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Assigning a classifying space to a saturated fusion system up to $F-$isomorphism

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Introduction and Definitions

The main problem in the topological side of the theory of $p-$local finite groups introduced by C. Broto, R. Levi and B. Oliver is to approximate the classifying space BG of a finite group G, at least up to $\mathbb{F}_p-$Cohomology via the $p-$local structure of the group G, which means the conjugacy relations of a Sylow $p-$subgroup S of G. The stable solution is due to K. Ragnarsson, see [Ragnarsson]. The aim here is to solve the problem up to $F-$isomorphism in the sense of Quillen.

A $p-$local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a finite $p-$group, $\mathcal{F}$ and $\mathcal{L}$ are finite categories which are modelled on the conjugacy relations in a finite group. The space $|\mathcal{L}|_p^\wedge$ is the classifying space of the $p-$local finite group where $(-)_p^\wedge$ denotes the $p-$completion functor in the sense of Bousfield and Kan [Bousfield-Kan].

Every finite group G gives rise to a $p-$local finite group $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ for each prime dividing the order of G and we obtain $H^*(BG; \mathbb{F}_p) \cong H^*(|\mathcal{L}|_p^\wedge; \mathbb{F}_p)$. However not every fusion system $\mathcal{F}$ is the fusion system of a finite group. This motivated the concept of an exotic fusion system.

Definition 1 *Let S be a finite p-group. A fusion system $\mathcal{F}$ on S is a category whose objects are all the subgroups of S, and which satisfies the following two properties for all $P, Q \leq S$:*

- $Hom_S(P,Q) \subseteq Hom_{\mathcal{F}}(P,Q) \subseteq Inj_S(P,Q)$
- *Each $\phi \in Hom_{\mathcal{F}}(P,Q)$ is the composite of an isomorphism followed by an inclusion.*

Definition 2 *A subgroup $P \leq S$ is called $\mathcal{F}-$centric if P and all subgroups P' which are $\mathcal{F}-$isomorphic to P contain their $S-$centralizers.*

When studying the cohomology ring of a saturated fusion system we will need the following two categories.

Definition 3 *Let $\mathcal{O}(\mathcal{F})$ be the orbit category of $\mathcal{F}$ with objects the same objects as $\mathcal{F}$ and morphisms the set $Mor_{\mathcal{O}(\mathcal{F})}(P,Q) = Mor_{\mathcal{F}}(P,Q)/Inn(Q)$. Define $\mathcal{O}^c(\mathcal{F})$ to be the full subcategory with objects the $\mathcal{F}-$centric subgroups of S.*

Definition 4 *Let $\mathcal{F}$ be a fusion system over a finite $p-$group S. A centric linking system associated to $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}-$centric subgroups of S together with a functor*

$$\pi : \mathcal{L} \to \mathcal{F}^c,$$

and "distinguished" monomorphisms $\delta_P : P \to Aut_{\mathcal{L}}(P)$ for each $\mathcal{F}-$centric subroup $P \leq S$ such that the following conditions are satisfied:

1. *π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects*
2. *For each $\mathcal{F}-$centric subgroup $P \leq S$ and each $x \in P$, $\pi(\delta_P(x)) = c_x \in Aut_{\mathcal{F}}(P)$.*
3. *For each $f \in Mor_{\mathcal{L}}(P,Q)$ and each $x \in P$, $f \circ \delta_P(x) = \delta_Q(\pi(x)) \circ f$.*

Throughout this entire discussion we omit the notion of saturation which is a technical condition modelled on the way a Sylow $p-$subgroup is embedded in a finite group.

Motivation and Further Background

Definition 5 *Let $\mathcal{F}$, $\mathcal{F}'$ be fusion system over finite $p-$group S, S' respectively. A morphism of fusion systems $\mathcal{F} \to \mathcal{F}'$ isa pair (α, Φ) consisting of a group homomorphism $\alpha : S \to S'$ and a covariant functor $\Phi : \mathcal{F} \to \mathcal{F}'$ with the following properties:*

- *for any subgroup Q of S we have $\alpha(Q) = \Phi(Q)$;*
- *for any morphism $\phi : Q \to R$ in $\mathcal{F}$ we have $\Phi(\phi) \circ \alpha|_Q = \alpha_R \circ \phi$.*

Definition 6 *This allows us to define the category of fusion systems over finite $p-$groups: $FUSION(p)$ with*

- *objects: fusion systems over finit $p-$groups and*
- *morphisms: morphisms between corresponding fusion systems.*

Definition 7 *Let G be a discrete group. A finite $p-$group $S \leq G$ is called a Sylow $p-$subgroup of G if all finite $p-$groups of G are subconjugate to S.*

Bemerkungen 1 *Infinite discrete groups need not have Sylow $p-$subgroups: An easy example is $C_p * C_p$.*

Definition 8 *Let p be a prime. Denote by $GROUP_{Syl_p}$ the full subcategory of groups which have a Sylow $p-$subgroup.*

The cohomology of a fusion system is defined as

$$H^*(\mathcal{F}) := \lim_{\mathcal{O}(\mathcal{F})} H^*(-) \cong \lim_{\mathcal{O}^c(\mathcal{F})} H^*(-) \cong H^*(|\mathcal{L}|) \cong H^*(|\mathcal{L}|_p^\wedge).$$

This generalizes the classical Theorem of Cartan and Eilenberg, see [Cartan-Eilenberg], that the cohomology of a finite group is given as the subring of stable elements of the cohomology ring of the Sylow.

Not every fusion system is the fusion system of a finite group: However, in 2007 G. Robinson and I. Leary together with R. Stancu independently constructed groups realising arbitrary fusion systems, see [Leary-Stancu]. Their models are iterated HNN constructions while Robinsons' models are iterated amalgams of finite groups, [Robinson].
Since it was our goal to associate a classifying space to a saturated fusion system, at least up to $\mathbb{F}_p-$Cohomology, it is a natural question to compare the cohomology of the group models realising a given fusion system to the cohomology of the fusion system. This will be done by constructing homology decompositions.

Definition 9 *A ring homomorphism $\gamma : A \to B$ is called an $F-$isomorphism in the sense of Quillen, see [Quillen], if every element in the kernel is nilpotent and for every element $b \in B$ there exist $k > 0$ such that $b^k \in Im(\gamma)$.*

Group Models for Fusion Systems

Theorem 1 *Let $p_1, ..., p_m$ be a collection of different primes, let $S_1, ..., S_m$ be a collection of p_i-groups respectively and $\mathcal{F}_{S_i}$ afusion system over S_i for $i = 1, ..., n$. Then there exists a group G such that $S_i \in Syl_{p_i}(G)$ for $i = 1, ..., n$ and $\mathcal{F}_{S_i}(G) = \mathcal{F}_{S_i}$ for $i = 1, ..., n$.*

Theorem 2 *Let $\mathcal{F}$ be a fusion system over the finite $p-$group S. Let G, G' be groups such that $S \in Syl_p(G)$, $S \in Syl_p(G')$, $\mathcal{F}_S(G) = \mathcal{F}_S(G')$. Let $\mathcal{G} = G \ast_S G'$. Then $S \in Syl_p(\mathcal{G})$ and $\mathcal{F} = \mathcal{F}_S(\mathcal{G})$.*

Theorem 3 *There is a covariant faithful functor $F : FUSION(p) \to GROUP$ which is injective on the set of objects and has the following two properties:*

1. *$\forall\ G \in GROUP_{Syl_p}$, $S \in Syl_p(G)$ we obtain $S \in Syl_p(F(\mathcal{F}_S(G)))$ and $\mathcal{F}_S(G) \cong \mathcal{F}_S(F(\mathcal{F}_S(G)))$*
2. *$\forall\ G, G' \in GROUP_{Syl_p}$, $S \in Syl_p(G)$, $S' \in Syl_p(G')$ the induced morphism $\mathcal{F}_S(G) \to \mathcal{F}_{S'}(G')$ and $\mathcal{F}_S(F(\mathcal{F}_S(G))) \to \mathcal{F}_{S'}(F(\mathcal{F}_{S'}(G')))$ are the same.*
3. *$H^*(B(F(\mathcal{F})); \mathbb{F}_p)$ is $F-$isomorphic in the sense of Quillen to $H^*(\mathcal{F})$ $\forall \mathcal{F} \in FUSION(p)$.*

Homology decompositions are powerful techniques which were developed to approximate a classifying space, at least up to $\mathbb{F}_p-$cohomology as a homotopy colimit of proper subspaces. We construct a homology decomposition to show the following theorem.

Theorem 4 *To every saturated fusion system $\mathcal{F}$ over a finite $p-$group S we can associate a $K(\mathcal{G}, 1)$ such that $S \in Syl_p(\mathcal{G})$, $\mathcal{F}_S(\mathcal{G}) = \mathcal{F}$, $B\mathcal{G}$ is $p-$good, and $H^*(B\mathcal{G})$ is F-isomorphic in the sense of Quillen to $H^*(\mathcal{F})$.*

We finish with some examples of models of Robinson and Leary-Stancu type.

1. Let $G = PSL_2(7)$ be the projective special linear group of rank 2 over the field of 7 elements. Then there exists $\mathcal{G}$ a model of Robinson type associated to the $2-$local finite group of G such that $H^*(B\mathcal{G}; \mathbb{F}_2) \cong H^*(|\mathcal{L}|; \mathbb{F}_2)$.
2. Let p be an odd prime and $(S, \mathcal{F}, \mathcal{L})$ be the $p-$local finite group associated to Σ_{p^2}. Then there does not exist a model of Robinson type associated to $\mathcal{F}$ such that $H^*(B\mathcal{G}; \mathbb{F}_p) \cong H^*(|\mathcal{L}|; \mathbb{F}_p)$.
3. Consider the fusion system over $(\mathbb{Z}/2)^2$ with two automorphisms and the associated model of Leary-Stancu type. Then the classifying space of this model is $p-$bad.

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