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Some computations on homology of moduli spaces of surfaces

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Factorable normed categories

Let $\mathscr{C}$ be a small category and $\mathscr{M}$ be its set of morphisms. A **norm** on $\mathscr{C}$ is a map $N : \mathscr{M} \to \mathbb{N}$ satisfying

(N1) $N(g) = 0 \iff g$ is an identity morphism
(N2) $N(g_1 \circ g_2) \leq N(g_1) + N(g_2)$, if g_1, g_2 are composable

For example, a normed group can be seen as a normed category with one object and with the group elements as morphisms. Our motivating example is the category of pairings $\mathbf{A}_p$. Its objects are all fixed-point-free involutions λ in the symmetric group $\mathfrak{S}_{2p}$; and for any two objects $\lambda_1, \lambda_2 \in \mathbf{A}_p$, there is exactly one morphism $\tau = \lambda_2\lambda_1^{-1} : \lambda_1 \longrightarrow \lambda_2$. We define the norm $N = N_{\mathbf{A}_p}$ on $\mathbf{A}_p$ to be $N_{\mathbf{A}_p}(\lambda_2\lambda_1) = \frac{1}{2}N_{\mathfrak{S}_{2p}}(\lambda_2\lambda_1)$, where $N_{\mathfrak{S}_{2p}}$ is the word length norm on $\mathfrak{S}_{2p}$ with respect to the set of all transpositions.

Consider the **bar complex** $B_*(\mathscr{C})$ of $\mathscr{C}$. $B_q(\mathscr{C})$ is the free $\mathbb{Z}$-module generated by q-tuples of composable morphisms $\gamma = (g_q, \dots, g_1)$, where the elements g_i are not identity morphisms.
We extend the norm N to generators of $B_*(\mathscr{C})$ by defining $N(\gamma) = \sum_{i=1}^{q} N(g_i)$. The **norm filtration** $\mathcal{F}_h B_*(\mathscr{C})$ is the free $\mathbb{Z}$-submodule of $B_*(\mathscr{C})$ generated by all γ with $N(\gamma) \leq h$. The complex of successive quotients $\mathcal{N}_*(\mathscr{C}; h) = \mathcal{F}_h B_*(\mathscr{C})/\mathcal{F}_{h-1}B_*(\mathscr{C})$ is called the **norm complex at norm** h of $\mathscr{C}$. In other words, it is the h-th column of the E^0-term of the spectral sequence corresponding to the norm filtration. The induced boundary operator ∂', that is, the differential d^0, is given by

$$d_i(g_q, \dots, g_1) = \begin{cases} (g_q, \dots, g_{i+1} \circ g_i, \dots, g_1) & \text{if } N(g_{i+1}\, g_i) = N(g_{i+1}) + N(g_i), \\ 0 & \text{if } N(g_{i+1}\, g_i) < N(g_{i+1}) + N(g_i). \end{cases}$$

To study the properties of $\mathcal{N}_*(\mathscr{C}; h)$, we generalize the concept of factorability for a normed group introduced by B. Visy ([V]) to normed categories: We call $\mathscr{C}$ **factorable** with respect to the norm N, if there is a map $\eta : \mathscr{M} \to \mathscr{M} \times \mathscr{M}$, $g \mapsto \eta(g) =: (\overline{g}, g')$, such that for all $g \in \mathscr{M}$ and $t \in \mathscr{M}$ with $N(t) = 1$

(F1) $N(g') = 1$ (F2) $\overline{g} \circ g' = g$ (F3) $N(\overline{g}) + N(g') = N(g)$

(F4) $N(\eta(g \circ t)) = N(g) + N(t) \iff N((\overline{g} \circ \overline{g' \circ t}, (g' \circ t)')) = N(g) + N(t)$

(F5) $N(\eta(g \circ t)) = N(g) + N(t) \implies (g \circ t)' = (g' \circ t)'$

Following the method from B. Visy on factorable normed groups, we have the following result:

Theorem. *If $\mathscr{C}$ is a factorable category with respect to the norm N, then the homology of the complex $\mathcal{N}_*(\mathscr{C}; h)$ is concentrated in the top degree h: $H_q(\mathcal{N}_*(\mathscr{C}; h)) = 0$, if $q < h$.*

Any factorable group is a factorable category, and any category $\mathscr{C}$ with the *constant norm*–i.e. the norm N with $N(g) = m > 0$ for every $g \neq$ identity–is factorable. Another example is given by the free category $F(Q)$ generated by a quiver Q, with the word length norm with respect to the set of arrows of Q. Our example, the category of pairings, is also factorable. Here for $\tau = \lambda_2\lambda_1^{-1}$,

$$\tau' := (\lambda_1(k), \lambda_1(\tau^{-1}(k)))(k, \tau^{-1}(k))$$

with k being the maximal element not fixed by τ.

Application to moduli spaces

The norm filtrations of $B_*(\mathfrak{S}_p)$ and $B_*(\mathbf{A}_p)$ have important connections with moduli spaces of surfaces and can be used to compute their homology groups. Denote by $\mathfrak{Mod} = \mathfrak{Mod}^m_{g,1}$ (resp. $\mathfrak{N} = \mathfrak{N}^m_{g,1}$) the moduli space of Riemann surfaces (resp. non-orientable (Kleinian) surfaces) of genus $g \geq 0$ with one boundary curve and $m \geq 0$ permutable punctures. Since the relation between $B_*(\mathfrak{S}_p)$ and $\mathfrak{Mod}$ is already better known, here we describe $\mathfrak{N}$ in more detail. Put $h = g + m + 1$ in this case.

Using the Hilbert uniformization method, Bödigheimer (see [E] and [Z]) found a finite bi-complex $\mathrm{NP} = \mathrm{NP}^m_{g,1}$ with a subcomplex NP' such that $\mathrm{NP} \smallsetminus \mathrm{NP}'$ is an open manifold of dimension $3h$ and homotopy equivalent to $\mathfrak{N}$. Surprisingly, the cells of NP are given by q-tuples $\Sigma = (\tau_q, \dots, \tau_1)$ of composable morphisms in the category of pairings $\mathbf{A}_p$, satisfying

(M1) τ_1 is a morphism from λ_0 (M2) $N_{\mathbf{A}_p}(\Sigma) \leq h$ (M3) $N_{\mathfrak{S}_{2p}}(\tau_q \cdots \tau_1 \cdot \lambda_0 \cdot J) \geq 2(p - m - 1)$,

where λ_0 and J denote the pairings $(2p-1, 2p) \cdots (2i-1, 2i) \cdots (1, 2)$ and $(2p-1, 2p-2) \cdots (2i, 2i+1) \cdots (2, 3)(1, 2p)$ respectively. For the parameters p and q we have $0 \leq p \leq 2h$ and $0 \leq q \leq h$. The cells violating equality in (M2) or (M3) correspond to "degenerate" surfaces. Note that NP/NP' is equivalent to the one-point-compactification of $\mathfrak{N}$; furthermore, $\mathfrak{N}$ is non-orientable.

The boundary operator $\partial = \partial' + (-1)^q\partial''$ of NP decomposes into a vertical part ∂' and a horizontal part ∂''. It turns out that ∂' is precisely the boundary operator of the norm complex of the category of pairings. Thus, the p-th column of the E^0-term of the spectral sequence associated to the double complex $\mathbb{NQ}_{\bullet\bullet}$ of cellular chains of $\mathrm{NP}_{\bullet\bullet}/\mathrm{NP}'_{\bullet\bullet}$ is exactly the norm complex $\mathcal{N}(\mathbf{A}_p; h)$.

Using the above theorem on the homology of $\mathcal{N}_*(\mathbf{A}_p; h)$, we conclude that the vertical homology $E^1_{p,q} = H_q(\mathbb{NQ}_{p,\bullet}, \partial')$ is concentrated in the top degree $q = h$; thus the E^1-term is a chain complex with differential induced by ∂'', and the spectral sequence collapses with $E^2 = E^\infty$. Furthermore, parallel to the case of $\mathfrak{Mod}$, under the orientation coefficient system $\mathcal{O}$, we have the following result:

Theorem. *The vertical homology $E^1_{p,q} = H_q(\mathbb{NQ}_{p,\bullet}; \mathcal{O})$ is concentrated in the top degree $q = h$.*

Again, the E^1-term is a chain complex with differential induced by ∂'', and the spectral sequence collapses with $E^2 = E^\infty$.

Let $V^N_p(h) = H_h(\mathbb{NQ}_{p,\bullet}, \partial') = \ker\big(\partial' : \mathcal{N}(\mathbf{A}_p; h) \to \mathcal{N}(\mathbf{A}_p; h-1)\big)$. This is a similar construction as the **Visy complex** for symmetric groups. Condition (M3) selects the correct summand of $\mathbb{NQ}_{\bullet\bullet}$, whose decomposition into summands corresponds to moduli spaces $\mathfrak{N}^m_{g,1}$, one for each (g, m) with given $h = g + m + 1$. The direct summand of $V^N_p(h)$ for a given $m \leq h$ is denoted by $V_\bullet(h, m)$. Due to Poincaré duality the $\mathbb{Z}_2$-homology of $V_\bullet(h, m)$ is the $\mathbb{Z}_2$-cohomology of the moduli space $\mathfrak{N}^m_{g,1}$. To obtain the integral (co)homology of $\mathfrak{N}^m_{g,1}$, we use the orientation coefficient system $\mathcal{O}$.

The chain complex $V_\bullet(h, m)$ is still large, but small enough for computations using the computer.

Homology computations (I)

The tables below show parts of the integral homology of $\mathfrak{M}$ for $h = 6$, and of $\mathfrak{N}$ for $h = 3, 4, 5$ and $\mathfrak{N}^0_{1,1}$. The list of the torsion summands of the forms $\mathbb{Z}_{2^k} (1 \leq k \leq 6)$, $\mathbb{Z}_{3^k} (1 \leq k \leq 4)$, $\mathbb{Z}_{5^k} (k = 1, 2)$, $\mathbb{Z}_7$, $\mathbb{Z}_{11}$, $\mathbb{Z}_{13}$ is complete.
In all tables, $\beta_n(\ell)$ is the n-th mod-ℓ Betti number and $* \in \{11, 13, 17, 19\}$, $\circledast \in \{7, 11, 13, 17, 19\}$, $\star \in \{3, 5, 7, 11, 13, 17, 19\}$—the corresponding Betti numbers are equal.

$\mathfrak{M}$	n	Torsion	$\beta_n(2)$	$\beta_n(3)$	$\beta_n(5)$	$\beta_n(7)$	$\beta_n(*)$
$\mathfrak{M}^0_{3,1}$	0		1	1	1	1	1
	1		0	0	0	0	0
	2	$\mathbb{Z}_2$	2	1	1	1	1
	3	$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_3$, $\mathbb{Z}_7$	4	2	1	2	1
	4	$\mathbb{Z}_2^2$, $\mathbb{Z}_3^2$	4	3	0	1	0
	5	$\mathbb{Z}_2$, $\mathbb{Z}_3$	4	4	1	1	1
	6	$\mathbb{Z}_2^3$	5	2	1	1	1
	7	$\mathbb{Z}_2$	4	0	0	0	0
	8		1	0	0	0	0
	9		1	1	1	1	1
$\mathfrak{M}^2_{2,1}$	0		1	1	1	1	1
	1	$\mathbb{Z}_2^2$, $\mathbb{Z}_5$	2	0	1	0	0
	2	$\mathbb{Z}_2^2$	5	1	2	1	1
	3	$\mathbb{Z}_2^4$	9	3	3	3	3
	4	$\mathbb{Z}_2^5$, $\mathbb{Z}_3^3$	10	4	1	1	1
	5	$\mathbb{Z}_2^4$, $\mathbb{Z}_3$	11	6	2	2	2
	6	$\mathbb{Z}_2^3$	9	3	2	2	2
	7	$\mathbb{Z}_2$	4	0	0	0	0
	8		1	0	0	0	0
$\mathfrak{M}^4_{1,1}$	0		1	1	1	1	1
	1	$\mathbb{Z}_2$	2	1	1	1	1
	2	$\mathbb{Z}_2^3$	4	0	0	0	0
	3	$\mathbb{Z}_2^3$	8	2	2	2	2
	4	$\mathbb{Z}_2^2$	8	3	3	3	3
	5	$\mathbb{Z}_2$	5	2	2	2	2
	6		2	1	1	1	1
$\mathfrak{M}^6_{0,1}$	0		1	1	1	1	1
	1		1	1	1	1	1
	2	$\mathbb{Z}_2$	1	0	0	0	0
	3	$\mathbb{Z}_2$	2	0	0	0	0
	4	$\mathbb{Z}_3$	1	1	0	0	0
	5		0	1	0	0	0

$\mathfrak{N}$	n	Torsion	$\beta_n(2)$	$\beta_n(3)$	$\beta_n(5)$	$\beta_n(\circledast)$
$\mathfrak{N}^0_{4,1}$	0		1	1	1	1
	1	$\mathbb{Z}_2^2$	2	0	0	0
	2	$\mathbb{Z}_2^4$	6	0	0	0
	3	$\mathbb{Z}_2^4$, $\mathbb{Z}_4^3$, $\mathbb{Z}_5$	12	1	2	1
	4	$\mathbb{Z}_2^8$, $\mathbb{Z}_4$, $\mathbb{Z}_3$	17	2	2	1
	5	$\mathbb{Z}_2^8$, $\mathbb{Z}_3$	17	2	0	0
	6	$\mathbb{Z}_2^2$	10	1	0	0
	7		3	1	1	1
$\mathfrak{N}^1_{3,1}$	0		1	1	1	1
	1	$\mathbb{Z}_2^4$	4	0	0	0
	2	$\mathbb{Z}_2^8$	12	0	0	0
	3	$\mathbb{Z}_2^8$, $\mathbb{Z}_4^3$	23	4	4	4
	4	$\mathbb{Z}_2^{11}$, $\mathbb{Z}_4$	29	6	6	6
	5	$\mathbb{Z}_2^8$	23	3	3	3
	6	$\mathbb{Z}_2$	9	0	0	0
	7		1	0	0	0
$\mathfrak{N}^2_{2,1}$	0		1	1	1	1
	1	$\mathbb{Z}_2^4$	4	0	0	0
	2	$\mathbb{Z}_2^7$	11	0	0	0
	3	$\mathbb{Z}_2^7$, $\mathbb{Z}_4$	20	5	5	5
	4	$\mathbb{Z}_2^7$	21	6	6	6
	5	$\mathbb{Z}_2^2$	11	2	2	2
	6		2	0	0	0
$\mathfrak{N}^3_{1,1}$	0		1	1	1	1
	1	$\mathbb{Z}_2^3$	4	1	1	1
	2	$\mathbb{Z}_2^5$	8	0	0	0
	3	$\mathbb{Z}_2^2$	10	3	3	3
	4	$\mathbb{Z}_2$	7	4	4	4
	5		2	1	1	1

Homology computations (II)

$\mathfrak{N}$	n	Torsion	$\beta_n(2)$	$\beta_n(\star)$
$\mathfrak{N}^4_{0,1}$	0		1	1
	1	$\mathbb{Z}_2$	2	1
	2	$\mathbb{Z}_2^2$	3	0
	3	$\mathbb{Z}_4$	3	0
	4		1	0
$\mathfrak{N}^0_{3,1}$	0		1	1
	1	$\mathbb{Z}_2^3$	3	0
	2	$\mathbb{Z}_2^5$	8	0
	3	$\mathbb{Z}_2^3$, $\mathbb{Z}_4^2$	12	2
	4	$\mathbb{Z}_2^2$	9	2
	5	$\mathbb{Z}_2$	4	1
	6		1	0

$\mathfrak{N}$	n	Torsion	$\beta_n(2)$	$\beta_n(\star)$
$\mathfrak{N}^1_{2,1}$	0		1	1
	1	$\mathbb{Z}_2^3$	3	0
	2	$\mathbb{Z}_2^4$	7	0
	3	$\mathbb{Z}_2^2$	9	3
	4	$\mathbb{Z}_2$	5	2
	5		1	0
$\mathfrak{N}^2_{1,1}$	0		1	1
	1	$\mathbb{Z}_2^3$	4	1
	2	$\mathbb{Z}_2^4$	7	0
	3		6	2
	4		2	2
$\mathfrak{N}^3_{0,1}$	0		1	1
	1	$\mathbb{Z}_2$	2	1
	2	$\mathbb{Z}_2$	2	0
	3		1	0

$\mathfrak{N}$	n	Torsion	$\beta_n(2)$	$\beta_n(\star)$
$\mathfrak{N}^0_{2,1}$	0		1	1
	1	$\mathbb{Z}_2^2$	2	0
	2	$\mathbb{Z}_2$	3	0
	3		2	1
$\mathfrak{N}^1_{1,1}$	0		1	1
	1	$\mathbb{Z}_2^2$	3	1
	2	$\mathbb{Z}_2$	3	0
	3		1	0
$\mathfrak{N}^2_{0,1}$	0		1	1
	1	$\mathbb{Z}_2$	2	1
	2		1	0
$\mathfrak{N}^0_{1,1}$	0		1	1
	1	$\mathbb{Z}_2$	2	1
	2		1	0

References

[A] **J. Abhau**, *Die Homologie von Modulräumen Riemannscher Flächen - Berechnungen für $g \leq 2$.* Diplom thesis, Bonn (2005).

[ABE] **J. Abhau, C.-F. Bödigheimer, R. Ehrenfried**, *The homology of the mapping class group $\Gamma_{2,1}$ for surfaces of genus 2 with a boundary curve.* Geometry & Topology Monographs, **14**, (2008), 1-25.

[E] **J. Ebert**, *Hilbert-Uniformisierung Kleinscher Flächen.* Diplom thesis, Bonn (2003).

[V] **B. Visy**, *Homology of Normed Groups and of Graph Complexes, applied to Moduli Spaces.* Ph.D. thesis, in preparation.

[Z] **M. Zaw**, *The moduli space of non-classical directed Klein surfaces.* Math. Proc. Camb. Phil. Soc., **136**, (2004), 599-615.

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