DOCTORAL STUDENT SEMINAR ON TMF
WINTER TERM 2012/2013

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One of the fundamental problems in algebraic topology is to calculate the stable homotopy groups of spheres. Though far from completely answering the question, there has been some remarkable recent progress using the chromatic approach.
We know that by the Pontryagin-Thom construction the stable homotopy groups of spheres correspond to framed bordism groups of manifolds, and that the framed bordism ring can be seen as kind of "inverse limit" of a tower

$$\Omega_*^O \longleftarrow \Omega_*^{SO} \longleftarrow \Omega_*^{Spin} \longleftarrow \Omega_*^{String} \longleftarrow \cdots \ \Omega_*^{fr}.$$

The part of the homotopy groups of spheres that is best understood geometrically is the part that is captured by KO-theory. The geometric interpretation relies on the fact that all of the corresponding framed cobordism invariants can be expressed in terms of invariants of *Spin*-cobordism. From the point of view of topology what makes this possible is the factorization

$$S^0 \longrightarrow MSpin \longrightarrow KO$$

of the unit of KO-theory through the Atiyah-Bott-Shapiro map $MSpin \longrightarrow KO$ [Hop02].
An obvious goal now was to find a ring spectrum E such that the unit map factors as

$$S^0 \longrightarrow MString \longrightarrow E.$$

It turned out that it is rather easy to produce maps from $MString$ into a complex oriented cohomology theory E if the formal group associated to E comes from an elliptic curve. The construction of $E = tmf$ is the outcome of the effort to produce a cohomology theory capturing all information emanating from elliptic formal groups. As there is no universal elliptic curve, the spectrum thus constructed cannot be elliptic and not even complex orientable.
Even though there is not yet a geometric interpretation of $tmf^0(X)$ as for example for KO, the homotopy of tmf has a nice interpretation in terms of modular forms. There is a natural map $\pi_{2*}(tmf) \longrightarrow MF_*$, which is not quite an isomorphism on the nose, but after tensoring with $\mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}\right]$. In particular it is this map which gives tmf its name: *topological modular forms*. It is remarkable that the above map has 2- and 3-torsion in its kernel and it is exactly at these primes where tmf approximates the sphere spectrum best.
The composition

$$\pi_{2*}(MString) \longrightarrow \pi_{2*}(tmf) \longrightarrow MF_*$$

reproduces (a refinement of) a genus which the physicist Ed Witten introduced in the late 1980's, and which is nowadays known as the Witten genus.

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Outline of the seminar.

The seminar can roughly be divided into three parts. The first six talks are designed to introduce the necessary notions and give a survey of the construction of TMF. Talks 7-10 deal more explicitly with the construction of the sheaf $\mathcal{O}^{top}$, in particular the $K(1)$-local part. The last four talks cover the descent spectral sequence and some computations in the homotopy of tmf.

We will start by introducing elliptic curves and modular forms. In the first talk we take the complex point of view, introducing both notions in the complex world requiring just some basic complex analysis. In this setting an elliptic curve is topologically just a torus. A modular form will turn out to be a function on the upper half space satisfying some modularity condition.
The second talk then shall generalize the notions of an elliptic curve and of modular forms in a sense appropriate for the rest of the seminar. This is done taking an algebro-geometric point of view, such that some background in the language of schemes or generally in algebraic geometry would be preferable, but is not necessary. The first two talks can easily be managed by novices.
Talk 3 continues the algebraic point of view, introducing formal groups, p-divisible groups and level structures on elliptic curves. We get to know the height of a formal group and shall see, at least for non-singular elliptic curves, what this height has to do with the height of the associated p-divisible group.
In spite of the perhaps scary name (containing the word "stack") talk four can also easily be managed by a non-expert. The main task is to introduce notions and give examples to convey a feeling for stacks.
In the next talk (talk 5) we briefly review the theory of periodic cohomology theories. The main task in this talk is to prove (or at least sketch the proof of) the fact that the map from the moduli stack of elliptic curves to the moduli stack of formal groups is flat. This will allow us to define elliptic cohomology at the end of this talk.
Talk 6 has a survey character in the sense that it should outline the construction of the spectrum TMF. One should therefore start reviewing the important facts of the previous talks, e.g. how an elliptic curve gives rise to a cohomology theory. Explain that what we have achieved so far is the structure of a presheaf $\mathcal{O}^{hom}$ of (co)homology theories on the moduli stack of elliptic curves on the site of affine schemes in the flat topology. What we would like to do is to take global sections, which is not possible since the site has no initial object and the category of (co)homology theories is not complete. State the solution in form of the Hopkins-Miller theorem and define TMF. Then proceed in outlining what we are going to do in the next four talks, which is the construction of the sheaf $\mathcal{O}^{top}$. In what follows we will focus on the $K(1)$-local part of the construction of that sheaf, but it would be great if one could discuss the $K(2)$-local part in this talk, too. This seems a tall order for one talk, but this is why it should have a survey character.
Talk 7 is devoted to the Igusa tower. It is a purely algebraic study of p^k-level structures, which allow to define a ring that later becomes the p-adic K-theory of TMF.
The next talk should be understood as an introduction to θ-algebras and E_∞-ring spectra. The p-adic K-theory of such an E_∞-ring spectrum naturally carries the

structure of a θ-algebra. One should provide examples, state the Goerss-Hopkins theorem, and say something about Andre-Quillen cohomology.
Talk 9 is concerned with the abstract properties of $K(1)$-local elliptic spectra needed for the construction of the $K(1)$-local part of the sheaf $\mathcal{O}^{top}$ to be introduced in talk 10.
In the talks 11 and 12 we shall try to construct the so called descent spectral sequence. As stated before we are primarily interested in the global sections $\mathcal{O}^{top}(\mathcal{M}_{ell})$ of the sheaf $\mathcal{O}^{top}$. Information of the global sections has now to be assembled from local information, i.e. from spectra $\mathcal{O}^{top}(U)$ for open subsets of the moduli stack. The descent spectral sequence will start with sheaf cohomology of the sheafification of this presheaf of local informations and converges to the homotopy groups of TMF.
The seminar concludes with some calculations in the homotopy of tmf.

Notational conventions.

- $\mathcal{M}_{ell}$ will denote the moduli stack of (smooth) elliptic curves over $Spec(\mathbb{Z})$.
- $\overline{\mathcal{M}_{ell}}$ will denote a moduli stack of generalized elliptic curves, where generalized means not necessarily smooth in the sense that we allow the fibers to be curves with a nodal singularity. $\overline{\mathcal{M}_{ell}}$ is the Deligne-Mumford compactification of $\mathcal{M}_{ell}$. In other words $\overline{\mathcal{M}_{ell}} = \mathcal{M}_{Weier} \setminus \{cuspidal\, curves\}$.
- $\mathcal{M}_{Weier}$ will denote the moduli stack of Weierstrass elliptic curves. The subscript "$Weier$" indicates that we allow the fibers to be curves with any kind of singularity (so this includes nodal and cuspidal curves).
- $TMF := \mathcal{O}^{top}(\mathcal{M}_{ell})$. So TMF is the global sections of $\mathcal{O}^{top}$ on the stack of (smooth) elliptic curves. Is sometimes referred to as the periodic version of tmf. The period is $24^2 = 576$.
- $Tmf := \mathcal{O}^{top}(\overline{\mathcal{M}_{ell}})$ is given as the global section of $\mathcal{O}^{top}$ on the (compactified) moduli stack of generalized elliptic curves.
- tmf itself would now morally be $\mathcal{O}^{top}(\mathcal{M}_{Weier})$, which we cannot make sense of, since the map $\mathcal{M}_{Weier} \longrightarrow \mathcal{M}_{FG}$ is representable but not flat. Thus tmf is given as the connective cover of Tmf.

List of talks

Talk 1 - Elliptic curves and modular forms over $\mathbb{C}$ (18.10.)
Define complex elliptic curves and classify isomorphism classes of those by classifying lattices in $\mathbb{C}$. Maybe you can say something about the embedding and elliptic functions. Introduce (the ring of) complex modular forms, mention important examples and tools as e.g. the Eisenstein series, the discriminant and the j-invariant, which will reappear in later talks. Make sure that you emphasize the connection between elliptic curves and modular forms. Finally you can say something on the (rational) Witten genus (rational here means as map $MString_* \otimes \mathbb{Q} \longrightarrow MF_* \otimes \mathbb{Q}$). Good references are given by the appropriate chapters in [FB93] (in german, presumably also available in english), [DS05], [Mil12], appendix B in [AHS01], [HBJ92] and maybe in [BvdGHZ08] or [Sil09]. The best reference for the Witten genus is [HBJ92]

Talk 2 - Generalized elliptic curves and modular forms I (18.10.)
Follow mainly appendix B in [AHS01] to introduce Weierstrass curves over a scheme and define generalized elliptic curves. Discuss singularities, the scheme $Spec(A)$ for $A = \mathbb{Z}[a_1, \ldots, a_6]$ and Weierstrass reparametrizations, i.e. scaling and translations, which turn out to be $\Gamma = A\left[r, s, t, u^{\pm 1}\right]$. Consider the line bundle ω as pullback of the fiber wise cotangent bundle along the identity section. For more motivation see the nice talk [Mau07]. Introduce modular forms as either sections of $\omega^{\otimes n}$ or as invariants under the (r, s, t)-transformations in the u^n-eigenspaces of A. State the presentation of the ring of integral modular forms. This is either in [AHS01] or in [Hop]. If possible say something about the q-expansion of modular forms (maybe as evaluation at the Tate curve). You may present an example of a Weierstrass curve. Prove Proposition $B.9$ of [AHS01].
References here are: Appendix B in [AHS01], [Mau07], [Hen07], ch. III in [Sil09], ch. 2 in [KM85].

Talk 3 - Elliptic curves and formal groups (8.11.)
Introduce formal groups associated to elliptic curves as formal completion at the identity section (cf. 2.8 of [Mei12] or appendix B in [AHS01]). Define the concept of a p-divisible group and introduce level structures on elliptic curves. You might consider the first chapter of [BL10], which concerns p-divisible groups. Say what is the height of a formal group and use this to establish the concepts of ordinary and super singular elliptic curves; you should mention the relative Frobenius here. [Hen07] is a good reference one can follow. Then you can quickly address the moduli problem. [KM85] is the standard reference for level structures and the corresponding moduli problems, but you should look at these nice slides [Sto]. You should cover all of chapter 3 in [Beh12], so state Serre-Tate. For general algebraic geometry see [Har77]

Talk 4 - Sheaves and stacks (8.11.)
In this talk you will present an expository on sheaves on stacks, basically following [Dou07] up to (including) 2.2. Make sure that you introduce flat, étale morphisms (cf. [Har77]) and the concept of an associated stack of [Hop] to produce examples. In the end $\mathcal{M}_{ell}$ and M_{Weier} are the examples we care about. Also discuss the substacks $\mathcal{M}_{ell}^{ord}$, $\mathcal{M}_{ell}^{ss}$ of $\mathcal{M}_{ell}$ as well as $\overline{\mathcal{M}_{ell}}\left[\Delta^{-1}\right]$ and $\overline{\mathcal{M}_{ell}}\left[c_4^{-1}\right]$ of $\overline{\mathcal{M}_{ell}}$.

Talk 5 - Presheaf of cohomology theories on $\mathcal{M}_{ell}$ (22.11.)
Review some basic material on periodic cohomology theories, formal groups and the Landweber exact functor theorem as e.g. in [Hoh07]. You might want to compare to [Hop] or [Rez07] as well. Then proceed using [Hoh07] to prove that the map $M_{ell} \longrightarrow M_{FG}$ is flat. Define elliptic cohomology theory (cf. def. 1.2 in [AHS01]). References are: [Hoh07], [Hop] and [Rez07]

Talk 6 - Construction of TMF (22.11.)
Recall everything done so far. Explain what we want (global sections of the sheaf of cohomology theories on M_{ell}) and why this is not possible (the sheaf takes values in the wrong category). Show up the solution in stating the Hopkins-Miller theorem and explain what it does for us (it upgrades the sheaf to E_∞-rings). The introduction of [Dou07] gives you a good overview, maybe [Lur09], [Goe10] or [Kon12]

are helpful as well. Then give an exposition on how the seminar continues in the next four talks, i.e. the construction of the sheaf $\mathcal{O}^{top}$ of E_∞-rings on $\mathcal{M}_{ell}$. Do this by following the introduction in [Beh12]. In the end cover/sketch as much as you can of the $K(2)$-local case in the construction of $\mathcal{O}^{top}$, since we will focus on the $K(1)$-local case (for more details on Lubin-Tate theory you may also consult [Rez98]).
References are: [Beh12], [Dou07],

Talk 7 - The Igusa tower (6.12.)
Cover chapter 5 in [Beh12]. In Lemma 5.2 ignore the cases $p = 2$ and 3 if you are short on time.

Talk 8 - θ-algebras and E_∞-rings (6.12.)
Give an introduction to the theory of θ-algebras and E_∞-ring spectra, so present definitions and examples. More precisely, define θ-algebras, the θ-structure on p-adic K-theory of an E_∞-ring spectrum and the cohomology of θ-algebras as in [GH], 2.2 and 2.4.3. In the end you should state theorem 7.1 in [Beh12].

Talk 9 - $K(1)$-local elliptic spectra (20.12.)
Cover the parts on *the p-adic K-theory of $K(1)$-local elliptic spectra* and *θ-compatible $K(1)$-local elliptic E_∞-ring spectra* of chapter 6 in [Beh12]. Furthermore cover chapter 7 up to Lemma 7.6.

Talk 10 - Construction of $\mathcal{O}^{top}_{K(1)}$ (20.12.)
Cover the rest of chapter 7 in [Beh12], but ignore the case $p = 2$.

Talk 11 - The descent spectral sequence I (17.1.)
This talk is joint with talk 12. It is recommended to cover homotopy limits and sheaves of spectra as in [Dou07], 2.3, and sheaf cohomology as in 3.1. References are: [Dou07]

Talk 12 - The descent spectral sequence II (17.1.)
This talk is joint with talk 11. You should cover the rest of chapter 3 of [Dou07].
References are: [Dou07]

Talk 13 - Calculations in the homotopy of tmf I (31.1.)
Discuss the relationship between the descent spectral sequence and the Adams–Novikov spectral sequence Tilman Bauer uses (since this is not well-documented in the literature you should consult Lennart Meier about these issues). Then discuss the change-of-rings isomorphism (as in chapter 2 of [Bau08]) and compute $\pi_* tmf$ at primes bigger than 3 as in chapter 4 of [Bau08].
References are: [Bau08], [Rez07], [Kon12]

Talk 14 - Calculations in the homotopy of tmf II (31.1.)
Compute $\pi_* tmf$ at the prime 3. References are: [Bau08], [Rez07]

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