HOMOTOPY & COHOMOLOGY

GK 1150

# Young Women in Topology

**Bonn, June 25 – 27, 2010**

# Strong monads and Morita theory

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## Introduction

A strong functor $T : \mathcal{A} \to \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$, tensored over a symmetric monoidal closed category $\mathcal{E}$, is a functor T equipped with a tensorial strength $\sigma_{X,A} : X \otimes TA \to T(X \otimes A)$ and satisfying some natural unit and associativity axioms. If $\mathcal{A}$ and $\mathcal{B}$ are enriched over $\mathcal{E}$, there is a correspondence between giving a strength on $T$, and giving an enrichment of $T$ over $\mathcal{E}$.

This correspondence extends to a 2-isomorphism between the 2-category of tensored $\mathcal{E}$-categories, $\mathcal{E}$-functors and $\mathcal{E}$-natural transformations, and the 2-category of tensored $\mathcal{E}$-categories, strong functors and strong natural transformations. In particular, we recover Anders Kock's correspondence between enriched monads and strong monads.

Two rings A and B are said to be Morita equivalent, if their module categories $Mod_A$ and $Mod_B$ are equivalent. The equivalences between two module categories always arise from certain bimodules via hom functors and tensor products. By Morita theory we call all results concernig equivalences of different kind of module categories such as derived categories, model categories etc.

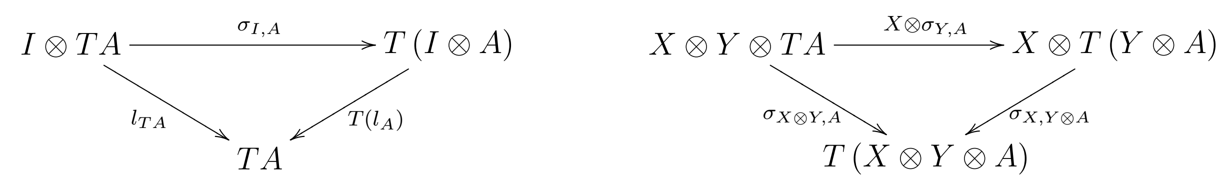
Using the context of a strong monad, categories equivalent to a module category for a monoid in $\mathcal{E}$ can be described intrinsically. We can formulate a homotopy-theoretical generalization using Quillen model categories: we indicate sufficient conditions for a strong monad $T$ on a monoidal model category $\mathcal{E}$ in order that the homotopy category of $T$-algebras be equivalent to the homotopy category of modules over a monoid in $\mathcal{E}$. This result gives us an exemple of Morita theory in enriched contect.

A result of this type has previously been obtained by Stefan Schwede for strong monads on the category of simplicial sets.

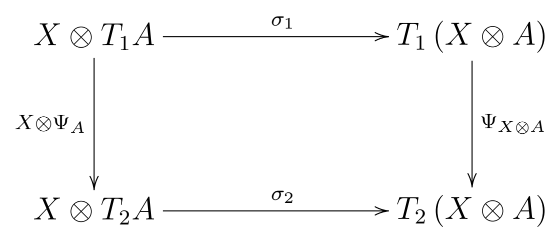
## Strong Functors

**Definition 1** *Let $\mathcal{E}$ be a symmetric monoidal closed category. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{E}$-categories tensored over $\mathcal{E}$. A strong functor $(T, \sigma)$ consists in giving:*

1. *A functor $T : \mathcal{A} \longrightarrow \mathcal{B}$;*
2. *A tensorial strength $\sigma_{X,A} : X \otimes TA \longrightarrow T(X \otimes A)$;*
3. *Axioms given by the commutativity of the following diagrams:*




**Definition 2** *Let $\mathcal{E}$ be a symmetric monoidal closed category. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{E}$-categories tensored over $\mathcal{E}$ and let $(T_1, \sigma_1)$,$(T_2, \sigma_2)$ be two strong functors such that $T_1, T_2 : \mathcal{A} \longrightarrow \mathcal{B}$. A strong natural transformation $\Psi : T_1 \longrightarrow T_2$ is given by the following commutatif diagram:*




**Proposition 1** *Let $\mathcal{E}$ be a symmetric monoidal closed category. Given two categories $\mathcal{A}$ and $\mathcal{B}$ tensored over $\mathcal{E}$ and a functor $T : \mathcal{A} \longrightarrow \mathcal{B}$, the following conditions are equivalent:*

1. *A functor $T$ extends to a strong functor $(T, \sigma)$*
2. *A functor $T$ extends to a $\mathcal{E}$-functor $(T, \varphi)$*

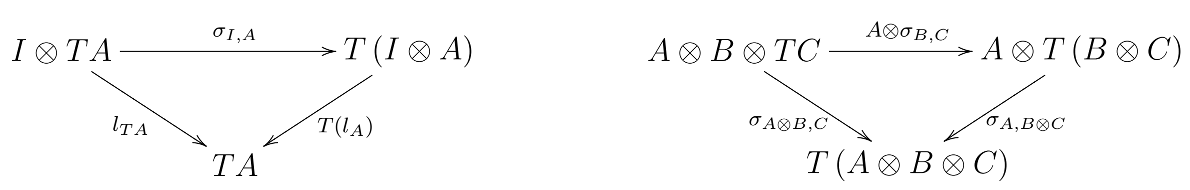
**Proposition 2** *Let $\mathcal{E}$ be a monoidal category. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{E}$-categories and $T_1, T_2 : \mathcal{A} \rightrightarrows \mathcal{B}$ two $\mathcal{E}$-functors. Given a natural transformation $\Psi : T_1 \longrightarrow T_2$, the following conditions are equivalent:*

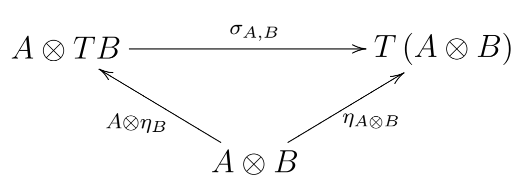
1. *A natural transformation $\Psi$ extends to a strong natural transformation*
2. *A natural transformation $\Psi$ extends to a $\mathcal{E}$-natural transformation*

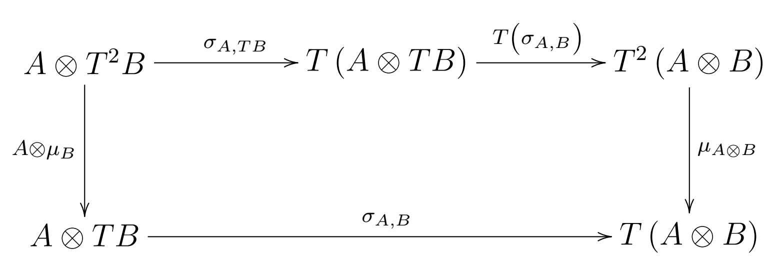
## Strong Monads

**Definition 3** *Let $\mathcal{E}$ be a monoidal category. A strong monad $(T, \mu, \eta, \sigma)$ in a category $\mathcal{E}$ consists in giving:*

1. *A monad $(T, \mu, \eta)$ in a category $\mathcal{E}$;*
2. *A tensorial strength $\sigma_{A,B} : A \otimes TB \longrightarrow T(A \otimes B)$;*
3. *Axioms given by the commutativity of the following diagrams:*




**Lemma 1** *Strong functors and strong natural transformations constitute the 1-cells and 2-cells of a 2-category of $\mathcal{E}$-tensored categories, written **CatStrong**.*

**Theorem 1** *A 2-category of strong functors and strong natural transformations of tensored $\mathcal{E}$-categories, called **CatStrong** is 2-isomorphic to a 2-category of $\mathcal{E}$-functors and $\mathcal{E}$-natural transformations of tensored $\mathcal{E}$-categories, called $\mathcal{E}$-**Cat**.*

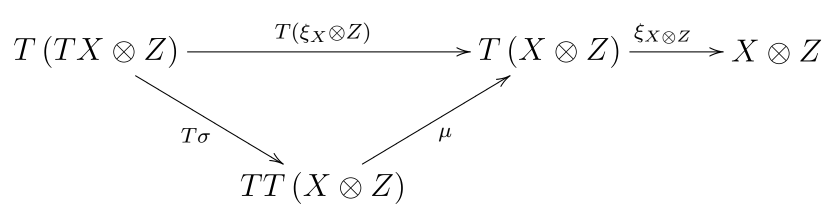
**Corollary 1** *Let $\mathcal{C}$ be a category. Given a monad $(T, \mu, \eta)$ in a category $\mathcal{C}$, the following conditions are equivalent:*

1. *A monad $(T, \mu, \eta)$ extends to a strong monad $(T, \mu, \eta, \sigma)$*
2. *A monad $(T, \mu, \eta)$ extends to a $\mathcal{E}$-monad $(T, \mu, \eta, \varphi)$*

## Enriched Morita Theorem

**Proposition 3** *Let $\mathcal{E}$ be a symmetric monoidal closed category with equalizers and $(T, \mu, \eta, \sigma)$ a strong monad on $\mathcal{E}$. Then the category $\mathbf{Alg}_T$ of $T$-algebras is enriched over $\mathcal{E}$.*

**Lemma 2** *Let $\mathcal{E}$ be a symmetric monoidal closed category with equalizers and $(T, \mu, \eta, \sigma)$ a strong monad on $\mathcal{E}$. Given a $\mathcal{E}$-category $\mathbf{Alg}_T$ with coequalizers, a category $\mathbf{Alg}_T$ has tensors and the tensor is given by the coequalizer:*




**Lemma 3** *Let $\mathcal{E}$ be a symmetric monoidal closed category with equalizers and $(T, \mu, \eta, \sigma)$ a strong monad on $\mathcal{E}$. Given a pair $(F_T, U_T)$ of adjoint functors $F_T : \mathcal{E} \leftrightarrows \mathbf{Alg}_T : U_T$, a functor $U_T$ becomes a $\mathcal{E}$-functor and has a left $\mathcal{E}$-adjoint $U_T$ i.e. a pair $(F_T, U_T)$ is a $\mathcal{E}$-adjunction.*

**Theorem 2** *Let $\mathcal{E}$ be a symmetric monoidal closed category with $I$ a cofibrant unit of $\mathcal{E}$. Let $\mathbf{Alg}_T$ be a category of $T$-algebras with a transferred model structure. Consider a strong monad $(T, \mu, \eta, \sigma)$ where a tensorial strength $\sigma_{X,Y} : X \otimes TY \to T(X \otimes Y)$ is a weak equivalence for $X, Y$ cofibrant objects of $\mathcal{E}$ and such that the unit $\eta : I \to TI$ is a cofibration.*
*Then $\mathbf{Alg}_T \cong \mathbf{Mod}_{M_T}$ is a Quillen equivalence for $M_T = T(I)$ a monoid on $\mathcal{E}$.*

## References


[Kock] **Kock, Anders**, *Strong functors and monoidal monads*, Arch. Math. (Basel), **23**, (1972), 113–120.

[Kock1] **Kock, Anders**, *Monads on symmetric monoidal closed categories*, Arch. Math. (Basel), **21**, (1970), 1–10.

[Schw] **Schwede, Stefan**, *Stable homotopy of algebraic theories*, Topology, **40**, (2001),1, 1–41.

[Lack] **Lack, Stephen and Street, Ross**, *The formal theory of monads. II*, J. Pure Appl. Algebra, **175**, (2002),1-3, 243–265.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