Morita theory in enriched context

Kruna Segrt

Laboratoire Jean – Alexandre Dieudonné
Université de Nice – Sophia Antipolis

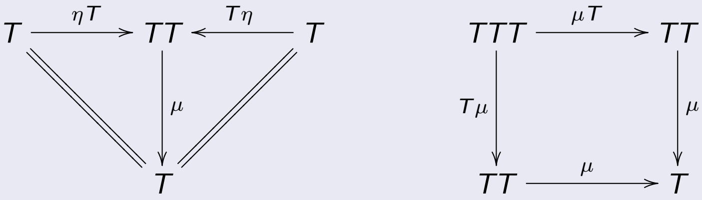
26 June 2010

Category of T-algebras

Monad

Let $\mathcal{C}$ be a category. A monad (T, μ, η) in a category $\mathcal{C}$ consists in giving:

1. A functor $T : \mathcal{C} \longrightarrow \mathcal{C}$;
2. Natural transformations $\eta : Id_{\mathcal{C}} \longrightarrow T$ and $\mu : TT \longrightarrow T$;
3. Axioms given by the commutativity of the following diagrams:



Category of T-algebras

T-algebra

Let $\mathcal{C}$ be a category and (T, μ, η) a monad on $\mathcal{C}$. An algebra on a monad (T, μ, η), written (C, ξ_C), consists in giving:

1. For every object C of $\mathcal{C}$, a functor $\xi_C : TC \longrightarrow C$;
2. Axioms given by the commutativity of the following diagrams:

$$\begin{array}{ccc} TTC & \xrightarrow{\mu_C} & TC \\ {\scriptstyle T(\xi_C)}\downarrow & & \downarrow{\scriptstyle \xi_C} \\ TC & \xrightarrow{\xi_C} & C \end{array} \qquad\qquad \begin{array}{ccc} C & \xrightarrow{\eta_C} & TC \\ & \searrow^{=} & \downarrow{\scriptstyle \xi_C} \\ & & C \end{array}$$

An algebra on a monad (T, μ, η) is also called a T-algebra.

Category of T-algebras

Morphism of T-algebras

Let $\mathcal{C}$ be a category and (T, μ, η) a monad on $\mathcal{C}$. Given two T-algebras (C, ξ_C) and (D, ξ_D) on $\mathcal{C}$, a morphism $f : (C, \xi_C) \longrightarrow (D, \xi_D)$ of T-algebras is a morphism $f : C \longrightarrow D$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{CD} TC @>{T(f)}>> TD \\ @V{\xi_C}VV @VV{\xi_D}V \\ C @>{f}>> D \end{CD}$$

Category of T-algebras

Proposition

Let $\mathcal{C}$ be a category and (T, μ, η) a monad on $\mathcal{C}$. A category of T-algebras, written $\mathbf{Alg}_T$ is such that:

1. A class of objects are T-algebras
2. A set of morphisms are the morphisms of T-algebras

The category $\mathbf{Alg}_T$ is also called the Eilenberg-Moore category of the monad.

Category of T-algebras

Proposition

Let (T, μ, η) be a monad on a category $\mathcal{C}$. Consider the forgetful functor U_T

$$U_T : Alg_T \longrightarrow \mathcal{C}$$

$$(C, \xi_C) \longrightarrow C$$

$$\Big((C, \xi_C) \xrightarrow{f} (D, \xi_D)\Big) \longrightarrow \Big(C \xrightarrow{f} D\Big)$$

Then

1. U_T is faithful;
2. U_T reflects isomorphisms;
3. U_T has a left adjoint F_T given by:

Category of T-algebras

$$F_T : \mathcal{C} \longrightarrow Alg_T$$

$$C \longrightarrow (TC, \mu_C)$$

$$\left(C \xrightarrow{f} C'\right) \longrightarrow \left((TC, \mu_C) \xrightarrow{T(f)} (TC', \mu_{C'})\right)$$

Model categories

Model category

A model category $\mathcal{C}$ consists in giving:

1. A category $\mathcal{C}$
2. Three distinguished classes of maps: weak equivalences, fibrations and cofibrations
   A map which is both a fibration (respectively cofibration) and a weak equivalence is called an acyclic fibration (respectively cofibration).
3. The following axioms

   MC1 Finite limits and colimits exist in $\mathcal{C}$;

   MC2 (2 out of 3) Given maps f and g in $\mathcal{C}$ such that fg is defined and if 2 out of 3 maps $f, g, andgf$ are weak equivalences faibles, then so is the third.

Model categories

Model category

MC3 (Retracts) Given maps f and g in $\mathcal{C}$ such that fg is a retract of g and g is a fibration, a cofibration or a weak equivalence, then so is f.

MC4 (Lifting) Acyclic cofibrations have a left lifting property with respect to fibrations and cofibrations have a right lifting property with respect to acyclic fibrations.

MC5 (Factorization) Any map f in $\mathcal{C}$ can be factored in two ways:

(i) $f = pi$, where i is a cofibration and p is an acyclic fibration
(ii) $f = pi$, where i is an acyclic cofibration and p is a fibration

Model categories

Monoidal model category

A monoidal model category $\mathcal{C}$ is a category which is at once:

1. A closed symmetric monoidal category
2. A closed model category
3. Such that the pushout-product axiom of Hovey is satisfied i.e. for any pair of cofibrations $f : X \to Y$ and $g : X' \to Y'$, the induced map

$$(X \otimes Y') \sqcup_{X \otimes X'} (Y \otimes X') \to Y \otimes Y'$$

is a cofibration.

Model categories

Quillen functor

Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories and $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ an adjoint pair, with F the left adjoint and G the right adjoint. We say that

1. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a left Quillen functor if F preserves cofibrations and acyclic cofibrations.
2. A functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ is a right Quillen functor if G preserves fibrations and acyclic fibrations.

Model categories

Quillen adjunction

Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories and $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ an adjoint pair, with F the left adjoint and G the right adjoint. We say that (F, G) is a Quillen adjunction if F is a left Quillen functor.

Quillen equivalence

Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories and $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ an adjoint pair that defines a Quillen adjunction, with F the left adjoint and G the right adjoint.

We say that F is a Quillen equivalence if for all cofibrant objects X in $\mathcal{C}$ and all fibrant objects Y in $\mathcal{D}$, a morphism $X \to GY$ is a weak equivalence in $\mathcal{C}$ if and only if the adjoint morphism $FX \to Y$ is a weak equivalence in $\mathcal{D}$.

Tensorial strength

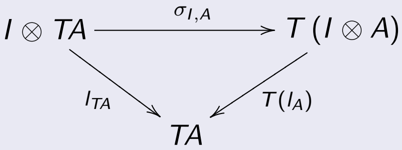
Strong functor

Let $\mathcal{E}$ be a symmetric monoidal closed category. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{E}$-categories tensored over $\mathcal{E}$. A strong functor (T, σ) consists in giving:

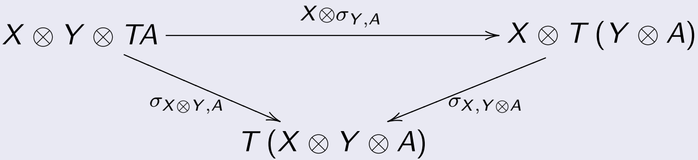
1. A functor $T : \mathcal{A} \longrightarrow \mathcal{B}$;
2. A tensorial strength $\sigma_{X,A} : X \otimes TA \longrightarrow T(X \otimes A)$;
3. Axioms given by the commutativity of the following diagrams:

Tensorial strength

Unit axiom



Associativity axiom



Tensorial strength

Strong natural transformation

Let $\mathcal{E}$ be a symmetric monoidal closed category. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{E}$-categories tensored over $\mathcal{E}$ and let (T_1, σ_1),(T_2, σ_2) be two strong functors such that $T_1, T_2 : \mathcal{A} \longrightarrow \mathcal{B}$.
A strong natural transformation $\Psi : T_1 \longrightarrow T_2$ is given by the following commutatif diagram:

$$\begin{array}{ccc} X \otimes T_1 A & \xrightarrow{\sigma_1} & T_1(X \otimes A) \\ {\scriptstyle X \otimes \Psi_A} \downarrow & & \downarrow {\scriptstyle \Psi_{X \otimes A}} \\ X \otimes T_2 A & \xrightarrow{\sigma_2} & T_2(X \otimes A) \end{array}$$

Tensorial strength

Lemma

Strong functors and strong natural transformations constitute the 1-cells and 2-cells of a 2-category of $\mathcal{E}$-tensored categories, written **CatStrong**.

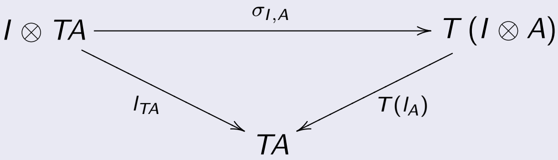
Strong monads

Strong monad

Let $\mathcal{E}$ be a monoidal category. A strong monad (T, μ, η, σ) in a category $\mathcal{E}$ consists in giving:

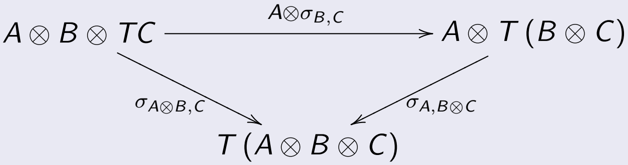
1. A monad (T, μ, η) in a category $\mathcal{E}$;
2. A tensorial strength $\sigma_{A,B} : A \otimes TB \longrightarrow T(A \otimes B)$;
3. Axioms given by the commutativity of the following diagrams:

Unit condition for σ

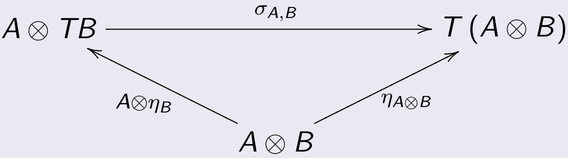


Strong monads

Associativity condition for σ



Strong naturality condition for η



Strong monads

Strong naturality condition for μ

$$\begin{array}{ccccc} A \otimes T^2 B & \xrightarrow{\sigma_{A,TB}} & T(A \otimes TB) & \xrightarrow{T(\sigma_{A,B})} & T^2(A \otimes B) \\ \downarrow{\scriptstyle A \otimes \mu_B} & & & & \downarrow{\scriptstyle \mu_{A \otimes B}} \\ A \otimes TB & & \xrightarrow{\qquad\qquad \sigma_{A,B} \qquad\qquad} & & T(A \otimes B) \end{array}$$

Strong monads

Theorem

A 2-category of strong functors and strong natural transformations of tensored $\mathcal{E}$-categories, called **CatStrong** is 2-isomorphic to a 2-category of $\mathcal{E}$-functors and $\mathcal{E}$-natural transformations of tensored $\mathcal{E}$-categories, called $\mathcal{E}$-**Cat**.

Strong monads

Corollary

Let $\mathcal{C}$ be a monoidal category. Given a monad (T, μ, η) in a category $\mathcal{C}$, the following conditions are equivalent:

1. A monad (T, μ, η) extends to a strong monad (T, μ, η, σ)
2. A monad (T, μ, η) extends to a $\mathcal{E}$-monad (T, μ, η, φ)

Morita theory in enriched context

Theorem

Let $\mathcal{E}$ be a monoidal model category that is cofibrantly generated and with a cofibrant unit. Given a strong monad (T, μ, η, σ) on $\mathcal{E}$, let $\mathcal{A}lg_T$ be a category of T-algebras that admits a model structure.

Consider that (T, μ, η, σ) is such that

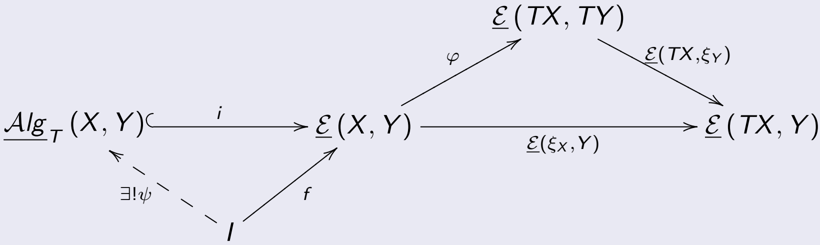
1. The tensorial strength $\sigma_{X,Y} : X \otimes TY \to T(X \otimes Y)$ is a weak equivalence for X,Y cofibrant in $\mathcal{E}$
2. The unit $\eta : I \to TI$ is a cofibration in $\mathcal{E}$

Then the monad morphism $\omega : - \otimes T(I) \to T$ induces a Quillen equivalence $\omega! : \mathcal{A}lg_T \rightleftarrows Mod_{T(I)} : \omega^*$.

Morita theory in enriched context

Proposition

Let $\mathcal{E}$ be a symmetric monoidal closed category with equalizers and (T, μ, η, φ) an enriched monad over $\mathcal{E}$. Then the category $\mathbf{Alg}_T$ of T-algebras is canonically enriched over $\mathcal{E}$. Moreover the $\mathcal{E}$-object $\underline{\mathcal{A}lg}_T(X, Y)$ is given by the equalizer:



Morita theory in enriched context

Proposition

Let $\mathcal{E}$ be a symmetric monoidal closed category with equalizers and (T, μ, η, σ) a strong monad on $\mathcal{E}$. Let $\mathbf{Alg}_T$ be a category of T-algebras with coequalizers. Then $\mathbf{Alg}_T$ is tensored over $\mathcal{E}$ and the tensor is given by the coequalizer:

$$
\begin{array}{ccccc}
T(TX \otimes Z) & \xrightarrow{T(\xi_X \otimes Z)} & T(X \otimes Z) & \xrightarrow{\xi_{X \otimes Z}} & X \otimes Z \\
& {\scriptstyle T\sigma} \searrow & & \nearrow {\scriptstyle \mu} & \\
& & TT(X \otimes Z) & &
\end{array}
$$

Morita theory in enriched context

Proposition

Let (T, μ, η, σ) be a strong monad. Then the object $T(I)$ has a structure of a monoid, namely it may be identified with $\mathcal{A}lg_T(T(I), T(I))$.

In fact, we have $T(I) \cong \mathcal{E}(I, T(I)) \cong Alg_T(T(I), T(I))$

Monoid axiom

Let $\mathcal{E}$ be a symmetric monoidal closed category and M_T a well pointed monoid on $\mathcal{E}$. Then a category Mod_{M_T} of modules over a monoid M_T admits a model structure.

Since the unit of a monad $\eta : I \to T(I)$ is a cofibration, a monoid $T(I)$ is well pointed. Therefore, the category $Mod_{T(I)}$ admits a model structure.

Morita theory in enriched context

We have a commutative diagram of forgetful functors:

$$\begin{array}{ccc} \mathcal{A}lg_T & \xrightarrow{\quad G \quad} & Mod_{T(I)} \\ & {\scriptstyle U_T}\searrow \quad \swarrow{\scriptstyle V} & \\ & \mathcal{E} & \end{array}$$

where $G = \mathcal{A}lg_T\left(T\left(I\right), -\right)$.

Morita theory in enriched context

Fibrations and weak equivalences in $Mod_{T(I)} \Longleftrightarrow$ fibrations and weak equivalences in $\mathcal{E}$.
Fibrations and weak equivalences in $\mathcal{A}lg_T \Longleftrightarrow$ fibrations and weak equivalences in $\mathcal{E}$.
The functor G preserves and even reflects fibrations and weak equivalences. Therefore, G is a right Quillen functor.

Morita theory in enriched context

Lemma

For each monoid M the endofunctor $-\otimes M$ has a canonical structure of a strong monad.

Proposition

For each strong monad (T, μ, η, σ) there is a canonical map of strong monads $-\otimes T(I) \to T$.
This map is an isomorphism if and only if the monad T is induced by a monoid.

Morita theory in enriched context

Since $Mod_{T(I)} = \mathcal{A}lg_{-\otimes T(I)}$,
By the adjoint lifting theorem, the monad morphism $\omega : - \otimes T(I) \to T$ induces the adjunction

$$\omega! : \mathcal{A}lg_T \rightleftarrows \mathcal{A}lg_{-\otimes T(I)} : \omega^*$$

Since the functor $\omega! = G$ preserves and reflects fibrations and weak equivalences, $(\omega^*, \omega!)$ is a Quillen equivalence if and only if for every cofibrant module M the unit of the adjunction is a weak equivalence.

Morita theory in enriched context

Let $X \otimes T(I)$ be a free module.
Then the unit of the adjunction

$$\eta_{adj} : X \otimes T(I) \to TX$$

is a weak equivalence for X cofibrant object.
Using the patching lemma, we extend this to all modules.

Morita theory in enriched context

Exemple

Suppose that $\mathcal{E}$ is a category of pointed simplicial sets.
Then a simplicial (reduced) Γ-ring gives rise to strong monad on pointed simplicial sets. If the underlying simplicial Γ-set is cofibrant in Bousfield-Friedlander sense, then the strong monad satisfies the axiom of our theorem and we recover a result of Stefan Schwede.

Bibliography

Kock, Anders, *Strong functors and monoidal monads*, Arch. Math. (Basel), **23**, (1972), 113–120.

Kock, Anders, *Monads on symmetric monoidal closed categories*, Arch. Math. (Basel), **21**, (1970), 1–10.

Schwede, Stefan, *Stable homotopy of algebraic theories*, Topology, **40**, (2001),1, 1–41.

Lack, Stephen and Street, Ross, *The formal theory of monads. II*, J. Pure Appl. Algebra, **175**, (2002),1-3, 243–265.