Configuration Spaces of Graphs

Liz Hanbury

University of Durham

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Outline

1. Introduction
2. Describing H_1 and H_2

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Definition of Configuration Space $F(\Gamma, n)$

- Let Γ be a graph.
- $F(\Gamma, n)$ is the collection of all n-tuples of distinct points in Γ.

Definition

$F(\Gamma, n) = \{(x_1, \ldots, x_n) \in \Gamma^n \mid x_i \neq x_j \text{ if } i \neq j\}$

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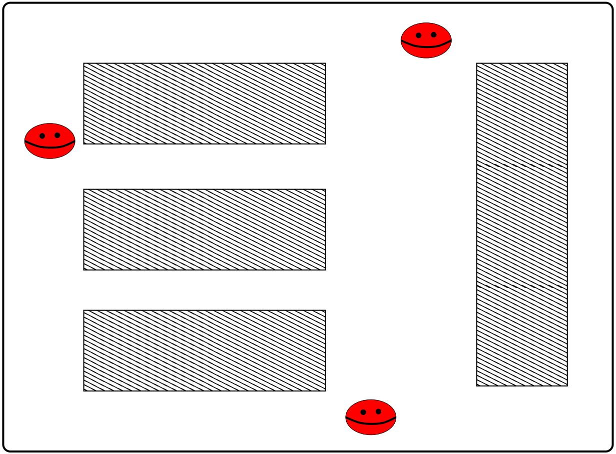

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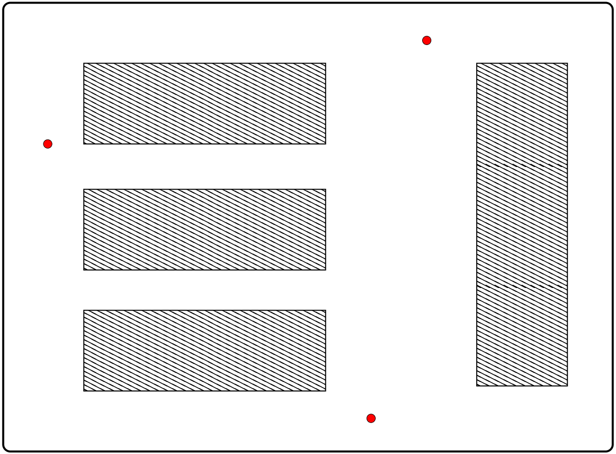

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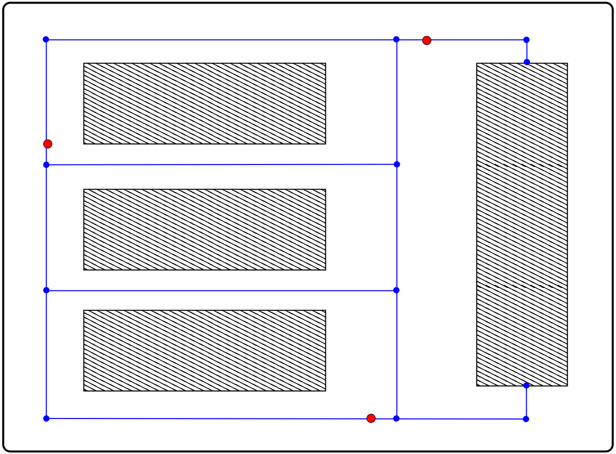

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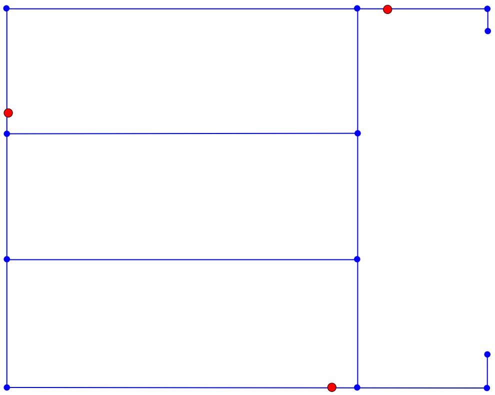

Examples of $F(\Gamma, 2)$

Also:

$$F(K_5, 2) \simeq \Sigma_6 \text{ (Copeland-Patty)}$$

$$F(K_{3,3}, 2) \simeq \Sigma_4 \text{ (Copeland-Patty)}$$

$$F(\text{Tree}, 2) \simeq \bigvee_N S^1 \text{ (Farber)}$$

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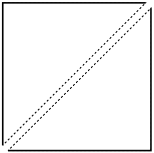

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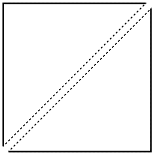

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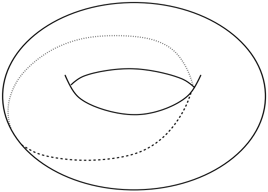

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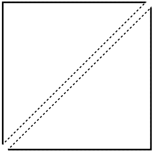

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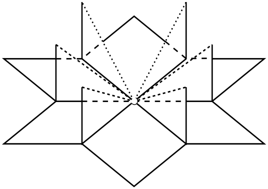

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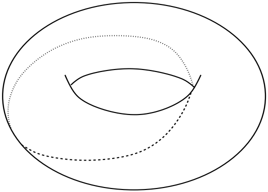

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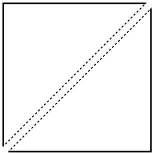

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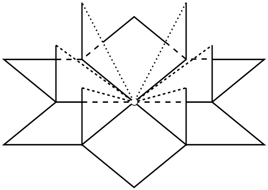

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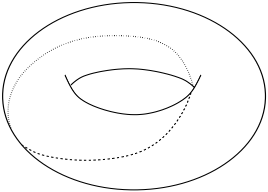

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$F(\Gamma, 2)$ is Usually Path-Connected

- $F([0,1], 2) \simeq 2$ points.
- $F(S^1, 2) \simeq S^1$.
- All other graphs Γ contain a vertex of valence ≥ 3.
- This implies that $F(\Gamma, 2)$ is path-connected.

Conclusion

$F(\Gamma, 2)$ is path-connected provided $\Gamma \not\simeq [0, 1]$.

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The Discrete Configuration Space $D(\Gamma, 2)$

For $x \in \Gamma$, let $\mathrm{supp}(x)$ denote the closure of the simplex containing x.

Definition

$D(\Gamma, 2) = \{(x, y) \in \Gamma \times \Gamma \mid \mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset\}$

- $D(\Gamma, 2)$ is a deformation retract of $F(\Gamma, 2)$.
- $D(\Gamma, 2)$ is a 2-dimensional cell complex.

$$\chi(D(\Gamma, 2)) = \chi(\Gamma)^2 + \chi(\Gamma) - \sum_{v \in V(\Gamma)} (\mu(v) - 1)(\mu(v) - 2).$$

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Our Aim

In summary:

- $F(\Gamma, 2)$ is path-connected for $\Gamma \not\cong [0, 1]$.
- $F(\Gamma, 2)$ is homotopy equivalent to a 2-dimensional cell complex $D(\Gamma, 2)$.
- We know $\chi(F(\Gamma, 2))$.

Our Aim:

- Describe $F(\Gamma, 2)$ for a large class of graphs Γ.
- Do this by studying $H_1(F(\Gamma, 2))$ and $H_2(F(\Gamma, 2))$.

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Long Exact Sequence of $(\Gamma \times \Gamma, F(\Gamma, 2))$

From now on, assume $\Gamma \ncong [0, 1], S^1$.

Lemma (Barnett-Farber)

The map $H_1(F(\Gamma, 2)) \to H_1(\Gamma \times \Gamma)$ is surjective.

- $H_2(F(\Gamma, 2)) \cong \ker \mathcal{I}$.
- $H_1(F(\Gamma, 2)) \cong H_1(\Gamma) \oplus H_1(\Gamma) \oplus \operatorname{coker} \mathcal{I}$.

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Planar Case

Barnett-Farber

- Find a basis for $\ker \mathcal{I} \cong H_2(F(\Gamma, 2))$.

Theorem (Barnett-Farber)

For planar graphs Γ with 'enough edges'

$$b_2(F(\Gamma,2)) = b_1(\Gamma)^2 - b_1(\Gamma) + 2 - \sum_{v \in V(\Gamma)} (\mu(v) - 1)(\mu(v) - 2),$$

$$b_1(F(\Gamma,2)) = 2b_1(\Gamma) + 1.$$

- For these graphs, $\operatorname{coker} \mathcal{I}$ has rank 1.
- For any planar graph, $\operatorname{rank}(\operatorname{coker} \mathcal{I}) \geq 1$.

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Non-planar Case

- For a large class of non-planar graphs, $\operatorname{coker} \mathcal{I} = 0$.
- We call these graphs mature.
- For a mature graph Γ we have

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Key Result

Theorem (H.-Farber)

Let u, v be vertices in Γ that are not joined by an edge. Assume that $\Gamma - \{u, v\}$ connected. Let Γ' be obtained from Γ by attaching an edge at u and v. If Γ is mature, then so is Γ'.

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Examples

H.-Farber

- The complete graph K_n is mature for all $n \geq 5$.

$$b_1(F(K_n, 2)) = (n-1)(n-2),$$

$$b_2(F(K_n, 2)) = \frac{1}{4}n(n-2)(n-3)(n-5)+1,$$

- The bipartite graph $K_{p,q}$ is mature for all $p, q \geq 3$.

$$b_1(F(K_{p,q}, 2)) = 2(p-1)(q-1),$$

$$b_2(F(K_{p,q}, 2)) = (p^2-3p+1)(q^2-3q+1).$$

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Examples
H.-Farber

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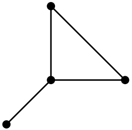
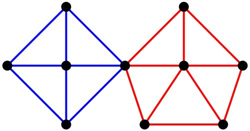
Non-Examples I
H.-Farber

The following imply that Γ is not mature:

1. Γ contains a univalent vertex.

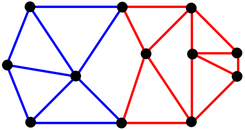


2. Γ decomposes as a wedge of two connected graphs Γ_1 and Γ_2.



Non-Examples II
H.-Farber

3. Γ decomposes as a double wedge of two connected graphs Γ_1 and Γ_2, each different from $[0, 1]$.



Summary

- Configuration spaces of graphs are relevant to motion planning in robotics.
- Barnett-Farber calculated $b_1(F(\Gamma, 2))$ and $b_2(F(\Gamma, 2))$ for a large class of planar graphs Γ.
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Let Γ be a metric graph and $r > 0$.

Definition

$F_r(\Gamma, 2) = \{(x, y) \in \Gamma \times \Gamma \mid d(x, y) \geq 2r\}$

He shows:

1. For $r > \frac{1}{2}\mathrm{diam}(\Gamma)$, $F_r(\Gamma, 2) = \emptyset$.
2. For $r < \epsilon_\Gamma$, $F_r(\Gamma, 2) \simeq F(\Gamma, 2)$.
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