Symmetric Squaring in Bordism

Denise Krempasky

Georg-August University Göttingen

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The aim of the talk

Theorem

Let (X, A) be a topological pair. Then the diagram

$$\begin{array}{ccc} \mathcal{N}_k(X,A) & \xrightarrow{\mu} & H_k(X,A,\mathbb{Z}_2) \\ \downarrow (\cdot)^{\mathsf{s}} & & \downarrow (\cdot)^{\mathsf{s}} \\ \check{\mathcal{N}}_{2k}((X,A)^{\mathsf{s}}) & \xrightarrow[\check{\mu}]{} & \check{H}_{2k}((X,A)^{\mathsf{s}},\mathbb{Z}_2), \end{array}$$

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$$(X, A)^{\mathsf{s}} := ((X \times X/\tau), pr(X \times A \cup A \times X \cup \Delta)),$$

where $pr\colon X \times X \to X \times X/\tau$ *denotes the canonical projection and* $\Delta := \{(x, x) | x \in X\} \subset X \times X$ *denotes the diagonal in* $X \times X$*.*

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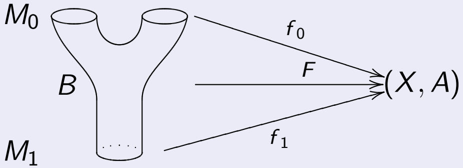
Idea of bordism.

Think of this as solid!



Unoriented singular bordism

bordism - more precisely

The singular manifolds $(M_0, \partial M_0; f_0)$ and $(M_1, \partial M_1; f_1)$ are called bordant iff the disjoint union $(M_0 \sqcup M_1, \partial(M_0 \sqcup M_1); f_0 \sqcup f_1)$ bords.

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"Bordant" is an equivalence relation and the set of equivalence classes is called $\mathcal{N}_n(X, A)$.

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- nice: It's complement is a compact, smooth, bounded manifold.
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- symmetric: It behaves well together with τ.

Čech bordism and homology

Definition

Define

$$\check{\mathcal{N}}_n(X,A)^{\mathbf{s}} := \varprojlim_{U \supset \Delta} \left\{ \mathcal{N}_n(X^{\mathbf{s}}, \overline{pr(X \times A \cup A \times X \cup U)}) \right\}$$

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Let (X, A) *be such that* X *is an ENR and* $A \subset X$ *is an ENR as well.*

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Let (X,A) be such that X is an ENR and $A \subset X$ is an ENR as well. Then

$$\check{\mathcal{N}}_*(X,A) \simeq \mathcal{N}_*(X,A) \text{ and } \check{H}_*(X,A) \simeq H_*(X,A,\mathbb{Z}_2).$$

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where g is a certain Morse function...

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The singular chain map

$$(\cdot)^{\mathsf{s}} \colon C_k(X, A, \mathbb{Z}_2) \to C_{2k}((X, A)^{\mathsf{s}}, \mathbb{Z}_2) \text{ by}$$

$$\sigma = \sum_{i=1}^{n} \sigma_i \mapsto \sigma^{\mathsf{s}} := \sum_{\substack{i<j \\ 1 \leq i,j \leq n}} pr_\sharp(\sigma_i \times \sigma_j),$$

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But what does that mean in the Čech context?

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Then $\sigma_{\mathbf{f}}^{\mathbf{s}} \in \check{H}_{2k}((B, \partial B)^{\mathbf{s}})$

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The aim of the talk

Theorem

Let (X, A) be a topological pair. Then the diagram

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is commutative.

- The vertical arrows are the "symmetric squaring" maps in unoriented bordism and in homology.
- The horizontal arrows represent the canonical map between bordism and homology.

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(\mu([B, \partial B, f]))^{\mathbf{s}} &= \left(H_k\left(f\right)\left(\sigma_{\mathbf{f}}\right)\right)^{\mathbf{s}}, \text{ with fundamental class } \sigma_{\mathbf{f}} \text{ of } (B, \partial B) \\
&= (H_k(f) \sum_i \sigma_i)^{\mathbf{s}}
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&= \check{\mu}\left[B \times B \setminus (\ldots)/\tau, \partial(-), f^{\mathbf{s}}\right], \text{ because } (\sigma_{\mathbf{f}})^{\mathbf{s}} \text{ is the} \\
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Theorem

Let k be even and (X, A) a topological pair. Then the following diagram commutes.

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\check{\Omega}_{2k}((X,A)^{\mathsf{s}}) & \xrightarrow[\check{\mu}]{} & \check{H}_{2k}((X,A)^{\mathsf{s}},\mathbb{Z}),
\end{array}$$

Thank you for your attention!

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