# An introduction to stable homotopy theory

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**"Abelian groups up to homotopy"**
spectra $\Longleftrightarrow$generalized cohomology theories

**Examples:**

1. **Ordinary cohomology:**

For $A$ any abelian group, $H^n(X; A) = [X_+, K(A, n)]$.

Eilenberg-Mac Lane spectrum, denoted $HA$.
$HA_n = K(A, n)$ for $n \geq 0$.

The coefficients of the theory are given by
$HA^*(\text{pt}) = \begin{cases} A & * = 0 \\ 0 & * \neq 0 \end{cases}$

2. **Hypercohomology:**
For $C.$ any chain complex of abelian groups,
$\mathbb{H}^s(X; C.) \cong \oplus_{q-p=s} H^p(X; H_q(C.))$.
Just a direct sum of shifted ordinary cohomologies.

$HC.^*(\text{pt}) = H_*(C.)$.

3. **Complex K-theory:**
$K^*(X)$; associated spectrum denoted $K$.

$$K_n = \begin{cases} U & n = \text{odd} \\ BU \times \mathbb{Z} & n = \text{even} \end{cases}$$

$$K^*(\text{pt}) = \begin{cases} 0 & * = \text{odd} \\ \mathbb{Z} & * = \text{even} \end{cases}$$

4. **Stable cohomotopy:**
$\pi_S^*(X)$; associated spectrum denoted $\mathbb{S}$.

$\mathbb{S}_n = S^n$, $\mathbb{S}$ is the *sphere spectrum.*

$\pi_S^*(\text{pt}) = \pi_{-*}^S(\text{pt}) =$ stable homotopy groups of spheres. These are only known in a range.

**“Rings up to homotopy”**
ring spectra $\Longleftrightarrow$gen. coh. theories with a product

1. For $R$ a ring, $HR$ **is a ring spectrum.**
The cup product gives a graded product:
$HR^p(X) \otimes HR^q(X) \to HR^{p+q}(X)$

Induced by $K(R, p) \wedge K(R, q) \to K(R, p + q)$.

2. For $A.$ a differential graded algebra (DGA),
$HA.$ **is a ring spectrum.** Product induced by
$\mu : A. \otimes A. \to A.$, or $A_p \otimes A_q \to A_{p+q}$.

The groups $\mathbb{H}(X; A.)$ are still determined by $H_*(A)$,
but the product structure is *not* determined $H_*(A)$.

3. $K$ **is a ring spectrum;**
Product induced by tensor product of vector bundles.

4. $\mathbb{S}$ **is a commutative ring spectrum.**

**Definition.** A "*ring spectrum*" is a sequence of pointed spaces $R = (R_0, R_1, \cdots, R_n, \cdots)$ with compatibly associative and unital products $R_p \wedge R_q \to R_{p+q}$.

**Definition.** A "*spectrum*" $F$ is a sequence of pointed spaces $(F_0, F_1, \cdots, F_n, \cdots)$ with structure maps $\Sigma F_n \to F_{n+1}$. Equivalently, adjoint maps $F_n \to \Omega F_{n+1}$.

**Example: $\mathbb{S}$ a commutative ring spectrum**

Structure maps: $\Sigma S^n = S^1 \wedge S^n \xrightarrow{\cong} S^{n+1}$.

Product maps: $S^p \wedge S^q \xrightarrow{\cong} S^{p+q}$.

Actually, must be more careful here. For example: $S^1 \wedge S^1 \xrightarrow{\text{twist}} S^1 \wedge S^1$ is a degree $-1$ map.

**History of spectra and $\wedge$**

*Boardman in 1965 defined spectra and $\wedge$.* $\wedge$ is only commutative and associative up to homotopy.

$A_\infty$ *ring spectrum* = best approximation to associative ring spectrum.

$E_\infty$ *ring spectrum* = best approximation to commutative ring spectrum.

*Lewis in 1991: No good $\wedge$ exists.*
Five reasonable axioms $\Longrightarrow$ no such $\wedge$.

*Since 1997, lots of monoidal categories of spectra exist!* (with $\wedge$ that is commutative and associative.)
1. 1997: Elmendorf, Kriz, Mandell, May
2. 2000: Hovey, S., Smith
3, 4 and 5 ... Lydakis, Schwede, ...

**Theorem.**(Mandell, May, Schwede, S. '01; Schwede '01)
All above models define the same homotopy theory.

## Spectral Algebra

Given the good categories of spectra with $\wedge$, one can easily do algebra with spectra.

## Definitions:

A *ring spectrum* is a spectrum $R$ with an associative and unital multiplication $\mu : R \wedge R \to R$ (with unit $\mathbb{S} \to R$).

An *R-module spectrum* is a spectrum $M$ with an associative and unital action $\alpha : R \wedge M \to M$.

$\mathbb{S}$-*modules* are spectra.
$S^1 \wedge F_n \to F_{n+1}$ iterated gives $S^p \wedge F_q \to F_{p+q}$.
Fits together to give $\mathbb{S} \wedge F \to F$.

$\mathbb{S}$-*algebras* are ring spectra.

## Homological Algebra vs. Spectral Algebra

| $\mathbb{Z}$ | $\mathbb{Z}$ (d.g.) | $\mathbb{S}$ |
|---|---|---|
| $\mathbb{Z}$-Mod $= \mathcal{A}b$ | d.g.-Mod $= \mathcal{C}h$ | $\mathbb{S}$-Mod $= \mathcal{S}pectra$ |
| $\mathbb{Z}$-Alg $=$ $\mathcal{R}ings$ | d.g.-Alg $=$ $\mathcal{D}GAs$ | $\mathbb{S}$-Alg $=$ $\mathcal{R}ing\ spectra$ |

| $\mathbb{Z}$ | $\mathbb{Z}$ (d.g.) | H$\mathbb{Z}$ | $\mathbb{S}$ |
|---|---|---|---|
| $\mathbb{Z}$-Mod | d.g.-Mod | H$\mathbb{Z}$-Mod | $\mathbb{S}$-Mod |
| $\mathbb{Z}$-Alg | d.g.-Alg | H $\mathbb{Z}$-Alg | $\mathbb{S}$-Alg |
| $\cong$ | quasi-iso | weak equiv. | weak equiv. |

*Quasi-isomorphisms* are maps which induce isomorphisms in homology.

*Weak equivalences* are maps which induce isomorphisms on the coefficients.

| $\mathbb{Z}$ | $\mathbb{Z}$ (d.g.) | H$\mathbb{Z}$ | $\mathbb{S}$ |
|---|---|---|---|
| $\mathbb{Z}$-Mod | d.g.-Mod | H$\mathbb{Z}$-Mod | $\mathbb{S}$-Mod |
| $\mathbb{Z}$-Alg | d.g.-Alg | H $\mathbb{Z}$-Alg | $\mathbb{S}$-Alg |
| $\cong$ | quasi-iso | weak equiv. | weak equiv. |
| | $\mathcal{D}(\mathbb{Z}) =$ $\mathcal{C}h[\text{q-iso}]^{-1}$ | $\mathcal{H}o(H\mathbb{Z}\text{-Mod})$ | $\mathcal{H}o(\mathcal{S}) =$ $\mathcal{S}pectra[\text{wk.eq.}]^{-1}$ |

**Theorem.** (Robinson ‘87; Schwede-S. ’03; S. ’07) Columns two and three are equivalent up to homotopy.

(1) $\mathcal{D}(\mathbb{Z}) \simeq_\Delta \mathcal{H}o(H\mathbb{Z}\text{-Mod})$.

(2) $\mathcal{C}h \simeq_{\text{Quillen}} H\mathbb{Z}\text{-Mod}$.

(3) Associative $\mathcal{D}GA \simeq_{\text{Quillen}}$ Assoc. $H\mathbb{Z}$-Alg.

(4) For $A.$ a DGA,
d.g. $A.$-Mod $\simeq_{\text{Quillen}} HA.$-Mod
and $\mathcal{D}(A.) \simeq_\Delta \mathcal{H}o(HA.\text{-Mod})$.

## Algebraic Models

**Thm.**(Gabriel)
Let $\mathcal{C}$ be a cocomplete, abelian category with a small projective generator $G$. Let $\mathcal{E}(G) = \mathcal{C}(G, G)$ be the endomorphism ring of $G$. Then

$$\mathcal{C} \cong \text{Mod-}\mathcal{E}(G)$$

Consider $\mathcal{C}(G, -) : \; X \to \mathcal{C}(G, X)$.

## Differential graded categories

**Defn:** $\mathcal{C}$ is a $\mathrm{Ch}_R$-*model category* if it is enriched and tensored over $\mathrm{Ch}_R$ in a way that is compatible with the model structures.

**Example:** differential graded modules over a dga.

Note, $\mathcal{E}(X) = \mathrm{Hom}_{\mathcal{C}}(X, X)$ is a dga.

**Defn:** An object $X$ is *small* in $\mathcal{C}$ if $\oplus[X, A_i] \to [X, \coprod A_i]$ is an isomorphism.

An object $X$ is a *generator* of $\mathcal{C}$ (or $\mathcal{H}o(\mathcal{C})$) if the only localizing subcategory containing $X$ is $\mathcal{H}o(\mathcal{C})$ itself. (A *localizing* subcategory is a triangulated subcategory which is closed under coproducts.)

**Example:** $A$ is a small generator of $A$-Mod.

**Thm:** If $\mathcal{C}$ is a $\mathrm{Ch}_R$-model category with a (cofibrant and fibrant) small generator $G$ then $\mathcal{C}$ is Quillen equivalent to (right) d.g. modules over $\mathcal{E}(G)$.

$$\mathcal{C} \simeq_Q \text{Mod-}\mathcal{E}(G)$$

**Example: Koszul duality**

Consider the graded ring $P_{\mathbb{Q}}[c]$ with $|c| = -2$.
Let tor $P$-Mod be d.g. torsion $P_{\mathbb{Q}}[c]$-modules.

$\mathbb{Q}[0]$ is a small generator of tor $P$-Mod.
Let $\widetilde{Q}$ be a cofibrant and fibrant replacement.

**Corollary:** There are Quillen equivalences:

$$\text{tor}\, P\text{-Mod} \simeq_Q \text{Mod-}\mathcal{E}(\widetilde{Q}) \simeq_Q \text{Mod-}\Lambda_{\mathbb{Q}}[x]$$

$$-\otimes_{\mathcal{E}(\widetilde{Q})}\widetilde{Q} : \text{Mod-}\mathcal{E}(\widetilde{Q}) \rightleftarrows \text{tor}\, P\text{-Mod} : \text{Hom}_{P[c]}(\widetilde{Q}, -)$$

$$\mathcal{E}(\widetilde{Q}) \to \widetilde{Q}$$

$$\mathcal{E}(\widetilde{Q}) \leftarrow \widetilde{Q}$$

Note $\mathcal{E}(\widetilde{Q}) = \text{Hom}_{P[c]}(\widetilde{Q}, \widetilde{Q})$
$\simeq \Lambda_{\mathbb{Q}}[x]$ with $|x| = 1$.

**Corollary:** Extension and restriction of scalars induce another Quillen equivalence:

$$- \otimes_{\mathcal{E}(\widetilde{Q})} \Lambda_{\mathbb{Q}}[x] : \text{Mod-}\mathcal{E}(\widetilde{Q}) \rightleftarrows \text{Mod-}\Lambda_{\mathbb{Q}}[x] : \text{res}.$$

**Spectral model categories**

**Defn:** Let Sp denote a monoidal model category of spectra. $\mathcal{C}$ is a Sp-*model category* if it is compatibly enriched and tensored over Sp. $\mathcal{E}(X) = F_{\mathcal{C}}(X, X)$ is a ring spectrum.

**Thm:** (Schwede-S.) If $\mathcal{C}$ is a Sp-model category with a (cofibrant and fibrant) small generator $G$ then $\mathcal{C}$ is Quillen equivalent to (right) module spectra over $\mathcal{E}(G) = F_{\mathcal{C}}(G, G)$.

$$\mathcal{C} \simeq_Q \text{Mod-}\mathcal{E}(G)$$

$$- \otimes_{\mathcal{E}(G)} G : \text{Mod-}\mathcal{E}(G) \rightleftarrows \mathcal{C} : F_{\mathcal{C}}(G, -)$$

**Rational stable model categories**

**Defn:** A Sp-model category is rational if $[X, Y]_{\mathcal{C}}$ is a rational vector space for all $X, Y$ in $\mathcal{C}$. In this case $\mathcal{E}(X) = F_{\mathcal{C}}(X, X) \simeq H\mathbb{Q} \wedge cF_{\mathcal{C}}(X, X)$.

**Rational spectral algebra $\simeq$ d.g. algebra:**

- There are composite Quillen equivalences
$$\Theta : H\mathbb{Q}\text{-Alg} \rightleftarrows \mathrm{DGA}_{\mathbb{Q}} : \mathbb{H}.$$
- For any $H\mathbb{Q}$-algebra spectrum $B$,
$$\text{Mod-}B \rightleftarrows \text{Mod-}\Theta B.$$

**Thm:** If $\mathcal{C}$ is a rational Sp-model category with a (cofibrant and fibrant) small generator $G$ then there are Quillen equivalences:

$$\mathcal{C} \simeq_Q \text{Mod-}\mathcal{E}(G)$$

$$\simeq_Q \text{Mod-}(H\mathbb{Q} \wedge c\mathcal{E}(G))$$

$$\simeq_Q d.g.\,\text{Mod-}\Theta(H\mathbb{Q} \wedge c\mathcal{E}(G)).$$

$\Theta(H\mathbb{Q} \wedge c\mathcal{E}(G))$ is a rational dga with
$$H_*\Theta(H\mathbb{Q} \wedge c\mathcal{E}(G)) \cong \pi_* c\mathcal{E}(G).$$