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The Homological Stability of Symmetric Groups with Twisted Coefficients

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Definitions & Statements

In 1960, Nakaoka showed the homological stability for symmetric groups. Later Kerz has given a simple proof for Nakaoka's theorem and we generalize it to twisted coefficients.

Theorem 1 *Let V be a coefficient system of degree k(see below). Then the inclusion map $\iota : \Sigma_{n-1} \to \Sigma_n$ induces an isomorphism on the homology for $m < n/2 - k$:*

$$\iota_* : H_m(\Sigma_{n-1}; V(\underline{n-1})) \xrightarrow{\cong} H_m(\Sigma_n; V(\underline{n})). \tag{1}$$

Definition 1 *Let $\mathcal{F}$ be the category of finite sets and injective maps. We define a functor $\mathbf{D} : \mathcal{F} \to \mathcal{F}$ which is a disjoint union with one point. For an arbitrary finite set T, the image $\mathbf{D}(T)$ will be denoted by $\dot{T}$ in order to simplify the notation.*

Definition 2 *We define a coefficient system V to be a functor $V : \mathcal{F} \to Ab$ where Ab is the category of abelian groups. The suspension sV of coefficient system V is defined by $sV = V \circ \mathbf{D}$, i.e., $sV(T) = V(\dot{T})$ for any finite set T.*

For any finite set T the inclusion $T \hookrightarrow \dot{T}$ induces a natural transformation $\mu^V(T) : V(T) \to sV(T)$. We denote the cokernel of the natural transformation μ^V by $\triangle V$.

Similar to van der Kallen [Ka], we introduce the notion of degree inductively.

Definition 3 *If the degree is $\leqslant -1$ then the coefficient system is the zero. We say that V is a coefficient system of degree $\leqslant k$ where $k \geq 0$ if sV is splitting and $\triangle V$ is a system of degree $k-1$. Here splitting means the splitting of the short exact sequence as Σ_T-modules:*

$$0 \to V(T) \to sV(T) \to \triangle V(T) \to 0 \tag{2}$$

Lemma 1 *V is coefficient system with degree $\leqslant 0$ if and only if V is constant, i.e. V is isomorphic to a functor which is given by $T \mapsto A$ on objects and $(S \hookrightarrow T) \mapsto id_A$ on morphisms, where A is a fixed abelian group.*

Example 1 *Let $\mathbb{F}$ be a field and T be an arbitrary finite set. If we define a coefficient system $V(T) := \mathbb{F}(T)$ where $\mathbb{F}(T)$ is the free abelian group generated by the elements of T, then this coefficient system has degree $\leqslant 1$.*

Definition 4 *For a finite set T, let X_T be a semi-simplicial set with all injective maps $\{0, 1, ..p\} \hookrightarrow T$ as p-simplices. We denote the augmented chain complex of X_T by $C_*(T)$.*

- The symmetric group Σ_T acts transitively on the simplices.
- In Σ_T, the stabilizer of a p-simplex σ is Σ_{T_σ} where T_σ is $T - \{\sigma(0), .., \sigma(p)\}$. Therefore one gets an isomorphism:

$$C_p(T) \cong \mathbb{Z}\Sigma_T \otimes_{\mathbb{Z}\Sigma_{T_\sigma}} \mathbb{Z}.$$

Sketch of the theorem part I

Theorem 2 *(Kerz [Ke]) The homology of $C(T)$ vanishes except in degree $|T| - 1$.*

We can extend this complex to an acyclic one, $\widehat{C}(T)$:

$$0 \to ker\ d \to C_{|T|-1}(T) \xrightarrow{d} C_{|T|-2}(T) \to \dots\dots \to C_0(T) \to C_{-1}(T) \to 0$$

For the proof of the Stability Theorem(1) we replace our permutation group Σ_n with permutation group Σ_T of an arbitrary set T, in order to use category theory.

Let $S \subset T$ be objects in $\mathcal{F}$ with $|T| = |S| + 1$. We denote the relative group homology $H_m(\Sigma_T, \Sigma_S; V(T), V(S))$ by $Rel^V_m(T, S)$. Because of the long exact sequence of relative group homology:

$$\dots \longrightarrow Rel^V_{m+1}(T, S) \longrightarrow H_m(\Sigma_S; V(S)) \xrightarrow{\iota_*} H_m(\Sigma_T; V(T)) \longrightarrow Rel^V_m(T, S) \longrightarrow \dots$$

the theorem reduces to prove $Rel^V_m(T, S) = 0$ for $m < |T|/2 - k + 1$.

We prove this by induction on the degree of the coefficient system and homological degree.

Induction beginning: degree -1 is obvious.

Inductive assumption Let V be a coefficient system of degree k_V then:

- $H_m(\Sigma_T, \Sigma_S; V(T), V(S)) = 0$, for $m < |T|/2 - k_v + 1$ with $k_V < k$;
- $H_s(\Sigma_T, \Sigma_S; V(T), V(S)) = 0$, for $s < m$ and $s < |T|/2 - k_V + 1$ with $k_V = k$.

For the sets $\dot{S} \subset \dot{T}$, the inclusion map $\iota : \Sigma_{\dot{S}} \to \Sigma_{\dot{T}}$ leads to a ι- linear map on the coefficient modules $V(\dot{S}) \to V(\dot{T})$. For the semi-simplicial sets $X_{\dot{S}}$ and $X_{\dot{T}}$, we have a bijection between the set of $\Sigma_{\dot{S}}$-orbits of p-simplices in $X_{\dot{S}}$ and the set of $\Sigma_{\dot{T}}$-orbits of p-simplices in $X_{\dot{T}}$ for $p \leqslant |\dot{S}| - 1$. Choose representatives of p-simplices. Then for the acyclic augmented chain complexes: $\widehat{C}_*(\dot{S}), \widehat{C}_*(\dot{T})$, there is a spectral sequence which in our case is:

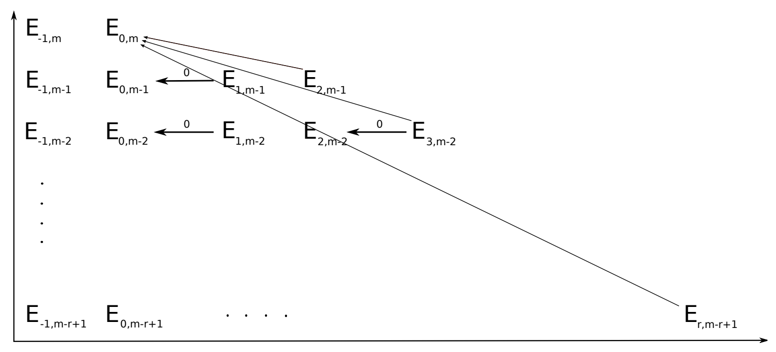
$$E^1_{pq} \cong H_q((\Sigma_{\dot{T}})_{\sigma_p}, (\Sigma_{\dot{S}})_{\sigma_p}; V(\dot{T}), V(\dot{S})) \Longrightarrow H_{p+q}(\Sigma_{\dot{T}}, \Sigma_{\dot{S}}, \widehat{C}(X^{\dot{T}}, V(\dot{T})), \widehat{C}(X^{\dot{S}}, V(\dot{S})) \tag{3}$$

where σ_p is the representative of p-simplices. The spectral sequence converges to zero since $\widehat{C}(\dot{T})$ and $\widehat{C}(\dot{S})$ are acyclic. Also one can compute that the differential $d^1_{pq} : E^1_{p,q} \to E^1_{p-1,q}$ is zero if p is odd.

The stabilizer of a 0-simplex in $\Sigma_{\dot{T}}$; $(\Sigma_{\dot{T}})_{\sigma_0}$ is Σ_T and we take the stabilizer of the (-1)-simplex in a group by itself. Hence the differential $d^1 : E^1_{0,m} \to E^1_{-1,m}$ is

$$H_m(\Sigma_T, \Sigma_S; V(\dot{T}), V(\dot{S})) \longrightarrow H_m(\Sigma_{\dot{T}}, \Sigma_{\dot{S}}; V(\dot{T}), V(\dot{S})).$$

Sketch of the theorem part II



Since the spectral sequence converges to zero, nothing survive to the E^∞-term. Moreover the groups $E^1_{r,m-r+1} = H_{m-r+1}((\Sigma_{\dot{T}})_{\sigma_r}, (\Sigma_{\dot{S}})_{\sigma_r}; V(\dot{T}), V(\dot{S})) \cong Rel^{s^{r+1}V}_{m-r+1}(\dot{T}_{\sigma_r}, \dot{S}_{\sigma_r})$ and $E^1_{r-1,m-r+1}$ are zero by induction hypothesis ($s^{r+1}V$ has the same degree with V). Hence one can easily see that $d^1 : E^1_{0,m} \to E^1_{-1,m}$ is an isomorphism:

Lemma 2 *Suppose the induction assumption holds and V is a coefficient system of degree k. Let $S \subset T$ be objects in $\mathcal{F}$ with $|T| = |S| + 1$. Then for $m < |T|/2 - k + 1$, $Rel^{sV}_m(T, S) \longrightarrow Rel^V_m(\dot{T}, \dot{S})$ is an isomorphism.*

The following lemma follows from a diagram chasing and the fact that the conjugation maps of permutation group induces identity in homology:

Lemma 3 *Consider an inclusion $Q \to R$ in the category $\mathcal{F}$ where $|R| = |Q| + 1$ and let $\dot{Q} = S$, $\dot{R} = T$. If the homomorphisms;*

$$g^{RQ}_m : Rel^V_m(R, Q) \longrightarrow Rel^V_m(T, S)$$
$$g^{TS}_m : Rel^V_m(T, S) \longrightarrow Rel^V_m(\dot{T}, \dot{S})$$

are surjective, then the relative group homology $Rel^V_m(\dot{T}, \dot{S})$ is zero.

Lemma 4 *Suppose the induction assumption holds, then the morphisms g^{RQ}_m and g^{TS}_m are surjective for $m < |T|/2 - k + 1$.*

We have a short exact sequence of coefficient systems; $0 \to V \to sV \to \triangle V \to 0$ which leads to a long exact sequence of relative homology;

$$\dots \to Rel^V_m(T, S) \xrightarrow{\mu_*} Rel^{sV}_m(T, S) \to Rel^{\triangle V}_m(T, S) \to \dots$$

By the induction assumption, the last group is zero: μ_* is surjective. If we compose μ_* with the surjective map $Rel^{sV}_m(T, S) \xrightarrow{d^1} Rel^V_m(\dot{T}, \dot{S})$ in Lemma(2), the composition $g^{TS}_m : Rel^V_m(T, S) \longrightarrow Rel^V_m(\dot{T}, \dot{S})$ will be surjective for $m < |T|/2 - k + 1$. We use a similar approach for g^{RQ}_m.

Proof of the Stability Theorem 1 The natural map $Rel^V_*(T, S) \longrightarrow Rel^{sV}_*(T, S)$ is injective because sV is splitting as in equation(2). Moreover from Lemma(2) for $m < |T|/2 - k + 1$,

$$Rel^{sV}_m(T, S) \longrightarrow Rel^V_m(\dot{T}, \dot{S})$$

is injective as well. Moreover $Rel^V_m(\dot{T}, \dot{S}) = 0$, as a corollary of Lemmas(3) and (4). Therefore considering the composition of those two injective maps, $Rel^V_m(T, S) = 0$.

Examples and Applications

Lemma 5 *Let V and W be coefficient systems of degree k and m, respectively. Then $V \oplus W$ is a coefficient system of degree $\leqslant max(k, m)$ and $V \otimes W$ is a coefficient system of degree $\leqslant k + m$.*

Lemma 6 *Suppose $\mathbb{F}$ is a field and X is a c-connected pointed space and let $V^X_m(T) := H_m(X^T; \mathbb{F})$, $m > 0$ then the degree of V^X_m is $\leqslant m - c$.*

As an application we study the Borel construction $B_n(X) := E\Sigma_n \times_{\Sigma_n} X^n$. There is a Leray-Serre spectral sequence of $B_n(X)$:

$$E^2_{pq} = H_p(\Sigma_n, H_q(X^n; \mathbb{F})) \Longrightarrow H_{p+q}(E\Sigma_n \times_{\Sigma_n} X^n)$$

and we have another one for $B_{n-1}(X)$ as well. Hence we can apply the comparison theorem of spectral sequences and we give a new proof of the following theorem:

Theorem 3 *$H_m(B_{n-1}(X)) \to H_m(B_n(X))$ is an isomorphism if $m < n/2$.*

Furthermore we hope to prove the homological stability of configuration spaces of smooth, connected, noncompact manifolds. Let $F_n(M)$ be the space of ordered configurations of n points. There is a free action of Σ_n on $F_n(M)$, permutes the points. $C_n(M) = F_n(M)/\Sigma_n$, there is a map $C_n(M) \to C_{n+1}(M)$ adding a point "near infinity" that is well-defined up to homotopy. Segal has shown that $C_n(M)$ satisfy homological stability but can one show it via Theorem(1)?

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