+1-m$ we can find $u \in \mathrm{Lev}_n(\bar{T}_{\trianglerighteq s})$ with $u(m) = \xi$ and $\max(u) > f_G(k)$. By our assumption $X_{s^\frown u} \cap f_G(k) = X_{f_G \upharpoonright (k+1)}$ and hence

$$X_{s^\frown\langle\xi\rangle} = X_{s^\frown u} \cap \xi = X_{f_G \upharpoonright (k+1)} \cap \xi.$$

By the preliminary observation this is all we had to show. ▪

4.2. **Arbitrary sets in the generic extension.**

Now we will apply the above to obtain a result about arbitrary sets in generic extensions by $\mathbb{P}_{\mathcal{U}}$ following [GKK10].

Theorem 4.5. *Let $V[G]$ be a generic extension by $\mathbb{P}_{\mathcal{U}}$ where $\mathcal{U}$ is sequence of pairwise distinct normal measures on κ. Then for every set $X \in V[G]$ of ordinals there exists $Y \subseteq \kappa$ such that $X \equiv_V Y$.*

Proof. Let G be a $\mathbb{P}_{\mathcal{U}}$-generic filter over V and $X \in V[G]$ a set of ordinals. We proceed by induction on the least ordinal γ having X as a subset. Since the case $\gamma \leqslant \kappa$ obviously holds, we may assume $\gamma > \kappa$.

Case 1: $\mathrm{cf}(\gamma) \leqslant \kappa$.

If γ is the successor of some ordinal α, then by the assumption $X \cap \alpha$ is constructibly equivalent over V to some subset of κ. But since X equals $(X \cap \alpha) \cup \{\alpha\}$ the same subset of κ works for X.

Let us assume that γ is a limit ordinal. In $V[X]$ fix an increasing cofinal sequence $\langle \gamma_\xi : \xi < \mathrm{cf}(\gamma)\rangle$ of ordinals in γ. For every $\xi < \mathrm{cf}(\gamma)$ consider the set $X \cap \gamma_\xi$ for which there is $Y_\xi \subseteq \kappa$ such that $X \cap \gamma_\xi \equiv_V Y_\xi$ by our assumption. Therefore we may also assume $\langle\, Y_\xi : \xi < \mathrm{cf}(\gamma)\rangle$ in $V[X]$. Then $X \cap \gamma_\xi \in V[Y_\xi]$ and thus there is a set of ordinals $z_\xi \in V$ such that $X \cap \gamma_\xi \in L[z_\xi, Y_\xi]$. Moreover, let α_ξ be an ordinal such that $X \cap \gamma_\xi$ is the α_ξ-th element in $L[z_\xi, Y_\xi]$ regarding $<_{L[z_\xi, Y_\xi]}$. Since $\mathbb{P}_{\mathcal{U}}$ has the κ^+-cc and $\mathrm{cf}(\gamma) < \kappa^+$ we can find sets $Z, A \in V$ which have size κ in V and approximate $\langle\, z_\xi : \xi < \mathrm{cf}(\gamma)\rangle$ and $\langle\, \alpha_\xi : \xi < \mathrm{cf}(\gamma)\rangle$, respectively, i.e., for every $\xi < \mathrm{cf}(\gamma)$ the pair $\langle z_\xi, \alpha_\xi\rangle$ is in $Z \times A$. But then there is a bijection $h : \kappa \to Z \times A$ in V. Furthermore, we denote by $\vartheta_X(\xi)$ the least ordinal ϑ such that the pair $\langle z, \alpha\rangle := h(\vartheta)$ has the property that in $V[X]$

$$X \cap \gamma_\xi \text{ is the } \alpha\text{-th element in } L[z, Y_\xi] \text{ regarding } <_{L[z,Y_\xi]}.$$

The pair $\langle z_\xi, \alpha_\xi\rangle \in Z \times A$ has this property and hence the function $\vartheta_X : \mathrm{cf}(\gamma) \to \kappa$ is well-defined and clearly an element of $V[X]$. Additionally, think of our sequence $\langle Y_\xi : \xi < \mathrm{cf}(\gamma)\rangle$ as $\tilde{Y} := \{\langle\langle \xi, \eta\rangle : \eta \in Y_\xi\} \subseteq \mathrm{cf}(\gamma) \times \kappa$. Now code ϑ_X and $\tilde{Y}$ into a subset Y of κ which lies in $V[X]$ then.
Eventually, ϑ_X, h and $\langle Y_\xi : \xi < \mathrm{cf}(\gamma)\rangle$ are elements of $V[Y]$ and hence so is the sequence $\langle X \cap \gamma_\xi : \xi < \mathrm{cf}(\gamma)\rangle$ and X as its union.

Note that we used nothing more but the fact that $\mathbb{P}$ satisfies the κ^+-cc in V.

Case 2: $\mathrm{cf}(\gamma) > \kappa$.

By the induction hypothesis for every $\xi < \gamma$ we have a subset of κ constructibly equivalent to $X \cap \xi$ over V and we may assume that either $X \cap \xi \in V$ or $X \cap \xi \equiv_V f_G$ by Theorem 4.2.

If there is $\xi < \gamma$ such that $X \cap \xi \equiv_V f_G$, then obviously $X \equiv_V f_G$. The remainder of the proof handles the case $X \cap \xi \in V$ for all $\xi < \gamma$.

Claim. Let $X \cap \xi \in V$ for all $\xi < \gamma$. Then $X \in V$.

Let $\dot{X}$ be a name for X. For every $\xi < \gamma$ we define

$$P_\xi := \{ p \in \mathbb{P}_{\mathcal{U}} \ : \ p \Vdash \dot{X} \cap \check{\xi} = \widecheck{(X \cap \xi)} \}$$

which is non-empty and in V. Note that $\xi \leqslant \zeta$ implies $P_\xi \supseteq P_\zeta$. We distinguish two cases in order to prove $X \in V$.

If $\bigcap_{\xi<\gamma} P_\xi \neq \varnothing$, let $p \in \bigcap_{\xi<\gamma} P_\xi$. Then for every $\alpha < \gamma$ let $\xi < \gamma$ be an ordinal greater than α and obtain

$$\begin{aligned} p \Vdash \check{\alpha} \in \dot{X} \quad &\text{iff} \quad p \Vdash \check{\alpha} \in \dot{X} \cap \check{\xi} \\ &\text{iff} \quad p \Vdash \check{\alpha} \in \widecheck{(X \cap \xi)} \quad \text{iff} \quad \alpha \in X. \end{aligned}$$

Hence X is definable in V.

If $\bigcap_{\xi<\gamma} P_\xi = \varnothing$, then construct a sequence $\langle p_\xi \ : \ \xi < \gamma \rangle$ which has an antichain as subsequence of length greater than κ in $V[X]$. This contradicts our assumption and hence case B cannot arise.

The definition of the sequence makes use of the whole sequence $\langle P_\xi \ : \ \xi < \gamma \rangle$ and therefore the construction takes place in $V[X]$. In case we have $P_{\xi+1} \subsetneq P_\xi$, pick $p \in P_\xi \setminus P_{\xi+1}$ and let $p_\xi \leqslant p$ such that

$$p_\xi \Vdash \dot{X} \cap \widecheck{(\xi+1)} \neq \widecheck{(X \cap (\xi+1))}.$$

Otherwise, let p_ξ be an arbitrary element of $\mathbb{P}_{\mathcal{U}}$. Since the unbounded set $\Xi := \{ \xi < \gamma \ : \ P_{\xi+1} \subsetneq P_\xi \}$ is in $V[X]$, also $\langle p_\xi \ : \ \xi \in \Xi \rangle \in V[X]$. The fact $\mathrm{cf}(\gamma) > \kappa$ tells us that $|\Xi| > \kappa$ and therefore it remains to prove that for distinct $\xi, \zeta \in \Xi$ the conditions p_ξ and p_ζ are incompatible in $\mathbb{P}_{\mathcal{U}}$.

Fortunately, this is an immediate consequence of the construction. If $\xi < \zeta$ are in Ξ, then $p_\zeta \in P_\zeta \subseteq P_{\xi+1}$ wich implies $p_\zeta \Vdash \dot{X} \cap \widecheck{(\xi+1)} = \widecheck{(X \cap (\xi+1))}$ and on the other hand we have $p_\xi \Vdash \dot{X} \cap \widecheck{(\xi+1)} \neq \widecheck{(X \cap (\xi+1))}$.

This part of the proof used only that $\mathbb{P}_{\mathcal{U}}$ satisfies the κ^+-cc in $V[X]$.

All in all we finished the proof of Theorem 4.5. ∎

5. Further Remarks

Let us return to the general setting, where the U_α are just κ-complete non-principal ultrafilters over κ. We explain why this is not sufficient for the minimality and state partial results in this setting.

5.1. **Why we require normality.**

We will now look at a situation in which a generic extension by $\mathbb{P}_{\mathcal{U}}$ (without normality), or more general by any forcing which singularizes a regular cardinal, has many proper intermediate models. For this we force over L^U which denotes the Silver model for the normal measure U. With our definition in Section 2 we have $L^U = L[U]$. Note that in L^U there is a sequence $\langle U_\alpha : \alpha < \kappa\rangle$ of κ-complete non-principal ultrafilters over κ such that there is as sequence of pairwise disjoint sets $\langle A_\alpha : \alpha < \kappa\rangle$ with $A_\alpha \in U_\alpha$, i.e., we only dropped the normality. To see the existence of such a sequence in L^U simply partition κ into κ-many parts of size κ such that the first one is in U. Then use appropriate bijections of κ onto itself to obtain the other ultrafilters as images of U.

On the other hand the Dodd-Jensen Covering Theorem for L^U tells us the following.

Theorem 5.1 (Covering Theorem for L^U for generic extensions of L^U, [DJ82])**.** *Let $L^U[G]$ be a generic extension of the Silver model L^U and let κ be the measurable cardinal in L^U. Then exactly one of following holds*

1. *$L^U[G]$ is covered by L^U, i.e., for every set $X \in L^U[G]$ there is a set Y in L^U such that $X \subseteq Y$ and $|Y| = \max\{|X|, \aleph_1\}$ in $L^U[G]$.*
2. *There exists an ω-sequence $f \in L^U[G]$ cofinal in κ which is a Prikry sequence for the classical Prikry forcing in L^U and $L^U[G]$ is covered by $L^U[f]$.*

Now, let G be $\mathbb{P}_{\mathcal{U}}$-generic over L^U. Then in $L^U[G]$ the Covering Theorem for L^U is false. To see this, note that κ is regular in L^U and moreover κ has cofinality ω in $L^U[G]$.
However, the above theorem now yields an ω-sequence $f \in L^U[G]$ cofinal in κ which is a Prikry sequence for the classical Prikry forcing in L^U. By the Theorem about the intermediate models of generic extensions by the classical Prikry forcing, which was cited in the introduction, $L^U[f] \subseteq L^U[G]$ has a great variety of intermediate models and therefore $L^U[G]$ cannot be minimal.

But this means that the forcing $\mathbb{P}_{\mathcal{U}}$ for the above sequence $\mathcal{U}$ adds a subset of κ which does not correspond to a subsequence of the Prikry sequence. Hence the behavior is much worse than for $\mathbb{P}_{\mathcal{U}}$ with $\mathcal{U}$ consisting of normal measures or the classical Prikry forcing.

5.2. **Partial results without normality.**

Even without normality it is possible to prove that every subsequence of the Prikry sequence constructs the whole Prikry sequence (see Corollary 4.4) and moreover it is possible to reduce the question about intermediate models to subsets of κ. This may be helpful for investigating the classical Prikry tree forcing.

If every subset of κ reduces to a subsequence of the Prikry sequence, then the proof in [GKK10] for showing that under this assumption every set in the generic extension by the classical Prikry forcing reduces to a subsequence of the Prikry sequence almost works. In the second part of this proof we lack in a characterization in the sense of the Mathias criterion for Prikry sequences coming from $\mathbb{P}_{\mathcal{U}}$. However, it is possible to prove some weak analog and together with Proposition 3.17 the proof works. For more details consult [Räs10].

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