EQUIVARIANT CATEGORIES AND FIXED LOCI OF HOLOMORPHIC
SYMPLECTIC VARIETIES

THORSTEN BECKMANN AND GEORG OBERDIECK

Abstract. Given a symplectic action by a finite group on the derived category of a
symplectic surface, we give a criterion for the equivariant category to be equivalent to
the derived category of a symplectic surface. We also describe the fixed loci of moduli
spaces of stable objects in terms of étale covers by moduli spaces of stable objects in
the equivariant category. This yields a general framework for describing fixed loci of
symplectic group actions on moduli spaces of stable objects on symplectic surfaces.
Various examples including the fixed locus of a (birational) involution on an irreducible
symplectic variety of O’Grady-10 type are discussed.

In the appendix we prove that for every distinguished stability condition on a K3
surface S after a GL+(2, R)-shift its heart A satisfies \( D^b(A) \cong D^b(S) \).

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1. Introduction

1.1. Equivariant categories. If a finite group \( G \) acts on a symplectic surface \( S \) and
preserves the symplectic form, then the quotient variety \( S/G \) has isolated ADE singularities
and admits a crepant resolution \( S' \) which is again symplectic. The derived McKay
correspondence [16] provides a natural equivalence between the bounded derived category of
\( G \)-equivariant sheaves on \( S \) and the bounded derived category \( D^b(S') \) of coherent sheaves
on \( S' \). The equivalence categorifies the classical McKay correspondence which relates rep-
resentation theoretic data associated to a \( G \)-action with the geometry of the resolution.

The (bounded) derived category of \( G \)-equivariant coherent sheaves is equivalent to the
equivariant category \( D^b(S)_G \) obtained from the action of \( G \) on \( D^b(S) \) by pullback of sheaves.

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In particular, to state the derived McKay correspondence one only needs to know the action on the derived category and not on the underlying surface. Hence one may ask what happens if we work more generally with an abstract action of a finite group $G$ on the derived category $D^b(S)$. Is the equivariant category, assuming reasonable conditions, again equivalent to the derived category of a symplectic surface $S'$? The study of this question and its applications are the topic of this paper.

Our setup is the following: Let $S$ be a non-singular complex projective surface which is symplectic, hence either a K3 surface or an abelian surface. Let $\sigma \in \text{Stab}^\dagger(S)$ be a Bridgeland stability condition in the distinguished connected component of the space of stability conditions of $D^b(S)$ constructed by Bridgeland [14, 15]. Let $\rho$ be the action of a finite group $G$ on $D^b(S)$ satisfying the following conditions:

(i) For every $g \in G$ the equivalence $\rho_g: D^b(S) \to D^b(S)$ is symplectic.

(ii) The stability condition $\sigma$ is fixed by every $\rho_g$.

(iii) The group $G$ acts faithfully, i.e. the equivariant category is indecomposable.

Here an equivalence is \textit{symplectic} if the induced action on singular cohomology preserves the class of the symplectic form. If we have $\rho_g \not\cong \text{id}$ for all $g \neq 1$, then the action $\rho$ is faithful. Moreover, for any non-faithful action the equivariant category decomposes as an orthogonal sum where each summand is determined by a faithful action on $D^b(S)$. No generality is lost by assuming (iii).

In the case of K3 surfaces, group actions satisfying these conditions have been classified by Gaberdiel, Hohenegger and Volpato [24] and Huybrechts [29] in terms of subgroups of the Conway group. Similar results for abelian surfaces have been obtained by Volpato [59]. In particular, there are many such group actions which do not arise from automorphisms of the surface even after deformation.

Write $\Lambda = H^{2*}(S, \mathbb{Z})$ for the even cohomology lattice and let

$$\Lambda_{\text{alg}} = \Lambda \cap (H^0(S, \mathbb{C}) \oplus H^{1,1}(S, \mathbb{C}) \oplus H^4(S, \mathbb{C}))$$

be its algebraic part. For every $E \in D^b(S)$ we define its Mukai vector by

$$v(E) = \text{ch}(E)\sqrt{\text{td}(S)} \in \Lambda_{\text{alg}}.$$  

The induced $G$-action on cohomology preserves the sublattice $\Lambda_{\text{alg}}$. We write $\Lambda^G_{\text{alg}}$ for the invariant sublattice. Let $M_\sigma(v)$ be the moduli space of $\sigma$-semistable objects with Mukai vector $v$. If $v$ is $G$-invariant, then we have an induced action of $G$ on $M_\sigma(v)$. Let $G^\vee = \text{Hom}(G, \mathbb{C}^*)$ be the group of characters of $G$.

\textbf{Theorem 1.1.} Let $v \in \Lambda^G_{\text{alg}}$ such that $M_\sigma(v)$ is a fine moduli space. If the fixed locus $M_\sigma(v)^G$ has a 2-dimensional $G$-linearizable connected component $F$, then there exists a connected étale cover $S' \to F$ of degree dividing the order of $G^\vee$ and an equivalence

$$D^b(S') \xrightarrow{\cong} D^b(S)_G$$

induced by the restriction of the universal family to $S' \times S$. 
We say here that a connected component of $M_{\sigma}(v)^G$ is $G$-linearizable if for some (or equivalently any) point on it the corresponding $G$-invariant object in $D^b(S)$ admits a $G$-linearization. By work of Ploog [51] the obstruction to finding such a linearization is an element in the second group cohomology $H^2(G, \mathbb{C}^*)$. Hence for groups where this cohomology vanishes, such as cyclic groups, the condition on $F$ to be $G$-linearizable is automatically satisfied.

Recall from [30] that the fine moduli space $M_{\sigma}(v)$ is smooth and inherits a symplectic form from the surface $S$. By assumption (i) the $G$-action preserves this symplectic form. Hence, its fixed locus is smooth and symplectic, so $S'$ is again a symplectic surface. We see that Theorem 1.1 provides the desired equivalence between the equivariant category and the derived category of a symplectic surface.

If the action of $G$ is induced by an action on the underlying surface $S$, then Theorem 1.1 recovers the derived McKay correspondence of [16] by taking the moduli space $\text{Hilb}^{|G|}(S)$ (the component $F$ is the closure of the locus of free orbits).

We state a version of Theorem 1.1 where we drop the condition on the moduli space to be fine. This is useful since not every group action on $D^b(S)$ induces an action on a fine moduli space.

**Definition 1.2.** Let $Z$ denote the central charge of $\sigma$. A vector $v \in \Lambda^G_{\text{alg}}$ is $(G, \sigma)$-generic if it is primitive and for every splitting $v = v_0 + v_1$ with $v_0, v_1 \in \Lambda^G_{\text{alg}} \setminus Z(v)$ the values $Z(v_0)$ and $Z(v_1)$ have different slopes.

Given any primitive vector $v \in \Lambda^G_{\text{alg}}$, one can show that after a small deformation of $\sigma$ along $G$-fixed stability conditions the class $v$ becomes $(G, \sigma)$-generic.

Let also $M_{\sigma}(v)$ denote the moduli stack of $\sigma$-semistable objects in class $v$.

**Theorem 1.3.** Let $v \in \Lambda^G_{\text{alg}}$ be $(G, \sigma)$-generic.

(a) The fixed stack $M_{\sigma}(v)^G$ has a good moduli space $\pi: M_{\sigma}(v)^G \to N$ which is smooth, symplectic and proper. The map $\pi$ is a $\mathbb{G}_m$-gerbe.

(b) If $N$ has a 2-dimensional connected component $S'$, then the restriction of the universal family induces an equivalence

$$D^b(S', \alpha) \cong D^b(S)^G$$

where $\alpha \in \text{Br}(S')$ is the Brauer class of the gerbe.

Here we let $D^b(S', \alpha)$ denote the derived category of $\alpha$-twisted coherent sheaves on $S'$. The notion of a good moduli space was introduced in [2]. The fixed stack is taken in the categorical sense of Romagny [52], see Section 3.1.

For the proof we use Orlov’s result on Fourier–Mukai functors [47] to construct an action of $G$ on the stack $\mathfrak{M}$ of universally gluable objects in $D^b(S)$ in the sense of Lieblich [36]. The fixed stack $\mathfrak{M}^G$ is precisely the stack of objects in the equivariant category $D^b(S)^G$. By transferring geometric properties from $\mathfrak{M}$ to its fixed stack, this yields a well-behaved moduli theory for objects in the equivariant category. The restriction of the universal family of $\mathfrak{M}^G$ to components, which are 2-dimensional and parametrize stable objects, then leads to a Fourier–Mukai kernel which induces the desired equivalence. The additional claims of Theorem 1.1 follow by a detailed analysis of the fixed stack of a trivial $\mathbb{G}_m$-gerbe.
1.2. Fixed loci. After having seen how fixed loci determine the equivariant category, we describe how conversely the equivariant category controls the fixed loci of moduli spaces of stable objects.

Consider an action of a finite group $G$ on $D^b(S)$ which satisfies conditions (i) and (ii), but not necessarily (iii). Assume that we have an equivalence

$$D^b(S', \alpha) \cong D^b(S)_G.$$ 

The surface $S'$ here is necessarily symplectic but can be disconnected since the action is not required to be faithful. Let

$$P: H^{2*}(S', \mathbb{Z}) \to H^{2*}(S, \mathbb{Z})$$

be the map induced from the composition $D^b(S', \alpha) \to D^b(S)_G \to D^b(S)$ where the latter map is the forgetful functor. Given an element $v \in \Lambda_{\text{alg}}^G$ we write

$$R_v = \{ v' \in \Lambda_{(S', \alpha),\text{alg}} | P(v') = v \}$$

where the algebraic part $\Lambda_{(S', \alpha),\text{alg}}$ of the lattice $H^{2*}(S', \mathbb{Z})$ is defined by the Hodge structure associated to the Brauer class $\alpha$.

By results of Macrì, Mehrotra, and Stellari the $G$-invariant stability condition $\sigma$ induces a stability condition, denoted $\sigma_G$, on $D^b(S)_G$ and hence on $D^b(S', \alpha)$. We write $M_{\sigma_G}(v')$ for the good moduli space of the stack $M_{\sigma_G}(v')$.

**Theorem 1.4.** Let $v \in \Lambda_{\text{alg}}^G$ such that $M_{\sigma}(v)$ is a moduli space of stable objects. Then there exists a degree $|G^V|$ étale morphism

$$\bigsqcup_{v' \in R_v} M_{\sigma_G}(v') \to M_{\sigma}(v)^G$$

whose image is the union of all $G$-linearizable connected components of $M_{\sigma}(v)^G$.

If $G$ is cyclic, or more generally, if the $G$-action on $D^b(S)$ factors through the action of a quotient $G \to Q$, such that $G$ is a Schur covering group of $Q$, then (1.1) is surjective.

We refer to Section 3.6 for a more general version of Theorem 1.4 which applies to any variety with a suitable stability condition and where we do not require the equivariant category to be equivalent to the derived category of some variety.

For general actions on $D^b(S)$ the map (1.1) may not be surjective. This issue is resolved by choosing a Schur covering group $\tilde{G} \to G$ which by definition is a maximal stem extension of $G$. It has the property that the restriction map

$$H^2(G, \mathbb{C}^*) \to H^2(\tilde{G}, \mathbb{C}^*)$$

vanishes, and so any $G$-invariant object becomes linearizable with respect to $\tilde{G}$. Hence, if we let $\tilde{G}$ act on $D^b(S)$ through $G$ and we take the equivariant category with respect to $\tilde{G}$, then (1.1) becomes surjective. This explains the second claim of the Theorem 1.4.

---

1. A basic example is given by the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ of 2-torsion points of an elliptic curve acting by translation: Every point in the fine moduli space $M$ of degree 2 line bundles is $G$-invariant (hence $M_G = M$), but none of them is $G$-linearizable, so the left hand side of (1.1) is the empty set, see also Remark 3.14.

2. An extension of groups $1 \to K \to E \to G \to 1$ is stem if $K$ is contained both in the commutator subgroup and the center of $E$. 

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With respect to stability conditions in the distinguished component moduli spaces of (twisted) stable objects on K3 surfaces are well understood (see [9] and the references therein): they are smooth, irreducible, and non-empty if and only if the Mukai vector has square at least \(-2\). The case of abelian surfaces is similar. Hence if the induced stability condition \(\sigma_G\) is distinguished on each component of \(S'\), then Theorem 1.4 completely describes the fixed locus \(M_G\) up to \(\acute{e}tale\) cover.

A map similar to (1.1) for the Enriques involution on K3 surfaces was used by Nuer to study the moduli space of stable objects on an Enriques surface [44].

If \(S'\) is a K3 surface and the equivalence is geometric, we can be more precise with our description of the fixed locus. The group \(G\) acts on the equivariant category \(\mathcal{D}_{G}(S')\) by twisting the linearization, see Section 2.1. The action induces an action on cohomology. Let

\[
\mathcal{R}_v \subset \Lambda(S', \alpha),_{alg}
\]

be a set of representatives of the coset \(R_v/G\).

**Theorem 1.5.** Let \(v \in \Lambda^G_{alg}\) such that \(M_\sigma(v)\) is a moduli space of stable objects. Suppose that \(G\) is cyclic and that we have an equivalence \(\mathcal{D}_b(S', \alpha) \rightarrow \mathcal{D}_b(S)_G\) for a K3 surface \(S'\) which is induced from a universal family as in Theorem 1.1 or Theorem 1.3. Then the induced stability condition \(\sigma_G\) lies in \(Stab^1(S')\) and we have an isomorphism

\[
M_{\sigma}(v)^G \cong \bigsqcup_{v' \in \mathcal{R}_v} M_{\sigma_G}(v').
\]

By combining work of Mongardi [41], Huybrechts [29] and Bayer–Macrì [9] we finally remark that symplectic actions of finite groups on moduli spaces of stable objects on K3 surfaces are always induced by actions on the derived category as considered above. Hence Theorems 1.4 and 1.5 in combination with Theorem 1.1 provide an effective method to determine the fixed locus of any such action.

**Proposition 1.6.** Let \(S\) be a K3 surface and let \(\sigma' \in Stab^1(S)\) be a stability condition. Let \(G\) be a finite group which acts faithfully and symplectically on a moduli space \(M\) of \(\sigma'\)-stable objects. Then the following holds:

(a) There exists a surjection \(G' \rightarrow G\) from a finite group \(G'\) and an action of \(G'\) on \(\mathcal{D}_b(S)\) which satisfies the conditions (i), (ii) of Section 1.1 (for some stability condition \(\sigma \in Stab^1(S)\)), and induces the given \(G\)-action on \(M\).

(b) If \(G\) is cyclic, then we can take \(G' = G\) in part (a).

1.3. Related work. Examples of symplectic group actions on the derived category of symplectic surfaces, in particular those which do not arise from symplectic automorphisms of the surface, can be obtained from two separate sources.

The first is the study of symplectic automorphisms of irreducible holomorphic symplectic varieties deformation equivalent to a moduli space of sheaves on a K3 surface or a generalized Kummer variety. For these varieties it has long been known that not every symplectic automorphism arises from an automorphism of the underlying surface, see [12] and the references therein. This is most evident for automorphisms of order 11, since every finite order
symplectic automorphism of a K3 surface has order at most 8, but many other examples are known, see [10, Sec. 4]. The classification of such automorphism groups, and finding geometric realizations and fixed loci are an active topic of research, see e.g. [41, 54].

Another rich source of examples is string theory. In physics the pair \((S, \sigma)\) of a symplectic surface and a distinguished stability condition corresponds to a non-singular sigma model on \(S\). Symplectic actions as we have considered above correspond to supersymmetry-preserving discrete symmetries. The equivariant categories are the orbifold sigma models. Physics predicts that the orbifold models should be again either K3 or torus (i.e. abelian surface) models. Relations to counting BPS states/dyons (see also Section 1.6 below) and to moonshine for Conway and other groups play a key role [48]. We do not venture further in this direction here, but only note that a complete classification of symplectic actions satisfying (i,ii,iii) has been obtained [24, 29, 59] by lattice methods. As has been observed by both Huybrechts [29] and Mongardi [41], not every of these symmetries does act on a smooth moduli space of sheaves of a K3 surface. Hence there are examples which can not be seen as automorphisms on holomorphic symplectic manifolds.

1.4. Examples. In order to illustrate our methods and the classification, let us consider some examples. We restrict ourselves to cyclic groups \(\mathbb{Z}_n\) acting on the derived category of a K3 surface. Given a variety \(X\) and an element \(g \in \text{Aut} H^*(X, \mathbb{C})\) of finite order \(n\) we define the frameshape of \(g\) as the formal symbol

\[
\pi_g = \prod_{a | n} a^{m(a)}
\]

that encodes the characteristic polynomial of \(g\) via

\[
\det(t \cdot \text{id} - g) = \prod_{a | n} (t^a - 1)^{m(a)}.
\]

Symplectic auto-equivalences of K3 surfaces of finite order preserving a stability condition are classified in terms of their frameshapes. It was shown in [22] that there are 42 frameshapes and at most 82 \(O_+(\Lambda)\) conjugacy classes which can occur. Their invariant lattices can be found in [50, App. C]. In order 2 there are three cases

\[
1^8 2^8, \quad 1^{-8} 2^{16}, \quad 2^{12}
\]

each in a unique conjugacy class. The case \(1^8 2^8\) corresponds to symplectic involutions of K3 surfaces, while the others are of derived nature. We shortly discuss one example in each class and describe the associated equivariant category. We refer to Section 7 for additional examples and further details.

1.4.1. Frameshape \(1^8 2^8\). Let \(S \to \mathbb{P}^2\) a K3 surface obtained as the double cover of the plane branched along a sextic curve, and let \(g: S \to S\) be a symplectic involution which fixes the hyperplane class \(H \in \text{Pic}(S)\). The derived McKay correspondence [10] (or Theorem 1.1) yields an equivalence \(D^b(S)_{\mathbb{Z}_2} \cong D^b(S')\) where \(S'\) is the symplectic resolution of \(S/\mathbb{Z}_2\). Theorem 1.5 then immediately yields the following description of the fixed locus of the moduli space \(M(0, H, 0)\) of Gieseker stable sheaves:

\[
M(0, H, 0)^G = (1 \text{ K3 surface}) \sqcup (28 \text{ points}).
\]

This matches perfectly the results of [24].
More interestingly, consider the singular moduli space \( M(0, 2H, 0) \) which admits an irreducible holomorphic symplectic resolution \( X \) of O’Grady type \([45, 4]\). The symplectic involution \( g \) lifts to a birational symplectic involution \( \tilde{g} \) of O’Grady \( \text{type} \) \([45, 4]\). Because \( g \) is only birational, the closure of the fixed locus of \( g \) does not need to be symplectic (and here it is not). Our methods yield the following:

**Proposition 1.7.** The closure of the fixed locus of the birational symplectic involution \( g : X \dashrightarrow X \) is smooth and the disjoint union of one connected component of dimension 6 containing 120 copies of \( \mathbb{P}^5 \), and 119 \( K3 \) surfaces of which 88 are derived equivalent to \( S' \).

1.4.2. **Frameshape** \( 1^{-8}\times 16 \). Let \( \text{Kum}(A) \) be the Kummer K3 surface of an abelian surface \( A \). The derived McKay correspondence \([16]\) provides an equivalence

\[
D^b(A)_{\mathbb{Z}_2} \cong D^b(\text{Kum}(A))
\]

where the group \( \mathbb{Z}_2 \) acts on \( A \) via multiplication with \(-1\). The action of the non-trivial character of \( \mathbb{Z}_2 \) defines a symplectic involution of frameshape \( 1^{-8}\times 16 \),

\[
Q : D^b(\text{Kum}(A)) \to D^b(\text{Kum}(A)),
\]

see also Section 7.1 for an explicit formula for \( Q \). Using Theorem 1.3 one finds that

\[
D^b(\text{Kum}(A))_{\mathbb{Z}_2} \cong D^b(A).
\]

1.4.3. **Frameshape** \( 2^{12} \). Let \( \tau : S \to S \) be a symplectic automorphism of a K3 surface of order 4 and let \( S' \) be the resolution of the quotient \( S/\langle \tau^2 \rangle \). Since we quotient out only by \( \tau^2 \), we have a residual involution

\[
\bar{\tau} : S' \to S'.
\]

As before, the McKay correspondence \( D^b(S') \cong D^b(S)_{\mathbb{Z}_2} \) provides the derived involution \( Q : D^b(S') \to D^b(S') \) by twisting with the non-trivial character of \( \mathbb{Z}_2 \). The equivalences \( \bar{\tau}^* \) and \( Q \) commute and are symplectic and the composition \( g = \bar{\tau}^* \circ Q \) is an involution of frameshape \( 2^{12} \). Then the involution \( g \) does not define an action of \( \mathbb{Z}_2 \) on the category, but defines instead a faithful(!) action of \( \mathbb{Z}_4 \). One has the following equivalence (see [11, Sec. 4.9] for details):

\[
D^b(S') \cong D^b(S')_{\mathbb{Z}_4}.
\]

In other words, the equivariant category under this action is equivalent to the category we started with. In particular, there does not exist a stable object which is \( G \)-invariant and \( G \) does not act on any fine moduli space of \( S' \).

1.5. **Open questions.** The main open question is the following:

Is the set of derived categories of (twisted) coherent sheaves on K3 and abelian

\((*)\) surfaces closed under taking equivariant categories by finite group actions satisfying (i, ii, iii)?

In this set we should also include deformations of these categories in the sense of [10] such as the Kuznetsov category of a cubic fourfold.

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4This also follows more abstractly by a result of Elagin, see [23, Thm. 1.3].

5This example first appeared in [22, Sec. 4.2] as a symmetry of K3 non-linear sigma models. We expect that the behaviour \( D^b(S)_{\mathbb{Z}_2} \cong D^b(S) \) is typical of the case where we have a 'failure of the level-matching condition', i.e. \( \lambda > 1 \) in [50, App. C].
We make two comments: (1) Equivariant categories can be taken successively, i.e. if \( H \subset G \) is a normal subgroup, then there is an equivalence \( \mathcal{D}_G \cong (\mathcal{D}_H)_{G/H} \). Hence it is enough to consider simple groups. When we restrict (*) to cyclic group actions (which are most relevant to applications), then the number of cases up to deformation is small enough that a case-by-case analysis may yield a full answer. (2) The parallel question in dimension 1 has an affirmative answer [11, Sec. 7].

Question (*) may be relevant to the classification of irreducible holomorphic symplectic varieties. Let \( X \) be such a variety and assume that it is the moduli space of stable objects in a Calabi–Yau 2-category \( \mathcal{C} \). Possibly after a deformation, let us further assume that \( \mathcal{C} \) admits a symplectic auto-equivalence of finite order which induces an action on \( X \) (one expects the existence of such equivalences to be governed by the Hodge theory of the category). If one can show that the fixed locus \( X^G \) has a 2-dimensional component, then the methods used in the proof of Theorem [1.1] yield an equivalence \( D^b(S) \to \mathcal{C}_G \) for a symplectic surface \( S \) given as the étale cover of this component. However, by a result of Elagin [23, Thm. 1.3] for a finite abelian group acting on a category \( \mathcal{D} \), one can recover \( \mathcal{D} \) from the equivariant category by taking the equivariant category with respect to the dual group \( G^\vee \). In this case this yields

\[
D^b(S)^{G^\vee} \cong (\mathcal{C}_G)^{G^\vee} \cong \mathcal{C}.
\]

We see that an affirmative answer to (*) would imply that \( \mathcal{C} \) is the derived category of a symplectic surface and hence that \( X \) is a holomorphic symplectic variety of the known kind. The philosophy is to use symplectic automorphisms to reconstruct a symplectic variety from its fixed locus. These and similar questions have been the motivation for this paper.

1.6. Donaldson–Thomas theory. Equivariant categories of K3 surfaces also appear naturally in the Donaldson–Thomas theory of (non-commutative) Chaudhuri–Hockney–Lykken Calabi–Yau threefolds, see [48] and [18] for an introduction in physical and mathematical terms respectively. We mention a basic result of the theory which may be viewed as a numerical version of Theorem [1.4].

Consider a symplectic auto-equivalence \( g: D^b(S) \to D^b(S) \) of finite order. Its framshape \( \pi_g = \prod_a a^{m(a)} \) determines a modular form by

\[
f_g(q) = \prod_a \eta(q^a)^{m(a)} = q + O(q^2)
\]

where \( \eta(q) = q^{1/24} \prod_{m \geq 1} (1 - q^m) \) is the Dedekind elliptic function. If \( g \) induces an automorphism of a moduli space \( M_\sigma(v) \) of stable objects, then one can show that the topological Euler characteristic of the fixed locus is

\[
(1.3) \quad e(M_\sigma(v)^G) = \text{Coefficient of } q^{v \cdot w/2} \text{ of } f_g(q)^{-1}
\]

where we write \( v \cdot w \) for the Mukai pairing, see Section [6.1]. If \( g \) is an automorphism of the surface and \( M \) is taken to be the Hilbert scheme, this result has been proven in [18], see also [17] for an extension to non-cyclic groups. The general case of (1.3) would be an easy consequence of Theorem [1.4] if a positive answer to (*) is known, but can be checked independently (details to appear elsewhere).
1.7. Plan of the paper. The paper consists of two parts. The first part can be read independently and deals with the construction of moduli spaces in the equivariant category with respect to induced stability conditions. In Section 2 we recall basic properties of equivariant categories and define natural pullback and pushforward functors under base change. In Section 3 we consider the relation between fixed stacks and the equivariant category. The fixed stack of a trivial $\mathbb{G}_m$-gerbe is studied in detail. The main result (given in Section 3.6) is an existence result for good moduli spaces of stacks of semistable objects in the equivariant category with respect to the induced stability condition.

The second part concerns equivariant categories of symplectic surfaces. In Section 4 we discuss Serre functors of equivariant categories and define equivariant Fourier–Mukai transforms. In Sections 5 and 6 we prove our main theorems. In Section 7 we discuss a series of examples illustrating the general theory.

In the Appendix we prove that for every distinguished stability condition on a K3 surface after a shift the heart generates the derived category.

1.8. Conventions. We always work over $\mathbb{C}$. A variety is connected unless specified otherwise. All functors are derived unless mentioned otherwise. The $K$-group of a triangulated category with finite-dimensional Hom-spaces is always taken numerically, i.e. modulo the ideal generated by the kernel of the Euler pairing. Given a smooth projective variety $X$ we let $D^b(X) = D^b(\text{Coh}(X))$ denote the bounded derived category of coherent sheaves on $X$. If $\pi: X \to T$ is a smooth projective morphism with geometrically connected fibers to a $\mathbb{C}$-scheme $T$, then $D(X)$ or $D(X/T)$ will stand for the full triangulated subcategory of $T$-perfect complexes of the unbounded derived category of $\mathcal{O}_X$-modules. We refer to Sections 2 and 8.1 of [10] for definitions and further references. If $T = \text{Spec}(\mathbb{C})$, then $D(X)$ is the bounded derived category of coherent sheaves as before.

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Part 1. Moduli spaces for the equivariant category

2. EQUIVARIANT CATEGORIES

2.1. Categorical actions. An action $(\rho, \theta)$ of a finite group $G$ on an additive $\mathbb{C}$-linear category $\mathcal{D}$ consists of

- for every $g \in G$ an auto-equivalence $\rho_g: \mathcal{D} \to \mathcal{D}$,
- for every pair $g, h \in G$ an isomorphism of functors $\theta_{g, h}: \rho_g \circ \rho_h \to \rho_{gh}$

such that for all $g, h, k \in G$ the following diagram commutes

\[
\begin{array}{ccc}
\rho_g \rho_h k & \xrightarrow{\rho_g \theta_{h, k}} & \rho_g \rho_h k \\
\downarrow{\theta_{g, h, k}} & & \downarrow{\theta_{g, h, k}} \\
\rho_{gh} k & \xrightarrow{\rho_{gh} k} & \rho_{gh} k.
\end{array}
\]

A $G$-functor $(f, \sigma): (\mathcal{D}, \rho, \theta) \to (\mathcal{D}', \rho', \theta')$ between categories with $G$-actions is a pair of a functor $f: \mathcal{D} \to \mathcal{D}'$ together with 2-isomorphisms $\sigma_g: f \circ \rho_g \to \rho_g' \circ f$ such that $(f, \sigma)$
intertwines the associativity relations on both sides, i.e. such that the following diagram commutes:

\[
\begin{array}{ccc}
f \rho_g \rho_h & \xrightarrow{\sigma_g \rho_h} & f' \rho_g \rho_h \\
\downarrow \rho_g \sigma_h & & \downarrow \rho_g \sigma_h \\
f \rho_g \rho_h & \xrightarrow{\sigma_{gh}} & f' \rho_g \rho_h \\
\end{array}
\]

A 2-morphism of \(G\)-functors \((f, \sigma) \to (\tilde{f}, \tilde{\sigma})\) is a 2-morphism \(t: f \to f'\) that intertwines the \(\sigma_g\), i.e. \(\tilde{\sigma}_g \circ t \rho_g = \rho'_g t \circ \sigma_g\).

**Definition 2.1.** Given a \(G\)-action \((\rho, \theta)\) on the category \(D\) the equivariant category \(D_G\) is defined as follows:

- Objects of \(D_G\) are pairs \((E, \phi)\) where \(E\) is an object in \(D\) and \(\phi = (\phi_g: E \to \rho_g E)_{g \in G}\) is a family of isomorphisms such that

\[
(2.2)
\begin{array}{ccc}
E & \xrightarrow{\phi_g} & \rho_g E \\
\downarrow & & \downarrow \\
gE & \xrightarrow{\rho_g f} & gE' \\
\end{array}
\]

commutes for all \(g, h \in G\).

- A morphism from \((E, \phi)\) to \((E', \phi')\) is a morphism \(f: E \to E'\) in \(D\) which commutes with linearizations, i.e. such that

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow & & \downarrow \\
gE & \xrightarrow{\rho_g f} & gE' \\
\end{array}
\]

commutes for every \(g \in G\).

For all objects \((E, \phi)\) and \((E', \phi')\) in \(D_G\) the group \(G\) acts on \(\text{Hom}_D(E, E')\) via \(f \mapsto (\phi'_g)^{-1} \circ \rho_g (f) \circ \phi_g\). By definition,

\[
\text{Hom}_{D_G}((E, \phi), (E, \phi')) = \text{Hom}_{D}(E, E')^{G}.
\]

The equivariant category comes equipped with a forgetful functor

\[
p: D_G \to D, \quad (E, \psi) \mapsto E
\]

and a linearization functor

\[
(2.3)
q: D \to D_G, \quad E \mapsto (\oplus_{g \in G} \rho_g E, \phi)
\]

where the linearization \(\phi\) is given by considering \(\theta_{h^{-1}g}^{-1}: \rho_g E \to \rho_h \rho_{h^{-1}g} E\) and then taking the direct sum over all \(g\),

\[
(2.4) \quad \phi_h = \oplus_g \theta_{h^{-1}g}^{-1} \circ \rho_g E \to \rho_h \left( \oplus_g \rho_{h^{-1}g} E \right) = \rho_h \left( \oplus_g \rho_{g} E \right).
\]

By [23 Lem. 3.8], \(p\) is both left and right adjoint to \(q\).

We discuss several properties of equivariant categories. We will often write \(g\) for \(\rho_g\).

**Example 2.2.** The trivial \(G\)-action on \(D\) is defined by \(\rho_g = \text{id}\) and \(\theta_{g,h} = \text{id}\) for all \(g, h \in G\). In this case the objects of \(D_G\) are pairs of an object \(x \in D\) and a homomorphism \(\phi: G \to \text{Aut}(x)\).
**Remark 2.3.** Consider the 2-category $G\text{-}\text{Cats}$ whose objects are categories with a $G$-action and whose morphisms are $G$-functors. The equivariant category $D_G$ satisfies the universal property that for all categories $\mathcal{A}$ we have the equivalence

$$\text{Hom}_{\text{Cats}}(\mathcal{A}, D_G) \cong \text{Hom}_{G\text{-}\text{Cats}}(\iota(\mathcal{A}), D)$$

where we let $\iota(\mathcal{A})$ denote the category endowed with the trivial $G$-action. In particular, any $G$-functor from $\iota(\mathcal{A})$ to $D$ factors over the forgetful functor $p$, see [25, Prop. 4.4] for more details.

If a triangulated category has a dg-enhancement, then the equivariant category is again triangulated [23, Cor. 6.10]. This is implied also more directly as follows.

**Proposition 2.4.** Let $D$ be a triangulated category with an action of a group $G$. Suppose there is a full abelian subcategory $\mathcal{A} \subset D$ such that $D^b(\mathcal{A}) = D$ and $G$ preserves $\mathcal{A}$, i.e. $\rho_g E \in \mathcal{A}$ for all $E \in \mathcal{A}$. Then the following holds.

(i) There exist a dg-enhancement $D_{dg}$ of $D$ together with an action of $G$ on $D_{dg}$ which lifts the action of $G$ on $D$.

(ii) The equivariant category $D_G$ is triangulated.

**Proof.** By [19, Sec. 1.2] the dg-quotient category

$$D_{dg}(\mathcal{A}) = C_{dg}(\mathcal{A})/\text{Acyclic}_{dg}(\mathcal{A})$$

of the dg-category of bounded complexes in $\mathcal{A}$ by the dg-category of acyclic bounded complexes in $\mathcal{A}$ defines a dg-enhancement of $D^b(\mathcal{A})$. By hypothesis $D^b(\mathcal{A}) \cong D$ hence $D_{dg}(\mathcal{A})$ is a dg-enhancement. Moreover, the $G$-action on $D$ induces a $G$-action on $\mathcal{A}$. Since $G$ preserves acyclic complexes we obtain a $G$-action on $D_{dg}(\mathcal{A})$ with the desired properties. This proves the first part.

For the second part we apply [21], see also [23, Thm. 7.1], to get

$$D_G = D^b(\mathcal{A})_G \cong D^b(\mathcal{A}_G)$$

and as a derived category the latter is naturally triangulated. □

**Remark 2.5.** If $X$ is a smooth projective variety, then $D^b(X)$ has (up to equivalence) a unique dg-enhancement [38].

The group of characters $G^\vee = \{\chi: G \to \mathbb{C}^* \mid \chi \text{ homomorphism}\}$ acts on the equivariant category $D_G$ by the identity on morphisms and by

$$\chi \cdot (E, \phi) = (E, \chi \phi)$$

on objects, where we let $\chi \phi$ denote the linearization $\chi(g) \phi_g: E \to \rho_g E$.

An object $E \in D$ is called $G$-invariant if for all $g \in G$ there exists an isomorphism $\rho_g E \cong E$. A $G$-linearization of $E$ is an element $\hat{E} \in D_G$ such that $p\hat{E} \cong E$. There is the following obstruction for a $G$-invariant simple object to be $G$-linearizable (which, since $H^2(\mathbb{Z}_n, \mathbb{C}^*) = 0$ for all $n$, is trivial for cyclic groups).

**Lemma 2.6 ([51, Lem. 1]).** Given a $G$-invariant simple object $E \in D$, there exists a class in $H^2(G, \mathbb{C}^*)$ which vanishes if and only if there exists a $G$-linearization of $E$. The set of (isomorphism classes) of $G$-linearizations of $E$ is a torsor under $G^\vee$. 


Example 3.14 below shows that this obstruction is effective.

Let \( \text{Aut} \, \mathcal{D} \) denote the group of isomorphism classes of equivalences of \( \mathcal{D} \). Every group action on \( \mathcal{D} \) yields a subgroup of \( \text{Aut} \, \mathcal{D} \). For the converse one has the following obstruction (which because of \( H^3(\mathbb{Z}_n, \mathbb{C}^*) = \mathbb{Z}_n \) is non-trivial even for cyclic groups).

**Lemma 2.7.** ([11 Sec. 2.2]) Assume that \( \text{Hom}(\text{id}_\mathcal{D}, \text{id}_\mathcal{D}) = \text{Cid} \) and let \( G \subset \text{Aut} \, \mathcal{D} \) be a finite subgroup.

(a) There exists a class in \( H^3(G, \mathbb{C}^*) \) which vanishes if and only if there exists an action of \( G \) on \( \mathcal{D} \) whose image in \( \text{Aut} \, \mathcal{D} \) is \( G \). Moreover, the set of isomorphism classes of such actions is a torsor under \( H^2(G, \mathbb{C}^*) \).

(b) There exists a finite group \( G' \) and a surjection \( G' \to G \) such that \( G' \) acts on \( \mathcal{D} \) and the induced map \( G' \to \text{Aut} \, \mathcal{D} \) is the given quotient map to \( G \).

(c) If \( G = \mathbb{Z}_n \), then we can take \( \mathbb{Z}_{n^2} \to \mathbb{Z}_n \) in (b).

### 2.2. Stability conditions.

A (Bridgeland) stability condition on a triangulated category \( \mathcal{D} \) is a pair \( (A, Z) \) consisting of

- the heart \( A \subset \mathcal{D} \) of a bounded \( t \)-structure on \( \mathcal{D} \) and
- a stability function \( Z : K(A) \to \mathbb{C} \)

satisfying several conditions, see [11]. Given an equivalence \( \Phi : \mathcal{D} \to \mathcal{D}' \) of triangulated categories the image of \( \sigma \) under \( \Phi \) is defined by

\[
\Phi \sigma = (\Phi A, Z \circ \Phi_*^{-1})
\]

where \( \Phi_* : K(\mathcal{D}) \to K(\mathcal{D}') \) is the induced map on \( K \)-groups. If \( \Phi : \mathcal{D} \to \mathcal{D} \) is an auto-equivalence, we say that \( \Phi \) preserves (or fixes) \( \sigma \) if \( \Phi \sigma = \sigma \).

Let \( X \) be a smooth projective variety together with an action of a finite group \( G \) on \( D^b(X) \) which fixes a stability condition \( \sigma = (A, Z) \). By [11] Lem. 2.16 \( \sigma \) induces a stability condition on \( D(X)_G \) defined by

\[
\sigma_G = (A_G, Z_G), \quad Z_G := Z \circ p_* : K(A_G) \to \mathbb{C}.
\]

**Lemma 2.8.** Let \( (E, \phi) \in A_G \). Then \( (E, \phi) \) is \( \sigma_G \)-semistable if and only if \( E \) is \( \sigma \)-semistable. If \( E \) is \( \sigma \)-stable, then \( (E, \phi) \) is \( \sigma_G \)-stable.

**Proof.** If an element \( E \in A_G \) is destabilized by \( F \), then \( p(E) \) is destabilized by \( p(F) \). Conversely, if \( p(E) \) is destabilized by \( F' \in A \), then the image of the adjoint morphism \( qF' \to E \) destabilizes \( E \). Hence an element in \( (E, \phi) \in A_G \) is \( \sigma_G \)-semistable if and only if \( E \in A \) is \( \sigma \)-semistable. A subobject of \( (E, \phi) \) is given by a subobject \( F \subset E \) such that \( \phi \) restricts to a linearization of \( F \). Hence any destabilizing subobject of \( (E, \phi) \) yields a destabilizing subobject of \( E \). This shows the second claim.

As in Definition 1.2 a class \( v \in K(A)^G \) is called \( (G, \sigma) \)-generic if it is primitive and for every splitting \( v = v_0 + v_1 \) with \( v_i \in K(A)^G \setminus \mathbb{Z} v \) the summands have different slopes.

**Lemma 2.9.** Let \( (E, \phi) \in A_G \) such that \( E \) is \( \sigma \)-semistable and its class \([E] \in K(A)^G \) is \((G, \sigma)\)-generic. Then \( (E, \phi) \) is \( \sigma_G \)-stable. In particular,

\[
\text{Hom}_{A_G}((E, \phi), (E, \phi)) = \text{Cid}.
\]
Proof. As explained above the object \((E, \phi)\) is \(\sigma_G\)-semistable. If it is not stable, then there exists a short exact sequence in \(\mathcal{A}_G\)
\[
0 \to (F_1, \phi) \to (E, \phi) \to (F_2, \phi) \to 0
\]
with \(F_1, F_2\) of the same phase as \(E\). Applying the forgetful functor we obtain
\[
0 \to F_1 \to E \to F_2 \to 0
\]
in \(\mathcal{A}\) with \(F_1, F_2\) of the same phase as \(E\). However, the classes \([F_i]\) are \(G\)-invariant which shows that \([E] = [F_1] + [F_2]\) is not \((G, \sigma)\)-generic. □

2.3. Fourier–Mukai actions. Let \(\pi: X \to T\) be a smooth projective morphism to a \(\mathbb{C}\)-scheme \(T\) with geometrically connected fibers. Let
\[p, q: X \times_T X \to X\]
bethe projections to the factors. The Fourier–Mukai transform \(FM_E: D(X) \to D(X)\) with kernel \(E \in D(X \times_T X)\) is defined by
\[
FM_E(A) = q_*(p^*(A) \otimes E).
\]
Using a push-pull argument we have isomorphisms
\[
FM_E(A \otimes \pi^* B) \cong FM_E(A) \otimes \pi^* B
\]
for all \(A \in D(X)\) and \(B \in D(T)\), functorial in both \(A\) and \(B\).

Definition 2.10. A Fourier–Mukai action of \(G\) on \(D(X)\) consists of
\[6\]
• for every \(g \in G\) a Fourier–Mukai kernel \(E_g \in D(X \times_T X)\),
• for every pair \(g, h \in G\) an isomorphism \(\theta_{g,h}: E_g \circ E_h \to E_{gh}\)
such that for all \(g, h, k\) the diagram (2.1) commutes with \(\rho_g\) replaced by \(E_g\).

For smooth projective varieties we have not defined anything new:

Lemma 2.11. ([11, Sec. 2.3]) Let \(X\) be smooth projective variety and let \(G\) be a finite group. Then any \(G\)-action on \(D^b(X)\) is induced by a unique Fourier–Mukai action.

Given a Fourier–Mukai action on the derived category of \(X/T\) our next goal is to define natural operations on the equivariant category. If \(G\) is induced by an action on \(X\), this is discussed in [10 Sec. 4]. Since our \(G\)-action does not have to preserve the tensor product or the structure sheaf, some care is needed in the general case.

2.3.1. Pushforward and pullback. Consider a fiber product diagram
\[
\begin{array}{ccc}
X' & \xrightarrow{\alpha} & X \\
\downarrow{\pi'} & & \downarrow{\pi} \\
T' & \xrightarrow{\beta} & T
\end{array}
\]
The pullback of the kernels of the \(G\)-action on \(X\),
\[
(\alpha \times \alpha)^* E_g \in D(X' \times_{T'} X'),
\]

\[6\]We write \(E \circ F\) to indicate the composition of correspondences \(E, F\).
Given an equivariant object \((F, \phi)\) in \(D(X)_G\) we define its pullback by

\[ \alpha^*(F, \phi) = (\alpha^*F, \phi') \in D(X')_G \]

where the \(G\)-linearization \(\phi'_g\) is the composition

\[ \alpha^*F \rightarrow \alpha^*(gE) = \alpha^*q_*((p^*(F) \otimes E_g)) \cong q'_*(\alpha \times \alpha)^*(p^*(F) \otimes E_g) \]

\[ \cong q'_*(p^*(\alpha^*F) \otimes (\alpha \times \alpha)^*E_g) = g\alpha^*(F) \]

with \(p', q': X' \times_T X' \rightarrow X'\) the projections. The pullback \(\alpha^*\) of an equivariant morphism is the pullback of the morphism in \(D(X)\) (one checks that the pullback morphism is \(G\)-invariant). Taken together this yields a functor

\[ \alpha^*: D(X)_G \rightarrow D(X')_G. \]

Similarly if \(\beta\) is proper and flat and \((E, \phi) \in D(X')_G\), we define the pushforward functor by

\[ \alpha_* (E, \phi) := (\alpha_* E, \phi') \]

where the \(G\)-linearization \(\phi'\) is obtained as the composition

\[ \alpha_* E \rightarrow \alpha_* gE = \alpha_* q_* ((p^*(E) \otimes (\alpha \times \alpha)^*E_g)) \]

\[ \cong q'_*(\alpha \times \alpha)^*(p^*(E) \otimes (\alpha \times \alpha)^*E_g) \cong q'_*(\alpha_* E \otimes E_g) = g\alpha_*(E). \]

The pushforward of an equivariant morphism is the pushforward of the underlying morphism. The pullback functor \(\alpha^*\) is left adjoint to \(\alpha_*\).

2.3.2. \(\text{Hom and tensor product.}\) Given a \(T\)-perfect object \(B \in D(T)\) and an equivariant object \((E, \phi) \in D(X)_G\) we define the tensor product by

\[ (E, \phi) \otimes \pi^* B := (\pi^* B \otimes E, \phi') \]

where the linearization \(\phi'\) is the composition

\[ E \otimes \pi^*(B) \xrightarrow{\phi \otimes \text{id}} \text{FM}_{E_g}(E) \otimes \pi^*(B) \xrightarrow{\cong} \text{FM}_{E_g}(E \otimes \pi^*(B)) = g(E \otimes \pi^*(B)). \]

More generally, if \(D(T)\) is equipped with the trivial \(G\)-action and \((B, \chi) \in D(T)_G\), we let

\[ (B, \chi) \otimes (E, \phi) := (\pi^* B \otimes E, \chi \phi') \]

Similarly, given two equivariant objects \((E, \phi)\) and \((F, \psi)\) in \(D(X)_G\) and an open subset \(U \subset T\) the group \(G\) acts on \(\text{Hom}_{D(X)_G}(E|_U, F|_U)\) by \(f \mapsto \phi_g|_U \circ \text{FM}_{E_g|_U}(f) \circ \psi_g^{-1}|_U\) where we use again that Fourier–Mukai actions induce actions after base change. Since this action is compatible with restrictions to smaller open subsets we obtain a \(G\)-action on \(\text{Hom}_\pi(E, F) := \pi_* \text{Hom}(E, F)\) and thus a bifunctor

\[ \text{Hom}_\pi: D(X)_G \times D(X)_G \rightarrow D(T)_G. \]

It satisfies the usual adjunctions with respect to the tensor product.

For any (closed or non-closed) point \(t \in T\) let \(i_t: X_t \rightarrow X\) be the inclusion of the fiber of \(X\) over \(t\). Given \((E, \phi) \in D(X)_G\) we write \((E, \phi)_t\) for the equivariant pullback \(i^*_t(E, \phi)\).
Lemma 2.12. Let \((E, \phi), (F, \psi)\) be objects in \(D(X)G\). Then
\[
t \mapsto \chi((E, \phi)_t, (F, \psi)_t) := \sum_i \dim \Ext^i_{D(X)_G}((E, \phi)_t, (F, \psi)_t)
\]
is locally constant in \(t\).

Proof. By a push-pull argument we have that
\[
\chi((E, \phi)_t, (F, \psi)_t) = \chi(k(t), \mathcal{H}om_G((E, \phi), (F, \psi))^G \otimes k(t)).
\]
Since \(\mathcal{H}om_G((E, \phi), (F, \psi))\) is perfect, the same holds for its invariant part which implies the claim. \(\Box\)

3. Moduli spaces

3.1. Group actions on stacks. Following [52] an action of a finite group \(G\) on a stack \(M\over C\) consists of

- for every \(g \in G\) an automorphism of stacks \(\rho_g : M \rightarrow M\)
- for every pair \(g, h \in G\) an isomorphism of functors \(\theta_{g,h} : \rho_g \rho_h \rightarrow \rho_{gh}\)

such that for all \(g, h, k \in G\) the diagram (2.1) commutes. In other words, if we view \(M\) as a category fibered in groupoids, then a \(G\)-action on \(M\) is precisely a \(G\)-action on the category \(M\) in the sense of Section 2.1 with the additional assumption that every \(\rho_g\) is a morphism of stacks. A morphism of stacks with \(G\)-actions (also called a \(G\)-equivariant morphism) is a \(G\)-functor \((f, \sigma)\) such that \(f\) is a morphism of stacks. A 2-morphism is a 2-morphism of \(G\)-functors.

Let \(\mathcal{S}t\) and \(G-\mathcal{S}t\) denote the 2-categories of stacks and stacks with a \(G\)-action respectively.

There is a functor \(\iota : \mathcal{S}t \rightarrow G-\mathcal{S}t\) which equips a stack with the trivial \(G\)-action. Let \(\mathcal{G}rp\mathcal{D}s\) be the category of groupoids.

Definition 3.1 ([52 Def. 2.3]). Let \(G\) be a finite group acting on a stack \(M\). The fixed stack is the functor \(M^G : \mathcal{S}t \rightarrow \mathcal{G}rp\mathcal{D}s\) defined by the equivalence
\[
\Hom_{\mathcal{S}t}(T, M^G) \cong \Hom_{G-\mathcal{S}t}(\iota(T), M).
\]

Hence there is a \(G\)-equivariant morphism \(\epsilon : \iota(M^G) \rightarrow M\) satisfying the following universal property: For any stack \(T\) and for any \(G\)-equivariant morphism \(f : \iota(T) \rightarrow M\) there exists a unique morphism \(\tilde{f} : T \rightarrow M^G\) such that \(\epsilon \circ \tilde{f} = f\).

Remark 3.2. As explained in [52 Proof of Prop. 2.5] the objects of \(M^G\) are pairs \((x, \{\alpha_g\}_{g \in G})\) of an element \(x \in M\) and maps \(\alpha_g : x \rightarrow g.x\) such that \(\theta_{g,h}^x \circ g \alpha_h \circ \alpha_g = \alpha_{gh}\) for all \(g, h \in G\). Morphisms are the morphisms in \(M\) which respect the linearizations. Hence, viewed as a category, the fixed stack \(M^G\) is the equivariant category \(M_G\) of the action \((\rho, \theta)\) in the sense of Definition 2.1.

This can be seen also more conceptually: By the universal property of the equivariant category (Remark 3.1) we have a functor \(M^G \rightarrow M_G\), but by the universal property of the fixed stack we also have an inverse.

Remark 3.3. By the universal property, if \((f, \sigma) : N \rightarrow M\) is a \(G\)-equivariant morphism such that \(f\) is a monomorphism (e.g. an open or closed immersion), then we have a fiber
Proposition 3.4. [52, Thm. 3.3, 3.6] Let $G$ be a finite group acting on an Artin stack $\mathcal{M}$ (locally) of finite type over $\mathbb{C}$. Then $\mathcal{M}^G$ is an Artin stack (locally) of finite type over $\mathbb{C}$ and the classifying morphism $\epsilon: \mathcal{M}^G \to \mathcal{M}$ is representable, separated and quasi-compact. If $\mathcal{M}$ has affine diagonal, then so does $\mathcal{M}^G$.

Furthermore, consider any property of morphisms of schemes that is satisfied by closed immersions and is stable under composition. Then, if the diagonal of $\mathcal{M}$ has this property, then $\epsilon$ has this property.

Proof. We prove that $\mathcal{M}^G$ has affine diagonal if $\mathcal{M}$ has. Everything else can be found in [52]. Assume that $\mathcal{M}$ has affine diagonal and consider the commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}^G & \xrightarrow{\Delta_M \circ \epsilon} & \mathcal{M} \times \mathcal{M} \\
\downarrow & & \downarrow \\
\mathcal{M} & \xrightarrow{\epsilon 	imes \epsilon} & \mathcal{M} \times \mathcal{M}.
\end{array}
\]

Since $\Delta_M$ is affine, $\epsilon$ is affine by the second part, hence so is the composition $\epsilon \circ \Delta$. Since $\epsilon \times \epsilon$ is separated, its diagonal is a closed immersion and hence affine. By the cancellation lemma it follows that $\Delta_{\mathcal{M}^G}$ is affine. \qed

If $G$ acts on a separated scheme, then the fixed stack is a closed subscheme and equal to the fixed locus defined in the usual way. However, in general the map $\epsilon: \mathcal{M}^G \to \mathcal{M}$ may behave quite subtle. For example, taking fixed stacks usually does not commute with passing to the good or coarse moduli space (if it exists).

3.2. The fixed stack of a trivial $G_m$-gerbe. Consider an action $(\rho, \theta)$ by the finite group $G$ on the stack $BG_m$ such that $\rho_g = id$ for all $g \in G$ but $\theta$ is arbitrary. According to Lemma 2.7 there is an associated class $\alpha(\theta) \in H^2(G, \mathbb{C}^*)$. A direct verification (see also [52]) shows the following:

\[
(BG_m)^G = \begin{cases} \\
\bigcup_{\chi \in G^\vee} B\mathbb{G}_m & \text{if } \alpha(\theta) = 0, \\
\emptyset & \text{if } \alpha(\theta) \neq 0.
\end{cases}
\]

In this section we consider the following generalization: Let $M$ be a complete variety, and consider the trivial $G_m$-gerbe

\[
\mathcal{M} = M \times B\mathbb{G}_m
\]

The projection and the section of the gerbe are denoted by

\[
p_1: \mathcal{M} \to M, \quad s = (id_M, t): M \to \mathcal{M}
\]

\footnote{We have stated Lemma 2.7 only for additive $\mathbb{C}$-linear category, but since $Aut(id_{BG_m}) = \mathbb{C}^* id$ on which $G$ acts trivially by conjugation, the result applies verbatim also in this case.}
where \( t : M \to BG_m \) corresponds to the trivial line bundle. We refer to [46] Def. 12.2.2 for a definition of gerbes and morphisms of gerbes.

**Lemma 3.5.** There is a 1-to-1 correspondence between the set of morphisms of \( \mathbb{G}_m \)-gerbes \( f : \mathcal{M} \to \mathcal{M} \) and the set of pairs \((F, \mathcal{L})\) where \( F : M \to M \) is an automorphism and \( \mathcal{L} \in \text{Pic}(M) \).

If the morphism \( f \) corresponds to \((F, \mathcal{L})\) and \( g \) corresponds to \((G, \mathcal{M})\), then \( f \circ g \) corresponds to \((F \circ G, \mathcal{L} \otimes F^*(\mathcal{M}))\).

**Proof.** Let \( f : \mathcal{M} \to \mathcal{M} \) be a morphism of gerbes. Define \( F = p_1 \circ f \circ s \) and let \( \mathcal{L} \) be the line bundle corresponding to \( p_2 \circ f \circ s : M \to BG_m \). By [46] Lem. 12.2.4 \( F \) is an automorphism.

Let \( L_{\text{univ}} \) be the universal line bundle on \( BG_m \). We write \( L_{\text{univ}} \) also for its pullback to \( M \times BG_m \). Since \( f \) is a morphism of gerbes we have\(^8\)
\[
f^* L_{\text{univ}} = (f^* L_{\text{univ}})|_M \otimes L_{\text{univ}} = p_1^*(\mathcal{L}) \otimes L_{\text{univ}}.
\]
Hence given \((F, \mathcal{L})\) we can recover \( f \) as the product of \( F \circ p_1 \) and the morphism associated to \( p_1^*(\mathcal{L}) \otimes L_{\text{univ}} \). This yields the 1-to-1 correspondence.

For the last claim, we have that
\[
g^* L_{\text{univ}} = (g^* L_{\text{univ}})|_M \otimes L_{\text{univ}} = p_1^*(\mathcal{M}) \otimes L_{\text{univ}}
\]
hence
\[
f^* g^* L_{\text{univ}} = p_1^* F^*(\mathcal{M}) \otimes f^* L_{\text{univ}}
\]
which gives the claim by restriction to \( M \).

Let \((\rho, \theta)\) be a \( G \)-action on \( \mathcal{M} \) such that for all \( g \in G \):

- the morphism \( \rho_g \) is a morphism of \( \mathbb{G}_m \)-gerbes,
- if \((F_g, \mathcal{L}_g)\) is the pair associated to \( \rho_g \), then \( F_g = \text{id} \).

For a \( \mathbb{C} \)-point \( p \in M \) the \( G \)-action \((\rho, \theta)\) induces an action \((\rho^p, \theta^p)\) on \( p \times BG_m \) such that for all \( g \in G \) we have \( \rho^p_g \cong \text{id}_{BG_m} \) (since \( \rho_g \) acts by gerbe morphisms). Hence as before we have an associated class
\[
\alpha(\theta^p) \in H^2(G, \mathbb{C}^*)
\]
The class \( \alpha(\theta^p) \) vanishes if and only if \((p \times BG_m)^G\) is non-empty. In this case we say that \( p \in M \) is \( G \)-linearizable.

By Remark 3.3 the fixed stack \( \mathcal{M}^G \) is non-empty if and only if \( M \) contains a \( G \)-linearizable point. Hence let \( p \in M \) be \( G \)-linearizable. The 2-isomorphisms \( \theta_{g,h} : \rho_g \rho_h \to \rho_{gh} \) induce isomorphisms
\[
(3.1) \quad \theta_{g,h} : \mathcal{L}_g \otimes \mathcal{L}_h \xrightarrow{\cong} \mathcal{L}_{gh}
\]
which satisfy the associativity relations \((2.1)\). In particular, up to isomorphism the line bundles \( \mathcal{L}_g \) only depend on the conjugacy class \( \bar{g} \) of \( g \) and we obtain a group homomorphism
\[
G_{ab} \to \text{Pic}(M), \bar{g} \mapsto [\mathcal{L}_g]
\]
where \( G_{ab} \) is the abelianization of \( G \), and \([\mathcal{L}]\) stands for the isomorphism class of a line bundle \( \mathcal{L} \).

---

\(^8\)The restriction to each \( m \times BG_m \) is equal to \( L_{\text{univ}} \) by hypothesis. Hence \( f^* L_{\text{univ}} = L_{\text{univ}} \otimes p_1^* L \) for some \( L \in \text{Pic}(M) \). Restricting to \( M \) yields the claim.

\(^9\)One can always reduce to this case by replacing \( \mathcal{M} \) with \( \mathcal{M} \times_M F \) for an irreducible component \( F \) of \( M^G \).
Claim. The $G$-action on $\mathcal{M}$ is isomorphic to an action which factors through $G_{ab}$ and such that the isomorphisms (3.1) are commutative, i.e. $\theta_{g,h} = \theta_{h,g}$ where we identify $\mathcal{L}_g \otimes \mathcal{L}_h$ with $\mathcal{L}_h \otimes \mathcal{L}_g$ by swapping the factors.

Proof of Claim. Let $H = [G,G]$ and choose representatives $\{g_1, \ldots, g_r\}$ for the cosets $G/H$ where we take the identity element for the unit coset. Given any element $g \in g_i H$ we set $\rho'_g = \rho_{g_i}$. The isomorphisms $\mathcal{L}_g \cong \mathcal{L}_{g_i}$ induced by (3.1) yield isomorphisms $t_g : \rho_g \cong \rho_{g_i} = \rho'_g$. Consider the action $(\rho'_g, \theta')$ on $\mathcal{M}$ where $\theta'$ is determined by the commutative diagram

$$
\begin{array}{ccc}
\rho_g \rho_h & \xrightarrow{\theta_{g,h}} & \rho_{gh} \\
\downarrow{t_g \rho_h} & & \downarrow{t_{gh}} \\
\rho'_g \rho'_h & \xrightarrow{\theta'_{g,h}} & \rho'_{gh}.
\end{array}
$$

By construction, $\rho'_g$ only depends on the image of $g$ in $G/H$. We need to show that we can further modify $\theta'$ such that it also only depends on the image in $G/H$, and is commutative. The key idea is that since $M$ is a complete variety, $\text{Hom}(\mathcal{L}_g, \mathcal{L}_h) = \mathbb{C}$, and hence we may find and check all the required relations by restricting to the point $p \in M$ where the action is trivial. Concretely, we may first choose an identification $\mathcal{L}_g|_p \cong \mathbb{C}$ for every $g$. Since $\alpha(\theta^p) = 0$ we may then modify $\theta'$ (i.e. replace $\theta'_{g,h}$ by $\lambda_{g,h} \theta'_{g,h}$ for some $\lambda_{g,h} \in \mathbb{C}$ which is the derivative of a 1-cycle) such that the restrictions

$$
\theta'_{g,h}|_p : \mathcal{L}_g|_p \otimes \mathcal{L}_h|_p \to \mathcal{L}_{gh}|_p
$$

are the identity maps under the given identification. Since $\mathcal{L}_g$ only depends on $G/H$ it follows that $\theta_{g,g'}$ only depends on the image of $g$ and $g'$ in $G/H$. (To spell this out: for any $g \in g_i H$, $g' \in g_j H$ and $h, h' \in H$ we have that $\theta_{g,g'}$ and $\theta_{g_i h, g_j h'}$ are both morphisms $\mathcal{L}_{g_i} \otimes \mathcal{L}_{g_j} \to \mathcal{L}_{g_i g_j}$ where $g_i g_j \in g_k H$; they agree after restriction to $p$ hence they must agree.) Similarly, the commutativity $\theta'_{g,g'} = \theta'_{g',g}$ follows by restriction. \[\Box\]

After replacing $(\rho, \theta)$ with an isomorphic action as in the Claim, we obtain a commutative $\mathcal{O}_M$-algebra

$$
\mathcal{A} = \bigoplus_{g \in G_{ab}} \mathcal{L}_g,
$$

where the multiplication is induced by $\theta$. Consider the étale cover

$$
\pi : Y \to M, \quad Y = \text{Spec}(\mathcal{A}).
$$

For every $g \in G$ the natural inclusion $\mathcal{L}_g \to \mathcal{A}$ yields a natural isomorphism

$$
(3.2) \quad \phi_g : \pi^*(\mathcal{L}_g) \cong \mathcal{O}_Y.
$$

The composition

$$
\pi^*(\mathcal{L}_g \otimes \mathcal{L}_h) \xrightarrow{\phi_g \otimes \text{id}_\mathcal{L}_h} \pi^*(\mathcal{L}_h) \xrightarrow{\phi_h} \mathcal{O}_Y
$$

is induced by $\mathcal{L}_g \otimes \mathcal{L}_h \to \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and hence isomorphic to

$$
\pi^*(\mathcal{L}_g \otimes \mathcal{L}_h) \xrightarrow{\pi^*(\theta_{g,h})} \pi^*(\mathcal{L}_{gh}) \xrightarrow{\phi_{gh}} \mathcal{O}_Y.
$$

We see that $\phi_g$ gives $s \circ \pi : Y \to M$ the structure of a $G$-equivariant morphism with respect to the trivial action on $Y$. This yields a morphism $Y \to \mathcal{M}^G$. 
Define the product
\[ \mathcal{Y} = Y \times B \mathbb{G}_m \]
and consider the morphism
\[ f = \pi \times \text{id}_{B \mathbb{G}_m} : \mathcal{Y} \to M. \]
As before, the tensor product of \( \phi_g \) with the identity on the universal bundle makes \( f \) equivariant with respect to the trivial action on \( \mathcal{Y} \). We obtain a morphism \( \mathcal{Y} \to M^G \). This yields the following description of the fixed stack.

**Proposition 3.6.** In the setting above, if \( M \) contains a \( G \)-linearizable point, then \( f : \mathcal{Y} \to M \) is the fixed stack of the \( G \)-action on \( M \).

**Proof.** We have seen above that there is a natural morphism \( \mathcal{Y} \to M^G \). Conversely, giving an equivariant morphism \( h : T \to M \times B \mathbb{G}_m \), where the scheme \( T \) carries the trivial \( G \)-action, is equivalent to a line bundle \( L \), a morphism \( h' = p_1 \circ h : T \to M \) and maps \( h'^* L_g \to \mathcal{O}_T \) satisfying the cocycle condition. The cocycle condition implies that the induced map
\[ h'^* (\oplus_{g \in G} L_g) \to \mathcal{O}_T \]
is an algebra homomorphism with respect to the algebra structure on \( \oplus_{g \in G} L_g \) defined by \( \theta \). Hence the map \( T \to M \) factors through \( Y \) and thus \( h \) factors through \( Y \times B \mathbb{G}_m \). This yields the inverse \( M^G \to \mathcal{Y} \). \( \square \)

### 3.3. Moduli spaces of equivariant objects

Let \( X \) be a smooth projective variety over \( \mathbb{C} \). Recall from \([36]\) the stack
\[ \mathcal{M} : \text{Sch}/\mathbb{C} \to \text{Grpds} \]
which associates to each scheme \( T \) the groupoid of \( T \)-perfect universally gluable objects in \( D(X \times T) \). As proven in loc. cit. \( \mathcal{M} \) is a quasi-separated algebraic stack locally of finite type over \( \mathbb{C} \) with affine diagonal, see also \([55, 0DPV]\) and \([10, \text{Sec. 8}]\).

Let \( G \) be a finite group which acts on \( D^b(X) \). By Lemma \([2.11]\) the action is given by Fourier–Mukai transforms. The pullback of the Fourier–Mukai kernels define a Fourier–Mukai action \( D^b(X \times T) \) such that the pullback morphisms are \( G \)-equivariant. This defines an action of \( G \) on \( \mathcal{M} \) in the sense of Section \([3.1]\).

**Remark 3.2** yields the following description of the fixed stack:

**Proposition 3.7.** The fixed stack \( \mathcal{M}^G \) is the stack of \( G \)-equivariant universally gluable perfect complexes in \( D(X) \), i.e. for every scheme \( T \) we have
\[ \mathcal{M}^G(T) = \{(E, \phi) \in D(X \times T)_{G \times 1} \mid E \text{ is universally gluable, } T\text{-perfect}\}. \]
The isomorphisms in \( \mathcal{M}^G(T) \) are the isomorphisms of objects in \( D(X \times T)_{G \times 1} \). The pullback is the equivariant pullback. The morphism \( \epsilon : \mathcal{M}^G \to \mathcal{M} \) is the map that forgets the \( G \)-linearization.

From now on let \( \sigma \) be a stability condition on \( D^b(X) \) which is preserved by the \( G \)-action. Let \( \mathcal{M}_\sigma(v) \) be the moduli stack of \( \sigma \)-semistable objects of class \( v \in K(A) \), i.e. for any scheme \( T \) we let

\[ \mathcal{M}_\sigma(v)(T) = \{ E \in D(X \times T) \mid \forall t \in T : E_t \text{ is } \sigma\text{-semistable with } [E_t] = v \}. \]
Since $G$ preserves $\sigma$-semistability, for any $G$-invariant $v \in K(A)$ we have an action
\[ G \times M_\sigma(v) \to M_\sigma(v). \]
The following result follows immediately from Proposition 3.7.

**Proposition 3.8.** We have
\[ M_\sigma(v)^G = \bigsqcup_{v' \in \mathcal{M}(A_G)} M_{\sigma_G}(v'), \]
where $M_{\sigma_G}(v')$ is the substack of $\mathfrak{M}^G$ defined by
\[ M_{\sigma_G}(v')(T) = \{ E \in D(X \times T)_{G \times 1} \mid \forall t \in T: E_t \text{ is } \sigma_G\text{-semistable}, [E_t] = v' \}. \]

3.4. **The fixed stack of a fine moduli space.** As in Section 3.3 consider a $G$-action on $D^b(X)$ which preserves a stability condition $\sigma$. Let $v \in K(D^b(X))$ be a $G$-invariant class such that $M_\sigma(v)$ has a fine moduli space $M_\sigma(v)$ which is smooth. The goal of this section is to determine the fixed stack $M_\sigma(v)^G$.

Write $M = M_\sigma(v)$ and $M = M_\sigma(v)$. By assumption there is a universal family $E \in D(M \times X)$, unique up to tensoring with a line bundle pulled back from the first factor. By the universal property of $M$ this yields a section $s_E : M \to M$ of the $\mathbb{G}_m$-gerbe $M \to M$. Hence $s_E$ defines a trivialization
\[ (3.3) \quad M_\sigma(v) \cong M_\sigma(v) \times \mathbb{G}_m. \]
The universal family $E_M \in D(M \times X)$ is identified under (3.3) with
\[ (p_1 \times \text{id}_X)^*(E) \otimes p_2^*(L_{\text{univ}}) \]
where $p_1, p_2$ are the projections to the factors.

Let $f : M \to M$ be a morphism of $\mathbb{G}_m$-gerbes and let
\[ F = p_1 \circ f \circ s_E, \quad \mathcal{L} = (p_2 \circ f \circ s_E)^*L_{\text{univ}} \]
be the associated automorphism and line bundle as in Lemma 3.5. We consider the difference of the pullbacks of the universal families under $F$ and $f$.

**Lemma 3.9.** In the situation above, we have
\[ ((f \times \text{id}_X)^*(E_M))|_M = (F \times \text{id}_X)^*(E) \otimes \mathcal{L}. \]

**Proof.** Under the identification (3.3) we have $E_M = (p_1 \times \text{id}_X)^*(E) \otimes L_{\text{univ}}$. Hence
\[ (F \times \text{id}_X)^*(E_M) = (p_1 \times \text{id}_X)^*((p_1 \times \text{id}_X)^*(E)) \otimes (f \times \text{id}_X)^*L_{\text{univ}} \]
\[ = (p_1 \times \text{id}_X)^*((F \times \text{id}_X)^*(E)) \otimes ((p_1 \times \text{id}_X)^*(\mathcal{L}) \otimes L_{\text{univ}}) \]
\[ = (p_1 \times \text{id}_X)^*((F \times \text{id}_X)^*(E) \otimes \mathcal{L}) \otimes L_{\text{univ}}. \]
Restricting to $M$ completes the claim. \qed

Consider the action of $G$ on $M$. For every $g \in G$ the morphism $\rho_g : M \to M$ commutes with the inclusion of the automorphism groups (in the derived category, we have $g(\lambda \text{id}) = \lambda g(\text{id}) = \lambda \text{id}$) and hence is a morphism of $\mathbb{G}_m$-gerbes. Let
\[ F_g : M \to M, \quad \mathcal{L}_g \in \text{Pic}(M) \]
be the associated pair constructed in Lemma 3.3. By Lemma 3.9 the line bundle $\mathcal{L}_g$ can also be described by

$$(1 \times g)(\mathcal{E}) = ((1 \times g)\mathcal{E}_M)|_M = ((\rho_g \times \text{id}_X)^*(\mathcal{E}_M)|_M = (\mathcal{F}_g \times \text{id}_X)^*(\mathcal{E}) \otimes \mathcal{L}_g.$$  

Let $F$ be a connected component of the fixed locus $M^G \subset M$ and let $L_g = \mathcal{L}_g|_F$ which only depends on the conjugacy class of $g$, see the discussion in Section 3.1. Consider further the associated étale cover

$$Y = \text{Spec} \left( \bigoplus_{g \in G_{ab}} L_g \right), \quad \pi: Y \to F$$

and define

$$\mathcal{Y} = Y \times B\mathbb{G}_m, \quad \epsilon: \mathcal{Y} \xrightarrow{\pi \times \text{id}_{B\mathbb{G}_m}} F \times B\mathbb{G}_m \to \mathcal{M}.$$  

**Proposition 3.10.** In the setting above, if $F$ contains a $G$-linearizable point, then $\mathcal{Y}$ is the union of the connected components of $M^G$ which map to $F$ and $\epsilon: \mathcal{Y} \to \mathcal{M}$ is the restriction of the classifying map $M^G \to \mathcal{M}$ to $\mathcal{Y}$.

The universal linearization of $\epsilon^*(\mathcal{E}_M)$ is pulled back from the canonical linearization of $(\pi \times \text{id}_X)^*(\mathcal{E}|_{F \times X})$.

By Proposition 3.7 a point $p \in F$ is $G$-linearizable if and only if the corresponding $G$-invariant object $\mathcal{E}_p$ is $G$-linearizable. Using Proposition 3.10 we see that there exists a $G$-linearizable point $p \in F$ if and only if every point on $F$ is $G$-linearizable. In this case we say that the connected component $F$ of $M^G$ is $G$-linearizable.

**Proof.** The first statement is Proposition 3.6. The second part follows since the linearization on $\mathcal{Y}$ is the pullback of the linearization on $Y$ given by (3.2).  

**□**

**Remark 3.11.** The action of $G^\vee$ on $D^b(X)_G$ by twisting the linearization preserves the stability condition $\sigma_G$. Moreover, for every $\chi \in G^\vee$ we have $p_\chi \chi \nu' = p_\chi \nu'$. Hence we have an induced action of $G^\vee$ on

$$\mathcal{M}_\sigma(v)^G = \bigcup_{p_\chi(v') = v} \mathcal{M}_{\sigma_G}(v').$$

In the setting of Proposition 3.10 by Lemma 2.6 we obtain a free action

$$\rho: G^\vee \times Y \to Y$$

such that $\pi \circ \rho_\chi = \pi$. Since any two $G$-linearizations of a $G$-invariant stable object in $D^b(X)$ differ by a character [5.1 Lem. 1], we have $Y/G^\vee = F$. In other words, $\pi: Y \to F$ is a principle $G^\vee$-bundle.

**Remark 3.12.** By working with twisted sheaves the results of this section can be generalized to the case when $\mathcal{M}_\sigma(v) \to \mathcal{M}_\sigma(v)$ is a non-trivial $\mathbb{G}_m$-gerbe. This case occurs precisely if $M_\sigma(v)$ is only a coarse moduli space of stable objects.

**Example 3.13.** Let $E$ be an elliptic curve and let $t_a: E \to E$ be the translation by a 2-torsion point $a \in E$. The group $G = \mathbb{Z}_2$ acts on $\text{Coh}(E)$ by $t_a^*$. Let $E' = E/t_a$. The equivariant category is $\text{Coh}(E)_G = \text{Coh}(E')$. Consider the moduli stack $\mathcal{M} = \mathcal{M}(1,0)$ of Gieseker stable sheaves with Chern characters $v = (1,0) \in H^2(E)$ or equivalently the moduli stack of degree 0 line bundles. It admits the fine moduli space $\mathcal{M} \cong E$ with universal
family the Poincaré bundle $\mathcal{P}$ on $E \times E$. Hence $\mathcal{M} \cong E \times B\mathbb{G}_m$. Since every degree 0 line bundle is translation invariant, the group $G$ induces the trivial action on $\mathcal{M}$. However, because of 

$$(1 \times t^*_a)(\mathcal{P}) = (\text{id} \times t_a)^*\mathcal{P} = \mathcal{P} \otimes p^*_a \mathcal{P},$$

the bundle $\mathcal{P}$ can not be linearized over $\mathcal{M}$. Indeed by Proposition 3.10 (with $L_g = \mathcal{P}_a$) one has $\mathcal{M}^G = \tilde{E} \times B\mathbb{G}_m$ where $\tilde{E}$ is the cover of $E$ defined by $\mathcal{P}_a$.

An alternative description of the fixed stack is also provided by Proposition 3.8. It shows that $\mathcal{M}^G = \mathcal{M}_E(1, 0) \cong E' \times B\mathbb{G}_m$. Since $E' \cong \tilde{E}$ these two presentations agree with each other.

**Example 3.14.** Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ be the subgroup of 2-torsion points of $E$ acting by translation. Let $\mathcal{M} = \mathcal{M}(1, 2)$ be the moduli space of degree 2 line bundles and let $\mathcal{M} \cong E$ be its fine moduli space. Then $\mathcal{M}^G = \mathcal{M}$ but $\mathcal{M}^G = \varnothing$, so $\mathcal{M}$ is not $G$-linearizable. Indeed, any $G$-linearization of a degree 2 line bundle $L$ is a descent datum for the quotient map $\pi: E \to E/G$. Hence there would exists a line bundle $L'$ on $E/G$ with $\pi^* L' = L$ which would imply that the degree of $L$ is divisible by 4.

### 3.5. The Artin–Zhang functor.

As before, consider an action of a finite group $G$ on $D^b(X)$ which preserves a stability condition $\sigma = (A, Z)$. In this section we further assume the following properties:

- $A$ is Noetherian
- $A$ satisfies the ‘generic flatness property’ of [1, Prop. 3.5.1].

The second condition implies that the subfunctor $\mathfrak{M}_A \subset \mathfrak{M}$ of objects, such that every geometric fiber lies in $A$, is open. By Remark 3.3 the open immersion $\mathfrak{M}_A \subset \mathfrak{M}$ yields the fiber diagram

$$
\begin{array}{ccc}
\mathfrak{M}^G & \longrightarrow & \mathfrak{M}^G \\
\downarrow & & \downarrow \\
\mathfrak{M}_A & \longrightarrow & \mathfrak{M}.
\end{array}
$$

By base change this shows that also $(\mathfrak{M}_A)^G \subset \mathfrak{M}^G$ is an open immersion.

Given a cocomplete, locally noetherian, $k$-linear abelian category $C$, let $N_C$ be the stack of finitely presented objects in $C$ as introduced by Artin and Zhang [5], see also [3, Def. 7.8]. Concretely, for a commutative ring $R$ let $C_R$ be the category of pairs $(E, \phi)$ with $E$ an object in $C$ and $\phi: R \to \text{End}_C(E)$ a morphism of $k$-algebras. Then $N_C(\text{Spec} R)$ is the groupoid of flat and finitely presented objects in $C_R$.

As discussed in [3, Ex. 7.20] our assumptions on $A$ imply that the stacks $\mathfrak{M}_A$ and $N_{\text{Ind}(A)}$ are equivalent, where $\text{Ind}(A)$ is the Ind-completion of $A$. Our first goal is to prove the parallel result for the equivariant abelian category $A_G$:

**Proposition 3.15.** $(\mathfrak{M}_A)^G \cong N_{\text{Ind}(A_G)}$.

We begin with two technical lemmata.

**Lemma 3.16.** If $A$ is a Noetherian $\mathcal{C}$-linear category, then every object in $\text{Ind}(A)$ can be written as a union of objects in $A$. 

Proof. Given objects $E \in \mathcal{A}$ and $F \in \text{Ind}(\mathcal{A})$ and an inclusion $F \subseteq E$ in $\text{Ind}(\mathcal{A})$ we first claim that $F \in \mathcal{A}$. Indeed, write $F = \lim_i F_i$ where the $F_i$ lie in $\mathcal{A}$. Then since $F \to E$ is a monomorphism we have $F'_i := \text{Im}(F_i \to F) = \text{Im}(F_i \to E)$ and thus this image lies in $\mathcal{A}$. Therefore, $F$ is a union of objects in $\mathcal{A}$ (namely the $F'_i$) which are subjects of $E$. Since $E$ is Noetherian, this union has to stabilize and since abelian categories contain finite colimits, $F \in \mathcal{A}$ as desired. Now, if $E \to F$ is a quotient in $\text{Ind}(\mathcal{A})$ with $E \in \mathcal{A}$ and $F \in \text{Ind}(\mathcal{A})$ then by the above the kernel lies in $\mathcal{A}$ and hence so does $F$. Therefore $\mathcal{A}$ is closed under quotients in $\text{Ind}(\mathcal{A})$. We conclude, that if $E = \lim_i E_i$ with $E_i \in \mathcal{A}$, then $E$ is the union of the $F_i = \text{Im}(E_i \to E)$. □

Lemma 3.17. Let $\mathcal{A}$ be a Noetherian abelian $\mathbb{C}$-linear category and $G$ a finite group. Then there exists a canonical isomorphism $\text{Ind}(\mathcal{A}_G) \cong \text{Ind}(\mathcal{A})_G$.

We refer to [49, Lem. 3.6] for a parallel result for $\infty$-categories.

Proof. If $\mathcal{A}$ is cocomplete (i.e. has all small filtered colimits) and $(E_i, \phi_i)$ is a direct system in $\mathcal{A}_G$, then the $\phi_i$ define a canonical $G$-linearization on $E = \lim_i E_i$. Hence $\mathcal{A}_G$ is also cocomplete.

Let $\mathcal{A}$ now be Noetherian. Applying the above argument to $\text{Ind}(\mathcal{A})$ we see that $\text{Ind}(\mathcal{A})_G$ is cocomplete. Hence by the universal property of $\text{Ind}$-completion, the inclusion $\mathcal{A}_G \to \text{Ind}(\mathcal{A})_G$ lifts to a functor $\text{Ind}(\mathcal{A}_G) \to \text{Ind}(\mathcal{A})_G$. By composing with the forgetful functor $\text{Ind}(\mathcal{A})_G \to \text{Ind}(\mathcal{A})$ one sees the functor is faithful. We check that the functor is essentially surjective and full.

Let $(E, \phi) \in \text{Ind}(\mathcal{A})_G$ where $E = \bigcup_i E_i$ is a union of objects $E_i$ in $\mathcal{A}$. By replacing $E_i$ by $\bigcup_{g \in G} \phi_g^{-1}(gE_i)$ if necessary we get that the restrictions $\phi_g|_{E_i} : E_i \to gE_i$ define $G$-linearizations on $E_i$. Moreover, after replacing the $E_i$ and $F_i$ suitably, any morphism $(E, \phi) \to (F, \psi)$ is the limit of a morphism $(E_i, \phi_i) \to (F_i, \psi_i)$. □

Proof of Proposition 3.15 Since $\mathcal{M}_\mathcal{A} = \mathcal{N}_{\text{Ind}(\mathcal{A})}$ we have that $\mathcal{M}_G^\mathcal{A}(\text{Spec } R)$ is the groupoid of pairs of $x \in \mathcal{N}_\mathcal{A}(R)$ together with linearizations $\phi_g : x \to gx$ satisfying the cocycle condition. Spelling this out this is the groupoid of triples of objects $E \in \text{Ind}(\mathcal{A})$, homomorphisms $\sigma : R \to \text{End}(E)$ and linearizations $\phi_g : E \to gE$ satisfying

$$\phi_g \circ \sigma_r = g \sigma_r \circ \phi_g,$$

or equivalently, the groupoid of pairs $(E, \phi) \in \text{Ind}(\mathcal{A})_G$ and $\sigma : R \to \text{End}_{\text{Ind}(\mathcal{A})_G}(E, \phi)$. However, $G$ finite implies that $\text{Ind}(\mathcal{A})_G = \text{Ind}(\mathcal{A}_G)$ (see Lemma 3.17) and hence this is precisely the groupoid $\mathcal{N}_{\text{Ind}(\mathcal{A}_G)}(\text{Spec } R)$. □

A stability condition $\sigma = (\mathcal{A}, Z)$ is called algebraic if $Z(K(\mathcal{A})) \subset \mathbb{Q} + i\mathbb{Q}$.

Theorem 3.18. In the above situation assume moreover that $\sigma$ is algebraic and that $\mathcal{M}_\sigma(v)$ is bounded for every $v \in K(D(X))$. Then for every $v' \in K(D^b(X)_G)$ the moduli stack $\mathcal{M}_{\sigma_G}(v')$ is an universally closed Artin stack of finite type over $\mathcal{C}$ which has a proper good moduli space. The inclusion $\mathcal{M}_{\sigma_G}(v') \to \mathcal{M}^G$ is an open embedding.

---

10We thank Eugen Hellman for providing this argument.
Proof. Let \( v = p_v v' \) and let \( \mathcal{M}_{A,v} \subset \mathcal{M}_A \) be the open and closed substack parametrizing objects of class \( v \). Invoking [3 Ex. 7.27], the stack \( \mathcal{M}_{A,v} \) has a \( \Theta \)-stratification whose open piece is \( \mathcal{M}_\sigma(v) \). This yields the fiber diagram
\[
\begin{array}{ccc}
\mathcal{M}_\sigma(v)^G & \longrightarrow & (\mathcal{M}_{A,v})^G \\
\downarrow \epsilon & & \downarrow \epsilon \\
\mathcal{M}_\sigma(v) & \longrightarrow & \mathcal{M}_{A,v},
\end{array}
\]
where the horizontal maps are open immersions. Since \( \mathcal{M}_{A,v} \subset \mathcal{M} \) is open and \( \mathcal{M} \) is an Artin stack locally of finite type with affine diagonal over \( \mathbb{C} \), applying Proposition 3.3 the same holds for \( (\mathcal{M}_{A,v})^G \). Moreover, both vertical morphisms \( \epsilon \) are affine. Since \( \mathcal{M}_\sigma(v) \) is of finite type, so is \( \mathcal{M}_\sigma(v)^G \).

By [3 Sec. 7] the stack \( \mathcal{M}_\sigma(v) \) is \( \Theta \)-reductive and \( S \)-complete. By [3 Prop. 3.20(1)] affine morphisms are \( \Theta \)-reductive and by [3 Prop. 3.42(1)] they are \( S \)-complete. Since both these properties are stable under composition, \( \mathcal{M}_\sigma(v)^G \) is \( \Theta \)-reductive and \( S \)-complete and hence by [3 Thm. A] admits a separated good moduli space.

It remains to show that \( \mathcal{M}_\sigma(v)^G \) is universally closed. For this recall from Proposition 3.15 the isomorphism \( (\mathcal{M}_A)^G \cong N_{\text{Ind}(A)} \). It follows from [3 Lem. 7.17] that \( \mathcal{M}_{A,G} \) satisfies the existence part of the valuative criterion of properness. Since \( \epsilon \colon (\mathcal{M}_{A,v})^G \to \mathcal{M}_{A,v} \) is affine, by [26 Prop. 1.19] the preimage of the \( \Theta \)-stratification of \( \mathcal{M}_{A,v} \) defines a \( \Theta \)-stratification of \( (\mathcal{M}_{A,v})^G \). By definition its open piece is the preimage of the stack of \( \sigma \)-semistable objects, which is precisely the stack of \( \sigma_G \)-semistable objects. By semistable reduction [3 Thm. B/C] we conclude that \( \mathcal{M}_\sigma(v)^G \) is universally closed and therefore that its good moduli space is proper. By Proposition 3.8 the stack \( \mathcal{M}_\sigma(v) \) is a closed and open substack of \( \mathcal{M}_\sigma(v)^G \), hence it satisfies the same conclusion.

We consider the deformation-obstruction theory of the functor \( \mathcal{M}_A^G \).

Proposition 3.19. Suppose that \( A \) is Noetherian, satisfies the generic flatness property and we have \( D^b(A) \cong D^b(X) \).

Let \( 0 \to I \to A' \to A \to 0 \) be a square zero extension of rings and let \( \iota \colon X \times \text{Spec } A \to X \times \text{Spec } A' \) be the natural inclusion. Let \( (E, \phi) \in \mathcal{M}_A^G(\text{Spec } A) \). Then there exists an obstruction class
\[
\omega(E, \phi) \in \text{Ext}^2(E, E \otimes I)^G_0
\]
which vanishes if and only if there exists a complex \( (E', \phi') \in \mathcal{M}_A^G(A') \) such that \( \iota^*(E', \phi') \cong (E, \phi) \). Moreover, in this case the set of extensions is a torsor over \( \text{Ext}^1(E, E \otimes I)^G \).

Here the subscript 0 stands for the traceless part defined by
\[
\text{Ext}^2(E, E)_0 = \text{Ker} \left( \text{Tr}: \text{Ext}^2(E, E) \to \text{H}^2(X, \mathcal{O}_X) \right).
\]

11 Since \( \epsilon \) is not proper in general (see Section 7.1 for an example where this fails) this does not follow directly from the fact that \( \mathcal{M}_\sigma(v) \) is universally closed. Instead we use the alternative description of the bigger stack \((\mathcal{M}_A)^G\).

12 The \( \Theta \)-stratification of \( \mathcal{M}_{A,v} \) corresponds to the Harder–Narasimhan filtration in \( A \). Given an equivariant object \((E, \phi)\) and a Harder–Narasimhan filtration \( E_i \) of \( E \) with respect to \( \sigma \) the restrictions \((E_i, \phi|_{E_i})\) define a Harder–Narasimhan filtration of \((E, \phi)\) which corresponds to the 'preimage' \( \Theta \)-stratification of \((\mathcal{M}_A)^G\).
Proof. By Proposition 3.15 we can use the deformation theory of the Artin–Zhang functor \( N_{\text{Ind}(A)} \). Since \( D^b(A) = D^b(X) \) for any \( (E, \phi) \in A \) we have

\[
\text{Ext}^i_{D^b(A)}((E, \phi), (E, \phi)) = \text{Ext}^i_{D^b(X)_G}((E, \phi), (E, \phi)) = \text{Ext}^i_{D^b(X)}(E, E)^G.
\]

Hence the existence of the obstruction class \( \omega(E, \phi) \in \text{Ext}^2(E, E \otimes I)^G \) follows from (3.7). The \((G\text{-invariant})\) trace map is the derivative to the determinant map on \( S \). Since the Picard stack is smooth, all obstructions to deforming \( \text{det}(E) \) vanishes. This shows that the obstruction class lies in the kernel of

\[
\text{Ext}^2(E, E)^G \xrightarrow{p^*} \text{Ext}^2(E, E) \xrightarrow{\text{Tr}} \mathbb{C}.
\]

\( \square \)

3.6. Summary. Let \( X \) be a smooth projective variety and let

\[
\text{Stab}^*(X) \subset \text{Stab}(X)
\]

be a connected component which contains an algebraic stability conditions \( \sigma = (A, Z) \) such that

- \( A \) satisfies the ‘generic flatness property’, and
- for all \( v \in K(A) \) the stack \( M_\sigma(v) \) is bounded.

Then by [53 Prop. 4.12] the same holds for all algebraic stability conditions in \( \text{Stab}^*(X) \). Moreover, as explained in [3 Ex. 7.27], for any \( v \in K(D^b(X)) \) and stability condition \( \sigma \in \text{Stab}^*(X) \) one can find an algebraic stability condition \( \sigma' \) such that \( M_\sigma(v) \) and \( M_{\sigma'}(v) \) define the same moduli functor.

The existence of components \( \text{Stab}^*(X) \) satisfying these condition is known for arbitrary curves and surfaces and abelian threefolds, as well as for certain Fano and Calabi–Yau threefolds, see for example [10 Rem. 26.4] and references therein.

The following summarizes the results of the last two sections.

**Theorem 3.20.** Let \( G \) be a finite group acting on the derived category \( D^b(X) \) of a smooth projective variety with a connected component \( \text{Stab}^*(X) \) as above (e.g. a surface). Let \( \sigma \in \text{Stab}^*(X) \) be a \( G \)-fixed stability condition.

(a) For every \( v' \in K(D^b(X)_G) \) the stack \( M_{\sigma_G}(v') \) is an universally closed Artin stack of finite type over \( \mathbb{C} \) which has a proper good moduli space.

(b) Let \( v \in K(D^b(X))^G \) such that \( M_\sigma(v) \) is a moduli stack of stable objects. Let \( M \) be its good moduli space and assume it is smooth. Then the natural morphism

\[
\bigcup_{v' \in K(D^b(X)_G)} \left| M_{\sigma_G}(v') \to M^G \right|
\]

is étale of degree \( |G'| \) with image the union of all \( G \)-linearizable connected components of \( M^G \). If \( H^2(G, \mathbb{C}^*) = 0 \) or, more generally, if the \( G \)-action on \( D^b(S) \) factors through the action of a quotient \( G \to Q \), such that \( G \) is a Schur covering group of \( Q \), then \( \text{Stab}^*(X) \) is surjective.

We will need the following \( G \)-invariant version of the argument in [3 Ex. 7.27].

**Lemma 3.21.** With \( X \) as above, let \( v \in K(D^b(X))^G \) and \( \sigma \in \text{Stab}^*(X)^G \). Then there exists an algebraic stability condition \( \sigma' \in \text{Stab}^*(X)^G \), such that \( M_\sigma(v) \) and \( M_{\sigma'}(v) \) define the same moduli functor.
Proof. We follow the arguments and notations from [3, Ex. 7.27]. Note also that the arguments from [39, Lem. 2.15] apply in our setting.

We restrict the decomposition of

\[ \mathcal{C}_{S'} = \left( \bigcup_{\gamma' \in S'} W_{\gamma'} \right) \setminus \bigcup_{\gamma' \notin S'} W_{\gamma'} \]

associated to \( v \) and \( \sigma \) to the set of invariant stability conditions \( \text{Stab}^* (X)^G \). Since we have \( \sigma \in \mathcal{C}_{S'} \), we conclude for all \( \gamma' \notin S' \) that the connected component of the submanifold \( \text{Stab}^* (X)^G \) containing \( \sigma \) is not entirely contained in \( W_{\gamma'} \). Then arguing as in [3, Ex. 7.27] for \( \mathcal{C}_{S'} \cap \text{Stab}^* (X)^G \) completes the proof. \( \square \)

Proof of Theorem 3.20. By Lemma 3.21 we may assume that \( \sigma \) is algebraic. Then part (a) follows from Theorem 3.18. For part (b), we will assume for simplicity that \( M \) is a fine moduli space. The case of a coarse moduli space of stable objects works parallel by using a twisted universal object instead.

By Proposition 3.8 we have the decomposition

\[ (3.8) \quad \mathcal{M}_\sigma (v)^G = \bigcup_{\substack{p, \psi = v}} \mathcal{M}_{\sigma G} (v'). \]

The map \( \epsilon : \mathcal{M}_{\sigma} (v)^G \to \mathcal{M}_{\sigma} (v) \) is induced from \( \epsilon : \mathcal{M}_{\sigma} (v)^G \to \mathcal{M}_{\sigma} (v) \) by passing to good moduli spaces. For every \( G \)-linearizable connected component \( F \subset M \), the scheme \( \text{Spec} (\oplus_{g \in G^a} \mathcal{L}_{g}) \) as defined in (3.5) is étale of degree \( |G^a| = |G'| \) over \( F \). By Proposition 3.10 it is the union of all connected components of (3.8) mapping to \( F \). Since every connected component maps to some \( F \) this shows the first claim.

If \( G \) factors through as a Schur cover \( G \to Q \), then we have \( M^G = M^Q \). Moreover for every connected component \( F \) and point \( p \in F \) the obstruction of being \( G \)-linearizable (as given by Lemma 2.6) is the pullback of a class in \( H^2 (Q, \mathbb{C}^*) \) and hence vanishes. This shows that every connected component of \( M^G \) is \( G \)-linearizable. \( \square \)

Part 2. Equivariant categories of symplectic surfaces

4. More on equivariant categories


Let \( \mathcal{D} \) be a \( \mathbb{C} \)-linear triangulated category with finite-dimensional Hom spaces. A Serre functor for \( \mathcal{D} \) is an equivalence \( S : \mathcal{D} \to \mathcal{D} \) together with a collection of bifunctorial isomorphisms

\[ \eta_{A,B} : \text{Hom} (A, B) \overset{\cong}{\to} \text{Hom} (B, SA)^{\vee} \]

for all objects \( A, B \in \mathcal{D} \). As discussed in [11, Sec. 5] given an action by a finite group \( G \) on \( \mathcal{D} \) the Serre functor \( S \) lifts to a Serre functor

\[ \tilde{S} : \mathcal{D}_G \to \mathcal{D}_G \]

which is of the form \( \tilde{S} (A, \phi) = (SA, \phi') \) for a certain linearization \( \phi' \). Moreover, for any objects \( (A, \phi) \) and \( (B, \psi) \) in \( \mathcal{D}_G \) the restriction of \( \eta_{A,B} \) to the \( G \)-invariant part defines bifunctorial isomorphisms

\[ \eta_{A,B} : \text{Hom} (A, B)^G \overset{\cong}{\to} (\text{Hom} (B, SA)^G)^{\vee} \]
EQUIVARIANT CATEGORIES

where the $G$-action on the left is defined by the linearizations $\phi, \psi$ and the $G$-action on the right is defined by the linearizations $\psi$ and $\phi'$.

We say that the category $\mathcal{D}$ is Calabi–Yau if there exists a 2-isomorphism

$$\text{id}_{\mathcal{D}} \cong S[-n]$$

for some integer $n$, called the dimension of $\mathcal{D}$.

**Remark 4.1.** The derived category $D^b(X)$ of a smooth projective $n$-dimensional variety $X$ has the Serre functor $S = (-) \otimes \omega_X[n]$. In this case we will usually denote the lifted functor $\tilde{S}$ also by $(-) \otimes \omega_X[n]$ where the action on the linearization is implicitly understood. So $(A, \phi) \otimes \omega_X[n]$ will stand for $\tilde{S}(A, \phi) = (A \otimes \omega_X[n], \phi')$.

**Remark 4.2.** The results discussed above work also in the relative case of a smooth projective morphism $\pi: X \to T$ with geometrically connected fibers as in Section 2.3. Given a Fourier–Mukai $G$-action on $D^b(X)$, the $\pi$-relative Serre functor lifts to a $\pi$-relative Serre functor of the equivariant category $D^b(X)_G$.

We have the following criterion for the equivariant category of a Calabi–Yau variety to be Calabi–Yau.

**Proposition 4.3.** ([11, Sec. 6.3, 6.4]) Let $X$ be a smooth projective variety which is Calabi–Yau, i.e. $\omega_X \cong \mathcal{O}_X$. Consider the action of a finite group $G$ on $D^b(X)$ which lifts to an action on the dg-enhancement $D_{dg}(X)$.

(i) If the induced action of $G$ on singular cohomology preserves the class of the Calabi–Yau form $[\omega_X] \in H^n(X, \Omega^n_X)$, then $D^b(X)_G$ is Calabi–Yau of dimension $n$.

(ii) Suppose that, moreover, we have an equivalence $D^b(X)_G \cong D^b(X')$ for a variety $X'$. The induced action of $G^\nu$ on $H^\ast(X', \mathbb{C})$ preserves the class of $\omega_{X'}$.

4.2. Equivariant Fourier–Mukai transforms. Let $X$ and $Y$ be smooth projective varieties and let $G$ be a finite group which acts on $D^b(X)$. By Lemma 2.11 this action is given by Fourier–Mukai transforms and hence defines an action by Fourier–Mukai transforms on $D^b(X \times Y)$, see Section 2.3.1. Since this action is pulled back from $X$, we often write $G \times 1$ for the group which acts on $D^b(X \times Y)$.

Consider the projections $X \leftarrow X \times Y \rightarrow Y$. The (equivariant) Fourier–Mukai transform

$$F_\mathcal{E}: D^b(Y) \to D^b(X)_G$$

with kernel $\mathcal{E} \in D^b(X \times Y)_{G \times 1}$ is defined by

$$F_\mathcal{E} A = \rho_\ast(\pi^\ast(A) \otimes \mathcal{E})$$

where the tensor product takes values in $D^b(X \times Y)_{G \times 1}$ and $\rho_\ast$ is the equivariant pushforward. Similarly, the (reverse) equivariant Fourier–Mukai transform $G_\mathcal{E}: D^b(X)_G \to D^b(Y)$ is defined by

$$G_\mathcal{E}(E, \phi) = \mathbb{H}om_{\mathcal{E}}(E, \rho_\ast(E, \phi))^G$$

where we used equivariant pullback and the $\pi$-relative Hom of Section 2.3.2.

---

13Take $\beta$ to be $Y \to \text{Spec}(\mathbb{C})$. 
Lemma 4.4. For any $\mathcal{E} \in D^b(X \times Y)_{G \times 1}$ let
\[ \mathcal{E}_L = \mathcal{E} \otimes \pi^* \omega_X^\vee[- \dim X], \quad \mathcal{E}_R = \mathcal{E} \otimes \pi^* \omega_Y^\vee[- \dim Y]. \]
Then $G_{\mathcal{E}_L}$ and $G_{\mathcal{E}_R}$ is the left and right adjoint of $F_\mathcal{E}$ respectively.

Proof of Lemma 4.4. For any $(A, \phi) \in D^b(X)$ and $B \in D^b(Y)$ we have
\[
\text{Hom}_{D^b(X)_G}((A, \phi), F_\mathcal{E} B) \\
\cong \text{Hom}_{D^b(X \times Y)_{G \times 1}}(\rho^*(A, \phi), \pi^* B \otimes \mathcal{E}) \\
\cong \text{Hom}_{D^b(X \times Y)}(\rho^* A, \pi^* B \otimes \mathcal{E})^G \\
\cong (\text{Hom}_{D^b(X \times Y)}(\pi^* B \otimes \mathcal{E}, \rho^*(A) \otimes \omega_{X \times Y}[\dim X + \dim Y]))^G \\
\cong (\text{Hom}_{D^b(Y)}(B, \mathcal{E} \otimes \rho^*(A) \otimes \omega_{X \times Y}[\dim X + \dim Y]))^G \\
\cong \text{Hom}_{D^b(Y)}(\mathcal{E} \otimes F_\mathcal{E} S(A), B)^G \\
\cong \text{Hom}_{D^b(Y)}(G_{\mathcal{E} \otimes \rho^* \omega_X^\vee[- \dim X]}(A), B).
\]
The other case is similar. \hfill \square

We have the following criterion when a Fourier–Mukai transform $F_\mathcal{E} : D^b(Y) \to D^b(X)_G$ is an equivalence.

Proposition 4.5. Let $\mathcal{E} \in D^b(X \times Y)_{G \times 1}$. Assume that
(i) $\text{Hom}_{D^b(X)_G}(\mathcal{E}_x, \mathcal{E}_y[i]) = \text{Hom}_{D^b(Y)}(\mathcal{E}_x, \mathcal{E}_y[i])$ for all $x, y \in Y$.
(ii) $D^b(X)_G$ is indecomposable.
(iii) The functor $F_\mathcal{E}$ commutes on objects with Serre functors, i.e. $\hat{S} F_\mathcal{E}(A) \cong F_\mathcal{E} S(A)$ for all $A \in D^b(Y)$.

Then $F_\mathcal{E}$ is an equivalence.

Proof. By Lemma 4.4 the functor $F_\mathcal{E} : D^b(Y) \to D^b(X)_G$ has both right and left adjoints. The assertion then follows from [16 Thm. 2.3]. \hfill \square

5. Proof of main results

Let $S$ be a symplectic surface, let $G$ be a finite group which acts on $D^b(S)$ and let $\sigma \in \text{Stab}^1(S)$ be a stability condition. Throughout this section we assume that this triple satisfies the conditions (i), (ii) and (iii) of Section 1.1.

5.1. Preliminaries. We have the following structure result.

Proposition 5.1. The equivariant category $D^b(S)_G$ is triangulated, indecomposable and Calabi–Yau of dimension 2.

Proof. Write $\sigma = (A, Z)$. Since the actions of $\hat{\text{GL}}^+(2, \mathbb{R})$ and $G$ on the stability manifold commute, by Proposition 4.1 we may assume that $D^b(A) \cong D^b(S)$.
Applying Proposition 2.4 we see that $D^b(S)_G$ is triangulated and that the $G$-action on $D^b(S)_G$ lifts to an action on the dg-enhancement. Hence by Proposition 4.3 and assumption (i) we find that $D^b(S)_G$ is Calabi–Yau. Since $G$ acts faithfully, $D^b(S)_G$ is indecomposable by definition. \hfill \Box

5.2. Moduli spaces. We consider moduli spaces of objects in $D^b(S)_G$. By work of Toda [56] the distinguished component $\text{Stab}^1(S)$ satisfies the assumptions of Theorem 3.20. Hence we have the following.

**Proposition 5.2.** Let $v' \in K(D^b(S)_G)$. Then $\mathcal{M}_{\sigma G}(v')$ is an universally closed Artin stack of finite type over $\mathbb{C}$ which admits a proper good moduli space.

We have the following for $(G, \sigma)$-generic Mukai vectors:

**Proposition 5.3.** If $v \in \Lambda^G$ is $(G, \sigma)$-generic, then $\mathcal{M}_{\sigma}(v)^G$ has a good moduli space $N$ which is smooth, symplectic and proper. The map $\pi: \mathcal{M}_{\sigma}(v)^G \to N$ is a $\mathbb{G}_m$-gerbe.

**Proof.** By arguing as in the proof of Lemma 3.21 we can deform the stability condition $\sigma$ inside $\text{Stab}^1(S)_G$ to an algebraic stability condition, without modifying the moduli functor $\mathcal{M}_{\sigma}(v)$. Together with Remark 3.19 this shows that we can assume that $\sigma$ is algebraic and that $D^b(A) \cong D^b(S)$.

Let $\pi: \mathcal{M}_{\sigma}(v)^G \to N$ be the good moduli space of $\mathcal{M}_{\sigma}(v)^G$. For every $x \in \mathcal{M}_{\sigma}(v)^G$ over a scheme $T$ corresponding to an equivariant object $(E, \phi)$ we have an inclusion $\mathbb{G}_m(T) \hookrightarrow \text{Aut}(x)$ by sending $f \in \mathbb{G}_m(T)$ to $f \cdot \text{id}_E$. Moreover, for every $\mathbb{C}$-point $p \in \mathcal{M}_{\sigma}(v)^G$ by Lemma 2.9 we have

$$\text{Aut}_{\mathcal{M}_{\sigma}(v)^G}(p) = \text{Aut}_{\mathcal{M}_{\sigma G}(v')}(p) = \text{Aut}_{\mathcal{A}_G}(E, \phi) = \mathbb{C}^*.$$ 

This shows that $\pi$ is a $\mathbb{G}_m$-gerbe.

Let $p \in \mathcal{M}_{\sigma}(v)^G$ be a $\mathbb{C}$-valued point corresponding to some object $(E, \phi) \in \mathcal{A}_G$. Let $v' \in K(A_G)$ be the class of $(E, \phi)$. Applying Lemma 2.9 again we have

$$\text{Hom}_{\mathcal{A}_G}((E, \phi), (E, \phi)) = \mathbb{C}.$$ 

Since $D^b(S)_G$ is Calabi–Yau of dimension 2, we find that

$$\text{Ext}^2_{\mathcal{A}_G}((E, \phi), (E, \phi)) = \text{Hom}_{\mathcal{A}_G}((E, \phi), (E, \phi)) \cong \mathbb{C}.$$ 

By Lemma 2.12 the Euler characteristic $\chi((E, \phi), (E, \phi))$ is locally constant and hence depends only on $v'$. We write $\chi(v', v')$ for its value. By Proposition 3.19 we conclude that the dimension of the tangent space of $N$ at $p$ is

$$\dim(T_{N, p} = \dim \text{Ext}^1_{\mathcal{A}_G}((E, \phi), (E, \phi)) = -\chi(v', v') + 2.$$ 

In particular, the dimension is locally constant in $p$. Moreover, from the $G$-invariant inclusion $\text{Cil} \subset \text{Hom}(E, E)$ we obtain via Serre duality a $G$-invariant surjection $\text{Ext}^2(E, E) \to \mathbb{C}$ which is precisely the trace map. This shows that the trace map is surjective on the $G$-invariant part and thus that the trace-free part vanishes:

$$\text{Ext}^2(E, E)^G = 0.$$ 

Using Proposition 3.19 again we find that all obstructions vanish and $N$ is smooth.

The symplectic form on $N$ can be constructed from the fact that it is a moduli space of stable objects in a 2-CY category. It can be seen also directly:
Recall from [30, Sec. 10] the anti-symmetric Yoneda pairing on $\mathcal{M}_\sigma(v)$,

\begin{equation}
\varepsilon x^1_{\rho}(\mathcal{E}, \mathcal{E}) \times \varepsilon x^1_{\rho}(\mathcal{E}, \mathcal{E}) \rightarrow \varepsilon x^2_{\rho}(\mathcal{E}, \mathcal{E}),
\end{equation}

where $\mathcal{E}$ is the universal family on $S \times \mathcal{M}_\sigma(v)$ and $\rho: S \times \mathcal{M}_\sigma(v) \rightarrow \mathcal{M}_\sigma(v)$ is the projection to the second factor. Restricting to the $G$-invariant part and pulling back (5.1) via $\epsilon: \mathcal{M}_\sigma(v') \rightarrow \mathcal{M}_\sigma(v)$ yields a pairing

\begin{equation}
\epsilon^*\varepsilon x^1_{\rho}(\mathcal{E}, \mathcal{E})^G \times \epsilon^*\varepsilon x^1_{\rho}(\mathcal{E}, \mathcal{E})^G \rightarrow \epsilon^*\varepsilon x^2_{\rho}(\mathcal{E}, \mathcal{E}).
\end{equation}

By Proposition 3.19 the sheaf $\epsilon^*\varepsilon x^1_{\rho}(\mathcal{E}, \mathcal{E})^G$ is the tangent bundle of $N$. Since the symplectic form is $G$-invariant, the image of (5.2) is the $G$-invariant part $\epsilon^*\varepsilon x^2_{\rho}(\mathcal{E}, \mathcal{E})^G = \mathcal{O}_N$. Equivariant Serre duality implies that the pairing (5.2) is non-degenerate and hence a symplectic form. □

5.3. **Proof of Theorem 1.1** Consider the étale morphism given in (3.7).

\begin{equation}
\bigcup_{\rho \cdot v' = v} \mathcal{M}_\sigma(v') \rightarrow M^G.
\end{equation}

Let

\[ S' \subset M_\sigma(v') \]

be a connected component which maps to the component $F \subset M^G$. By Remark 3.11 the degree of the projection $S' \rightarrow F$ divides the order of $G^\nu$.

By the second part of Proposition 3.10 the moduli space $\mathcal{M}_\sigma(v')$ is fine, i.e. there is a universal equivariant object on $\mathcal{M}_\sigma(v') \times S$. Let

\[ \mathcal{E} = (E, \phi) \in D^b(S' \times S)_{1 \times G}. \]

be its restriction to $S' \times S$. We will check that the induced Fourier–Mukai transform

\[ F_\mathcal{E}: D^b(S') \rightarrow D^b(S)_G \]

is an equivalence.

For any $x \in S'$ we have

\[
\text{Hom}_{D^b(S)_G}(\mathcal{E}_x, \mathcal{E}_x) = \text{Hom}_{D^b(S)}(\mathcal{E}_x, \mathcal{E}_x)^G = \mathbb{C}
\]

\[
\text{Ext}^1_{D^b(S)_G}(\mathcal{E}_x, \mathcal{E}_x) = \text{Ext}^1_{D^b(S)}(\mathcal{E}_x, \mathcal{E}_x)^G \cong T_{S', x} \cong \mathbb{C}^2
\]

\[
\text{Ext}^2_{D^b(S)_G}(\mathcal{E}_x, \mathcal{E}_x) = \text{Hom}_{D^b(S)_G}(\mathcal{E}_x, \mathcal{E}_x)^{\vee} \cong \mathbb{C}.
\]

The first line follows from the stability of $\mathcal{E}_x$. The second line follows from Proposition 3.19 the smoothness of $S'$, and since $F$ and hence $S'$ are 2-dimensional. The third line follows since the equivariant category is Calabi–Yau. In particular, we have $\chi(\mathcal{E}_x, \mathcal{E}_x) = 0$, and using Lemma 2.12 this yields

\[ \chi(\mathcal{E}_x, \mathcal{E}_y) = 0 \quad \text{for all } x, y \in S'. \]

For all distinct $x, y \in S'$ by the stability of $\mathcal{E}_x$ and $\mathcal{E}_y$ we have

\[
\text{Hom}_{D^b(S)_G}(\mathcal{E}_x, \mathcal{E}_y) = 0
\]

\[
\text{Ext}^2_{D^b(S)_G}(\mathcal{E}_x, \mathcal{E}_y) = \text{Hom}_{D^b(S)_G}(\mathcal{E}_y, \mathcal{E}_x)^{\vee} = 0.
\]
Hence from the Euler characteristic calculation we also get $\text{Ext}^1(E_x, E_y) = 0$. We have therefore proven that for all $x, y \in S'$ we have

$$\text{Hom}_{D^b(S')} (C_x, C_y[i]) = \text{Hom}_{D^b(S)G} (E_x, E_y[i]).$$

By Proposition 5.1 the category $D^b(S)G$ is indecomposable and Calabi–Yau of dimension 2. Applying Proposition 1.5 we conclude that $F_E$ is an equivalence. □

5.4. **Proof of Theorem 1.3.** Part (a) follows from Proposition 5.3.

For the second part we argue similarly to the proof of Theorem 1.1. Since $\pi$ is a $\mathbb{G}_m$-gerbe with Brauer class $\alpha$, the universal equivariant object on $M_{\sigma G}(v) \times S$ restricted to $\pi^{-1}(S') \times S$ descends to an $\alpha \times 1$-twisted $1 \times G$-equivariant universal family $\mathcal{E}$ on $S' \times S$. Arguing as in Theorem 1.1 shows that the associated Fourier–Mukai transform $F_E : D^b(S', \alpha) \to D^b(S)G$ is an equivalence. □

5.5. **Proof of Theorem 1.4.** The claim follows from Theorem 3.20. □

5.6. **Proof of Theorem 1.5.** By Proposition 6.1 below the induced stability $\sigma_G$ lies in $\text{Stab}^\dagger(S)$. Since $S'$ is a K3 surface and $\sigma_G$ is distinguished, for every $v' \in R_v$ the moduli space $M_{\sigma_G}(v')$ is an irreducible holomorphic symplectic variety. The étale map

$$(5.4) \quad M_{\sigma_G}(v') \to M^G$$

is the quotient map for the faithful action of the stabilizer of $v'$ in $G^\vee$ on $M_{\sigma_G}(v')$. By the second part of Proposition 1.3 the stabilizer acts symplectically and thus must have a fixed point. However, since the quotient map is étale, this can only be possible if the the stabilizer is trivial, or equivalently if $\chi$ is an isomorphism onto its image. Hence $\text{Stab}^\dagger(S)$ is a trivial Galois cover. Further, since $G$ is cyclic, every point of $M^G$ is $G$-linearizable. Moreover, every point of $M^G$ has precisely $G\vee$ preimages. This shows the claim. □

6. Existence and properties of auto-equivalences

Let $S$ be a symplectic surface. In this section we tie up some loose ends in order to make the theorems we proved in the last section effective in practice. After some preliminary notation, we will consider the following topics:

(i) Given a $G$-fixed distinguished stability condition $\sigma \in \text{Stab}^\dagger(S)$ we will show that the induced stability condition is distinguished, at least if the equivalence arises from a universal family. This is useful, because for distinguished stability conditions the moduli spaces of objects are well-understood.

(ii) We will prove that any symplectic action on a moduli space of stable objects on $S$ is induced by an action on the derived category (Proposition 1.6).

6.1. **Mukai lattice.** The even cohomology of the symplectic surface $S$,

$$\Lambda = H^2(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}),$$

admits a non-degenerate pairing, called the **Mukai pairing**, defined by

$$\langle (r_1, D_1, n_1), (r_2, D_2, n_2) \rangle = -r_1n_2 - r_2n_1 + \int_S D_1 \cup D_2.$$

We will also write $\alpha \cdot \beta$ for $\langle \alpha, \beta \rangle$. For any $E, F \in D^b(S)$ we have

$$v(E) \cdot v(F) = -\chi(E, F).$$
6.2. Stability conditions. Given a stability condition \( \sigma = (A, Z) \in \text{Stab}^\dagger(S) \) in the distinguished component we will identify the stability function
\[
Z: \Lambda_{\text{alg}} \to \mathbb{C}
\]
with the corresponding element in \( \Lambda_{\text{alg}} \otimes \mathbb{C} \) under the Mukai pairing.

Let \( \mathcal{P}(S) \subset \Lambda_{\text{alg}} \otimes \mathbb{C} \) be the open subset of elements whose real and imaginary part span a positive-definite 2-plan, let \( \mathcal{P}^+(S) \subset \mathcal{P}(S) \) be the connected component which contains \( e^{i\omega} \) for an ample class \( \omega \), and let
\[
\mathcal{P}^+_0(S) = \mathcal{P}^+(S) \setminus \bigcup_{\delta \in \Lambda_{\text{alg}}} \delta \perp \delta = -2 \delta.
\]
Bridgeland [15] proved that
\[
\pi: \text{Stab}^\dagger(S) \to \mathcal{P}^+_0(S), \quad \sigma = (A, Z) \mapsto Z
\]
is a covering map. His results were generalized to the twisted case in [31].

6.3. Induced stability conditions. Let \( \sigma \in \text{Stab}^\dagger(S) \) be a stability condition and let \( G \) be a finite group which acts on \( D^b(S) \). We assume the conditions (i), (ii) and (iii) of Section 1.1 are satisfied. Suppose we are given an equivalence
\[
F_E: D^b(S', \alpha) \to D^b(S)_G
\]
induced from a universal family \( E \) as in Theorem 1.1 or Theorem 1.3.

Proposition 6.1. We have \( F_E^{-1}(\sigma_G) \in \text{Stab}^\dagger(S') \).

We begin with a description how the Mukai lattices \( \Lambda \) and \( \Lambda' \) of the surfaces \( S \) and \( S' \) interact. Consider the composition of the forgetful and linearization functors with the equivalence \( F_E \):
\[
\text{FM}_{p(E)} = p \circ F_E, \quad \text{FM}_{p(E)^{\dagger}[2]} = F_E^{-1} \circ q,
\]
where we have also written \( p \) for the forgetful functor of \( D^b(S' \times S)_{1 \times G} \). Passing to cohomology this yields morphisms
\[
p: \Lambda' \to \Lambda, \quad q: \Lambda \to \Lambda'
\]
which are both left and right adjoints of each other. The composition is \( pq = \oplus_g g \). Let
\[
L \subset \Lambda'
\]
denote the saturation of the sublattice \( q(\Lambda) \).

Given a lattice \( M \) we write \( M(n) \) for the lattice obtained by multiplying the intersection form with the integer \( n \).

Lemma 6.2. We have the finite-index sublattices
\[
\Lambda^G \oplus (\Lambda^G)^\perp \subset \Lambda, \quad L \oplus L^\perp \subset \Lambda'.
\]
The map \( p \) vanishes on \( L^\perp \) and defines an embedding of lattices \( p: L([G]) \hookrightarrow \Lambda^G \). The map \( q \) vanishes on \( (\Lambda^G)^\perp \) and defines an embedding of lattices \( q: \Lambda^G([G]) \hookrightarrow L \).
Proof. The isomorphism of correspondences

$$\rho_g \circ p(\mathcal{E}) = (\text{id} \times \rho_g)(p(\mathcal{E})) \cong p(\mathcal{E}),$$

shows that the image of $p : \mathcal{N} \to \Lambda$ lies in the invariant lattice $\Lambda^G$. By adjunction it follows that $q$ vanishes on $(\Lambda^G)^\perp$. In particular, for all $v', w' \in L$ we can write $v' = q(v)$ and $w' = q(w)$ where $v, w \in \Lambda^G \otimes \mathbb{Q}$. We obtain

$$\langle v', w' \rangle = \langle qv, qw \rangle = \langle v, pw \rangle = |G|\langle v, w \rangle.$$

Since $\Lambda^G$ is non-degenerate, this shows that $L$ is non-degenerate and we have the finite-index sublattice $L \oplus L^\perp \subset \mathcal{N}$. It also shows that $q$ defines an embedding $\Lambda^G(|G|) \hookrightarrow L$. Moreover, with the same notation as above we have

$$\langle pv', pv' \rangle = \langle pqv, pqw \rangle = |G|\langle v, pqw \rangle = |G|\langle qv, qw \rangle = |G|\langle v', w' \rangle.$$

We find that $p$ defines an embedding $L(|G|) \hookrightarrow \Lambda^G$. For every $w' \in L^\perp$ we have $\langle pw', v \rangle = \langle w', qv \rangle = 0$ for all $v \in \Lambda$, which shows that $pw' = 0$.

If $G$ is abelian, then one can show that $L$ is the invariant lattice for the action of the dual group on $D^b(S')$, that is $L = (\mathcal{N})^G$.

Proof of Proposition 6.2. To ease the notation we assume that the Brauer class $\alpha$ vanishes and hence that we work with the usual derived category $D^b(S')$. The case with non-trivial Brauer class works parallel.

Let $\tau = F^\perp_\pi(\sigma_G)$. By construction the functor $F_\pi$ is induced from a universal family $\mathcal{E} \in D^b(S^t \times S)_{1 \times G}$ of $\sigma_G$-stable objects. Since $\mathcal{E}_x$ is $\sigma_G$-stable for all $x \in S'$, the skyscraper sheaves $\mathcal{C}_x$ are $\tau$-stable for all $x \in S'$.

Let us consider the central charge $Z_\tau$ of the stability condition $\tau$. By definition, it is given by the composition

$$Z_\tau : \mathcal{N} \xrightarrow{p} \Lambda_{alg}^G \subset \Lambda_{alg} \xrightarrow{Z} \mathbb{C}.$$

By Lemma 6.2 the central charge $Z_\tau$ factors over $L$ and the real and imaginary part of $Z_\tau$ span a positive-definite 2-plane, because $\Re(Z)$ and $\Im(Z)$ do so.

We want to apply now the reasoning of the proof of [15, Prop. 10.3]. As in [15, Sec. 10], there is a unique $g \in \text{GL}^+(2, \mathbb{R})$ such that the central charge of $g\tau$ is of the form $\exp(\beta + i\omega)$ for some $\beta, \omega \in \text{NS}(S')$ with $\omega > 0$, and such that the sheaves $\mathcal{C}_x$ have phase 1. Then as in the first step in [15, Prop. 10.3] we apply [15, Lem. 10.1] to conclude that for any curve $C \subset S'$ and torsion sheaf $\mathcal{E}$ supported on $C$ satisfies $\Im(Z_\tau(\mathcal{E})) > 0$ which implies $\Im(\omega \cdot [C]) > 0$. Combining this with $\omega^2 > 0$ we find that the class $\omega$ is ample.

Invoking again [15, Lem. 10.1] we find further that the heart $\mathcal{B}$ of $g\tau$ is the tilt of the torsion pair $(\mathcal{T}, \mathcal{F})$, where $\mathcal{T} = \text{Coh}(S') \cap \mathcal{P}(0, 1]$ and $\mathcal{F} = \text{Coh}(S') \cap \mathcal{P}(-1, 0]$ and $\mathcal{P}$ is the slicing corresponding to $g\tau$ (for more on tilting we refer to Appendix A or 27). Arguing as in the second step of the proof of [15, Prop. 10.3] we deduce that the torsion pair $(\mathcal{T}, \mathcal{F})$ coincides with the torsion pair $(\mathcal{T}_\omega, \mathcal{F}_\omega)$ associated with the classes $\omega, \beta$ which is constructed in [15, Sec. 6]. With the notation of loc. cit. this yields that $\mathcal{B} = \mathcal{A}(\omega, \beta)$ and therefore $g\tau = \sigma_{\omega, \beta}$. In particular, $\tau \in \text{Stab}^b(S')$ and the proof is finished. \qed
6.4. **Proof of Proposition 1.6.** Let $S$ be a K3 surface with a stability condition $\sigma' = (A',Z') \in \text{Stab}^\dagger(S)$. Let $M$ be a fine moduli space of $\sigma'$-stable objects of Mukai vector $v \in \Lambda$ and let $G$ be a finite group which acts symplectically on $M$. Consider the Hodge isometry
\[
\Lambda \supset v^\perp \cong H^2(M,\mathbb{Z}).
\]

By [41, Thm. 26] the induced action of $G$ on $H^2(M,\mathbb{Z})$ acts trivially on the discriminant lattice. Hence, the action lifts to an action on $\Lambda$ which fixes the vector $v$ and acts by Hodge isometries. Since $G$ acts symplectically on $M$, the action on $\Lambda$ preserves the class of the symplectic form.

Let $H \in H^2(M,\mathbb{Z})$ be a $G$-invariant ample class (obtained for example by averaging any ample class over its images under $G$). Recall the wall and chamber decomposition of $\text{Stab}^\dagger(S)$ associated to $v$ [15, Sec. 9] and denote by $C$ the chamber which contains $v$. From [8, Thm. 1.2] we infer that there exists a stability condition $\sigma = (A,Z) \in C$ such that the associated divisor class $\ell_\sigma$ equals the class $H$ (for the construction and properties of the divisor classes $\ell_\sigma$ we refer to [9]). By definition the central charge $Z$ is contained in the $C$-vector space $\text{Span}_C\langle H,v \rangle \subset \Lambda \otimes \mathbb{C}$ and hence fixed by $G$. Moreover, since $\sigma$ and $\sigma'$ lie in the same chamber, the moduli functors $M_\sigma(v)$ and $M_{\sigma'}(v)$ agree. This proves $M = M_\sigma(v)$.

Hence we have obtained a subgroup $G \subset O(\Lambda)$ which acts by Hodge isometries, preserves the class of the symplectic form and $Z$. An application of [29, Prop. 1.4] shows that this action on $\Lambda$ is induced by a subgroup
\[
G \subset \text{Aut} \, D^b(S)
\]
which preserves $\sigma$ and acts symplectically. Using part (b) of Lemma 2.7 there is a surjection $\tilde{G} \to G$ from a finite group $\tilde{G}$ which acts on $D^b(S)$ with image $G$ in $\text{Aut} \, D^b(S)$. To conclude, observe that by construction the action of $\tilde{G}$ preserves $\sigma$ and $v$ and hence induces an action on $M = M_\sigma(v)$. Since the restriction map $\text{Aut}(M) \to O(H^2(M,\mathbb{Z}))$ is injective [40, Lem. 7.1.3], the action of $\tilde{G}$ on $M$ factors through the given action by $G$. This proves the first part.

For the second part, assume that $G \subset \text{Aut} \, M$ is cyclic. Then the action of $\mathbb{Z}_n$ on $M$ has at least one fixed point which corresponds to a $\mathbb{Z}_n$-invariant simple object $F$. Hence the claim follows from [11, Sec. 4.8].

\[\square\]

7. **Examples**

7.1. **The dual action of a geometric involution.** Let $\iota : S \to S$ be a symplectic involution with at least one fixed point and let $G = \mathbb{Z}_2$ be the group generated by $\iota$. Hence we are in one of the following two cases:

(i) $S$ is an abelian surface and $\iota$ is multiplication by $(-1)$, or
(ii) $S$ is a K3 surface and $\iota$ is a **Nikulin involution** [54].

---

14 The case of a coarse moduli space works similarly.
The number \( r \) of fixed points of \( G \) is 16 and 8 respectively, and in both cases the minimal resolution \( S' \) of \( S/\mathbb{Z}_2 \) is a K3 surface. In the fiber diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{\alpha} & S' \\
\downarrow{\beta} & & \downarrow \\
S & \xrightarrow{} & S/\mathbb{Z}_2
\end{array}
\]

the map \( \beta \) is the blowup at the fixed points and \( \alpha \) identifies \( S' \) with the fixed locus \( \text{Hilb}^2(S)^G \).

By [16] (or Theorem 1.1) we have the equivalence

\[
\Phi = \beta_* \alpha^*: \text{Db}(S') \to \text{Db}(S/G).
\]

Let \( Q: \text{Db}(S') \to \text{Db}(S') \) be the involution given by the action of the dual group \( G' \) . By applying both sides to skyscraper sheaves one finds

\[
Q = T\sigma_s(-\delta) \circ \prod_{i=1}^r \text{ST}_{\sigma_{E_i}(-2)}
\]

where \( T_{\mathcal{L}}(E) = E \otimes \mathcal{L} \) is the twist by a line bundle \( \mathcal{L} \), and

\[
\text{ST}_{E}(F) = \text{Cone}(\text{Hom}^*(E, F) \otimes E \to F)
\]

is the spherical twist by the spherical object \( E \) . The \( E_i \) are the exceptional divisors of the resolution \( S' \) and

\[
\delta = \frac{1}{2} \sum_{i=1}^r E_i.
\]

The involution \( Q \) fixes skyscraper sheaves of points not on the exceptional divisor and sends \( \mathcal{O}_{S'} \) to \( \mathcal{O}_{S'}(\delta) \) as well as \( \mathcal{O}_{E_i}(-1) \) to \( \mathcal{O}_{E_i}(-2)[1] \) . For \( x \in E_i \) the action exchanges the two distinguished triangles

\[
\begin{align*}
\mathcal{O}_{E_i}(-1) & \to C_x \to \mathcal{O}_{E_i}(-2)[1] \\
\mathcal{O}_{E_i}(-2)[1] & \to Q(C_x) \to \mathcal{O}_{E_i}(-1).
\end{align*}
\]

The frameshape of \( Q \) is

\[
\pi_\sigma = \begin{cases} 
1^{-8}2^{16} & \text{if } S \text{ is an abelian surface,} \\
1^82^8 & \text{if } S \text{ is a K3 surface.}
\end{cases}
\]

As an example of a fixed stack computation, consider the moduli space

\[
\mathcal{M} = \mathcal{M}_{\sigma_G}(0, 0, 1)
\]

where \( \sigma_G \) is induced by a \( G \)-fixed stability condition on \( \text{Db}(S) \) which is equivalent to Gieseker stability for the Mukai vector \( v = (0, 0, 1) \) . Then the \( \mathcal{C} \)-points of \( \mathcal{M} \) correspond to the objects

\[
\begin{align*}
\mathcal{C}_x & \text{ for all } x \in S', \\
Q(\mathcal{C}_x) & \text{ for all } x \in E_i, \\
\mathcal{O}_{E_i}(-1) & \oplus \mathcal{O}_{E_i}(-2)[1].
\end{align*}
\]

In this list the \( \mathcal{C}_x \) for all \( x \notin E_i \) and the \( \mathcal{O}_{E_i}(-1) \oplus \mathcal{O}_{E_i}(-2)[1] \) are invariant under \( Q \) . Every \( \mathcal{C}_x \) for \( x \notin E_i \) admits two distinct \( G' \)-linearizations, while \( \mathcal{O}_{E_i}(-1) \oplus \mathcal{O}_{E_i}(-2)[1] \) admits

\[15\text{ See also } [35] \text{ for a related discussion of this involution.}
\]

\[16\text{ On the Mukai lattice the involution } Q \text{ acts by}
\]

\[
(1, 0, 0) \mapsto (1, -r/4), \quad (0, E_i, 0) \mapsto (0, -E_i, 1), \quad (0, 0, 1) \mapsto (0, 0, 1).\]
only one. We find that the good moduli space of \( M \) is the quotient variety \( S/\mathbb{Z}_2 \) and the good moduli space of \( M^{G'} \) is \( S \). The forgetful map \( \epsilon: M^{G'} \to M \) induces the quotient map \( S \to S/\mathbb{Z}_2 \) on good moduli spaces. Applying Theorem 1.3 we obtain the equivalence

\[
D^b(S) \to D^b(S')_{G'}
\]

where the cocycle \( \alpha \) is trivial since \( S/\mathbb{Z}_2 \) is a fine moduli space away from the singularities.

Among other things this example shows that while the good moduli space of \( M \) may be singular, its fixed stack has a smooth proper good moduli space (as guaranteed by part (a) of Theorem 1.3). We also see that \( \epsilon \) is not proper, because it does not satisfy the valuative criterion of properness.

7.2. Involutions on a genus 2 K3 surface. Let \( \pi: S \to \mathbb{P}^2 \) be a K3 surface branched over a sextic curve and let \( g: S \to S \) be a symplectic involution preserving the pullback \( H \) of the hyperplane class. The involution descends to an involution \( g_{\mathbb{P}^2} \) of \( \mathbb{P}^2 \) which can be chosen to act by \( (x, y, z) \mapsto (-x, y, z) \), see [54, Sec. 3.2]. The fixed locus of \( g_{\mathbb{P}^2} \) is \( p = (1, 0, 0) \) and the line \( x = 0 \). Let \( C_0 \) be the preimage under \( \pi \) of the line \( x = 0 \) and let \( C_1 \) be the preimage of a generic line of the form \( \lambda y + \mu z \). Let also \( C \in |O(2H)| \) be a curve that is preserved under \( g \) but disjoint from the fixed points \( p_i \). These curves are preserved by \( g \) and contain 6, 2 and 0 fixed points respectively. Consider the quotients

\[
C_0' = C_0/\mathbb{Z}_2, \quad C_1' = C_1/\mathbb{Z}_2 \quad \text{and} \quad C' = C/\mathbb{Z}_2
\]

which are rational, elliptic, and of genus 3 respectively. After reordering the exceptional divisors one has in \( \text{Pic}(S') \) the relations

\[
C_0' = \frac{1}{2} C' - \frac{1}{2} (E_3 + \ldots + E_6),
\]

\[
C_1' = \frac{1}{2} C' - \frac{1}{2} (E_1 + E_2).
\]

Suppose that \( S \) is of minimal Picard rank 9. Then by [54, Lem. 1.10] the Picard group of \( S' \) has the \( \mathbb{Z} \)-basis \( C_1', \delta, E_2, \ldots, E_8 \). The map on cohomology

\[
P: H^*(S', \mathbb{Z}) \to H^*(S, \mathbb{Z})
\]

induced by the composition \( D^b(S') \xrightarrow{\Phi} D^b(S)^G \to D^b(S) \) is given by

\[
1 \mapsto 1 - p, \quad p \mapsto 2p, \quad E_i \mapsto p, \quad \delta \mapsto 4p, \quad C' \mapsto 2H, \quad C_1' \mapsto H - p
\]

where we let \( p \) denote the class of a point on both \( S \) and \( S' \).

Let \( \sigma \) denote a generic \( G \)-fixed stability condition on \( S \) which for vectors \((0, kH, 0)\) is equivalent to Gieseker stability. We are interested here in calculating the fixed locus of the good moduli spaces \( M_\sigma(0, H, 0) \) and \( M_\sigma(0, 2H, 0) \).

Since \( H \) is irreducible on \( S \), the coarse moduli space \( M_\sigma(0, H, 0) \) is smooth. Hence by Theorem 1.5 (and using the notation given there) we have

\[
M_\sigma(0, H, 0)^G = \bigcup_{v' \in R_H} M_{\sigma'}(v').
\]

\[\text{We denote the class in the Picard group with the same symbol as the underlying curve.}\]
A direct calculation shows that there is a unique vector in \( \overline{R}_H \) of square 0 given by \( C'_1 + E_1 \), and 28 vectors of square \(-2\). Therefore,

\[
M_\sigma(0, H, 0)^G = \tilde{S} \sqcup (28 \text{ points})
\]

where \( \tilde{S} = M_{\sigma_G}(0, C'_1 + E_1, 0) \) is a smooth K3 surface.

We consider \( M_\sigma(0, 2H, 0) \). The set \( R_{2H} \) is given by vectors of the form

\[
v' = C' + \sum_{i=1}^{8} a_i E_i + cp
\]

where all the \( a_i \) are either integers or half-integers, \( \sum_i a_i \) is even and \( c = -\sum_i a_i/2 \). Moreover, only vectors satisfying

- \((v')^2 \geq -2\) (equivalently \( \sum_i a_i^2 \leq 3 \)), or
- \( v' = v_1 + v_2 \) with \( v_i \in R_H \)

contribute to \( R_{2H} \). One finds that \( \overline{R}_{2H} \) (i.e. modulo \( \mathbb{Q} \)) consists of the following:

(i) The vector \( C' \) of square 4. It can be decomposed in 28 different ways as a sum \( v_1 + v_2 \) with \( v_1, v_2 \in R_H \) both of square \(-2\), and in a unique way as \( v_1 + v_2 \) with \( v_1, v_2 \in R_H \) both of square 0 (given as \( C'_1 + E_1 \)). The moduli space \( M_{\sigma_G}(C') \) is of dimension 6. Its singular locus is the disjoint union of the product variety \( \tilde{S} \times \tilde{S} \) and 28 isolated points.

(ii) 63 vectors of square 0. Each vector can be written in 6 different ways as a sum of two \((-2)\)-vectors in \( R_H \). The moduli space in each case is a K3 surface with 6 singularities of type \( A_1 \).

(iii) 56 vectors of square 0, each written uniquely as \( v_1 + v_2 \) where \( v_1 \) is of square 0 (equal to \( C'_1 + E_1 \)) and \( v_2 \) is of square \(-2\). In each case we have \( M_{\sigma_G}(v') = M_{\sigma_G}(v_1) = \tilde{S} \).

(iv) 1 vector of square 0 obtained as \( 2v_1 \), where \( v_1 = C'_1 + E_1 \in R_H \) is of square 0. The good moduli space \( M_{\sigma_G}(2v_1) \) is \( \text{Sym}^2 M_{\sigma_G}(v_1) = \text{Sym}^2 \tilde{S} \).

(v) 378 vectors of square \(-4\) written uniquely as \( v_1 + v_2 \) where \( v_1, v_2 \in R_H \) are both of square \(-2\). The good moduli space is a point.

(vi) 28 vectors of square \(-8\) obtained as \( 2v \), where \( v \in R_H \) is of square \(-2\). The good moduli space is a point.

Note that since \( G \) is cyclic, the image of \( \bigsqcup_{v' \in \overline{R}_{2H}} M_{\sigma_G}(v') \) in \( M_\sigma(0, 2H, 0) \) is precisely the fixed locus we are interested in. A basic sublocus of this fixed locus is

\[
\text{Sym}^2 (M_\sigma(0, H, 0)^G) \subset M_\sigma(0, 2H, 0)^G.
\]

The scheme \( \text{Sym}^2 M_\sigma(0, H, 0)^G \) consists of

(a) 1 copy of \( \text{Sym}^2(\tilde{S}) \),

(b) 28 copies of \( \tilde{S} \) corresponding to sheaves \( E \oplus F \) with \( E \in \tilde{S} \) and \( F \) corresponding to one of the 28 fixed points and

(c) \( \text{Sym}^2(28 \text{ points}) \) consisting of 378 + 28 points corresponding to the direct sum of distinct and identical stable sheaves respectively.

Given distinct \( G \)-invariant stable sheaves \( E, F \) of the same slope, the direct sum \( E \oplus F \) admits precisely \( |G'|^2 \) many \( G \)-linearizations. Moreover, if distinct \( E, F \in M_\sigma(0, H, 0) \) are isolated points of the fixed locus, then no equivariant lift of \( E \oplus F \) has class \( C' \) (since
otherwise \((E, \phi) = Q(F, \phi)\) so \(E = F\). We see that the 378 points in (c) are the image of the points (v), but also of the \(6 \cdot 63\) singular points on the K3 surfaces in (ii).

Similarly, the 28 K3 surfaces in (b) are the image of the 56 K3 surfaces in (iii). Since there are precisely 4 linearizations, these K3 surfaces can not appear in the image of other components, and so yield connected components of \(M_{\sigma}(0, 2H, 0)^G\). A direct sum \(E \oplus E\) of a stable object \(E\) admits precisely \(|\text{Sym}^2(G^*)| = \left(G^*\right)^{2} / 2^{2}\) many linearizations (here 3). Hence the 28 remaining points in (c) are the image of the 28 points in (vi) and the 28 isolated singularities in (i). Moreover, if \(v_1 \in R_H\) of square 0, then \(M_{\sigma_G}(2v_1) = \text{Sym}^2 M_{\sigma_G}(v_1)\) maps to the same locus as the inclusion

\[
M_{\sigma_G}(v_1) \times M_{\sigma_G}(Qv_1) \subset M_{\sigma_G}(0, C', 0).
\]

Hence the image of \(M_{\sigma_G}(2v_1)\) lies in the image of the main component \(M_{\sigma_G}(0, C', 0)\). The 63 moduli spaces in (ii) contain stable points and since we have already taken the coset modulo \(Q\), they must embed into \(M_{\sigma}(0, 2H, 0)^G\) as isolated components. We conclude that

\[
M_{\sigma}(0, 2H, 0)^G = Y \sqcup (28 \text{ smooth K3s}) \sqcup (63 \text{ K3s with 6 nodes})
\]

where \(Y\) is the image of \(M_{\sigma_G}(0, C', 0)\) and hence 6-dimensional.

We turn to the proof of Proposition 1.7 and the O’Grady 10 resolution

\[
X \to M_{\sigma}(0, 2H, 0)
\]

as constructed in [4]. Recall from [54] that Pic\((S) = ZH \oplus E_8(-2)\). Hence there exists 240 vectors \(\alpha \in E_8(-2)\) of square \(-4\). The involution \(g\) acts on these vectors by \(g\alpha = -\alpha\). Let \(A \subset E_8(-2)\) be a list of representatives of the orbits of the \((-4)\)-vectors under this action. The singular locus of \(M_{\sigma}(0, 2H, 0)\) is the locus of polystable sheaves, and therefore given by

\[
M_{\sigma}(0, 2H, 0)^{\text{sing}} = \text{Sym}^2 M_{\sigma}(0, H, 0) \sqcup \bigsqcup_{\alpha \in A} (M_{\sigma}(H + \alpha) \times M_{\sigma}(H - \alpha)).
\]

The resolution \(X\) is obtained by a blowup of \(M_{\sigma}(0, 2H, 0)\) along \(\text{Sym}^2 M_{\sigma}(0, H, 0)\), followed by a resolution of the 120 isolated points. The fiber of \(X\) over each of these 120 points is a \(\mathbb{P}^5\). The automorphism \(g\): \(M_{\sigma}(0, 2H, 0) \to M_{\sigma}(0, 2H, 0)\) natural lifts to the blowup (by universal property), but it is not clear a priori whether it lifts along the resolution of the 120 points. Hence we only obtain a birational involution \(g’: X \dashrightarrow X\) defined away from 120 disjoint copies of \(\mathbb{P}^5\).

Proposition 1.7 follow now from the above and a local analysis of \(g\) along \(M_{\sigma}(0, 2H, 0)^{\text{sing}} \cap M_{\sigma}(0, 2H, 0)^G\) using the local description of the moduli spaces given in [33 Sec. 2] and [4 Sec. 3]. This is straightforward and we just highlight the main points:

- The 120 isolated singular points of \(M_{\sigma}(0, 2H, 0)\) lie in \(Y\). They are the images of the stable points of \(M_{\sigma_G}(C')\) corresponding to \(q(E_\alpha)\) where \(E_\alpha\) is the unique stable object in class \(H + \alpha\). The map \(g’\) does not extend to the resolution and the closure of the fixed locus of \(g’\) contains the whole exceptional \(\mathbb{P}^5\).
- The 63 K3 surfaces with 6 nodes described in (ii) meet the singular locus of \(M_{\sigma}(0, 2H, 0)\) at the singularities. The corresponding component in the fixed locus of \(g’\) is the proper transform and smooth.
The 28 smooth K3 surfaces in \( M_6(0, 2H, 0)^G \) corresponding to (iii) lie completely in the singular locus \( M_6(0, 2H, 0)^{sing} \). The corresponding component in the fixed locus of \( g' \) is a trivial \( 2 : 1 \) cover of this locus and hence given by 56 K3 surfaces.

The K3 surfaces in (iii) and precisely 32 of the K3 surfaces in (ii) arise as moduli spaces of semistable objects on \( S' \) for a Mukai vector \( w \) which satisfies \( \langle w, \Lambda' \rangle = \mathbb{Z} \). Hence all of them are derived equivalent to \( S' \).

### 7.3. An order 3 equivalence.

Let \( E, F \) be elliptic curves defined by cubic equations \( f, g \) respectively and consider the cubic fourfold \( X \subset \mathbb{P}^5 \) defined by the equation \( f(x_0, x_1, x_2) + g(x_3, x_4, x_5) = 0 \). As in [42] Ex. 1.7(iv) define a \( G = \mathbb{Z}_3 \)-action on \( X \) by letting the generator act by

\[
(x_0, \ldots, x_5) \mapsto (x_0, x_1, x_2, \zeta x_3, \zeta x_4, \zeta x_5),
\]

where \( \zeta \) is a non-trivial third root of unity. The induced action of \( G \) on the Fano variety of lines on \( X \) has fixed locus \( F(X)^G = E \times F \). Since \( F(X) \) is a moduli space of stable objects in the Kuznetsov component \( \mathcal{A} \) of \( D^b(X) \), using arguments parallel to the proof of Theorem 1.1 shows that \( \mathcal{A}_G \cong D^b(A) \) for some connected \( \acute{e} \)tale cover \( A \to E \times F \) of degree 1 or 2. In particular, \( A \) is an abelian surface.

### 7.4. Order 11 equivalences.

Let \( g : D^b(S) \to D^b(S) \) be a symplectic auto-equivalence of a K3 surface \( S \) of order 11 fixing a stability condition \( \sigma \in \text{Stab}^1(S) \). The associated action on cohomology is one of three possible conjugacy classes, each with invariant lattice of rank 4 [53] App. C]. This implies that the pairs \( (S, g) \) are isolated points in their moduli space. By [29] each such \( g \) induces automorphisms on moduli spaces of stable objects \( M \). If we want to determine the equivariant category \( D^b(S)_{\mathbb{Z}_n} \) through Theorem 1.1 we would need to find a 2-dimensional component of the fixed locus in some \( M \). This seems difficult in this case without studying the concrete geometry. From [13] we can at least read of the Euler characteristic of the fixed locus: If \( M \) is of dimension \( 2n \), then \( e(M^g) \) is the coefficient of \( q^{n-1} \) of the series

\[
\frac{1}{\eta(q^2) \eta(q)^{11}} = \frac{1}{q^2} + 2 + 5q + 10q^2 + 20q^3 + 36q^4 + 65q^5 + 110q^6 + O(q^7).
\]

Since the Euler characteristic of a K3 surface is 24, we hence should expect 2-dimensional fixed components only in cases where \( \dim M \geq 10 \).

### APPENDIX A. HEARTS ON SYMPLECTIC SURFACES

Let \( S \) be a smooth projective symplectic surface and recall the notation from Section 6.2. The goal of this section is to prove the following result:

**Proposition A.1.** Let \( \sigma \in \text{Stab}^1(S) \) be a stability condition. Then there exists an element \( g \in \text{GL}^+(2, \mathbb{R}) \) such that \( g \sigma = (A, Z) \) satisfies

\[
D^b(A) \cong D^b(S).
\]

Let us first recall from [15] how the component \( \text{Stab}^1(S) \) is built up. First one considers the set \( V(S) \) of stability conditions \( \sigma_{\omega, \beta} = (A_{\omega, \beta}, Z_{\omega, \beta}) \) with central charge \( Z_{\omega, \beta} = (\exp(\beta + i\omega), \cdot) \) where \( \beta, \omega \in NS(S) \otimes \mathbb{R} \) with \( \omega \) ample. The heart \( \mathcal{A}_{\omega, \beta} \) is obtained from the torsion pair \( (T_{\omega, \beta}, F_{\omega, \beta}) \) of \( \text{Coh}(S) \) by tilting, see [15] Sec. 6. Next, let \( U(S) \) be the orbit of \( V(S) \)
under the free action of $\tilde{\GL}^r(2, \mathbb{R})$ on $\Stab^b(S)$. Elements in $U(S)$ are characterized as those stability conditions in $\Stab^b(S)$ such that all skyscraper sheaves are stable of the same phase. Finally, a detailed analysis of the boundary $\partial U(S)$ [13, Thm. 12.1] yields that any $\sigma \in \Stab^b(S)$ can be mapped into $\overline{U}(S)$ using (squares of) spherical twists. If $S$ is an abelian surface, then we even have $U(S) = \Stab^b(S)$ [13, Thm. 15.2].

We start the proof by considering the set of geometric stability conditions $V(S)$.

**Lemma A.2.** For all $\sigma = (A, Z) \in V(S)$ we have $D^b(A) \cong D^b(S)$.

**Proof.** Recall that a torsion pair $(T, F)$ of an abelian category $C$ is called cotilting, if for all $E \in C$ there is a surjection $F \to E$ with $F \in F$. By [13, Prop. 5.4.3], which is a refined version of [27], for any cotilting torsion pair $(T, F)$ one has $D^b(C') \cong D^b(C)$, where $C'$ is the tilt along $(T, F)$.

If $\sigma_{\omega, \beta} \in V(S)$, then its heart $A_{\omega, \beta}$ is obtained from $\text{Coh}(S)$ by tilting along the torsion pair $(T_{\omega, \beta}, F_{\omega, \beta})$. Huybrechts proved in [28, Prop. 1.2] that this torsion pair is cotilting. □

**Proposition A.3.** Let $\sigma \in V(S)$ and let $P$ be the associated slicing. Then for all $a \in \mathbb{R}$ there is a natural derived equivalence $D^b(P(a, a+1)) \cong D^b(S)$.

Since Lemma A.2 proves the assertion for $a = 0$ and the property is preserved by shifts, we only need to consider the case $a \in (0, 1)$. Write $\sigma = (A_{\omega, \beta}, Z_{\omega, \beta})$ and $A := P(a, a+1)$. Then

$$A \subset \langle A_{\omega, \beta}, A_{\omega, \beta}[1]\rangle$$

and $A$ is a tilt of $A_{\omega, \beta}$ for the torsion pair $T = A_{\omega, \beta} \cap A = P(a, 1]$ and $F = A_{\omega, \beta} \cap A[-1] = P(0, a]$. There is a natural exact functor

$$\Phi: D^b(A) \to D^b(A_{\omega, \beta}) \cong D^b(S)$$

of triangulated categories [43, Sec. 7.3]. The proof given below shows that this functor defines a derived equivalence.

**Proof of Proposition A.3**. The main idea in the proof is to show that $\Phi$ is essentially surjective. For this we make first some observations.

Take a very ample line bundle $O(1)$. The line bundle $O(-i)$ will lie in $F_{\omega, \beta}$ for $i \gg 0$. Recall from [13, Sec. 6] that the central charge $Z_{\omega, \beta}$ of the stability condition $\sigma_{\omega, \beta}$ sends an object $E \in D^b(S)$ with Mukai vector $v(E) = (r, l, s)$ to

$$Z_{\omega, \beta}(E) = -s + \frac{r}{2}(\omega^2 - \beta^2) + l\beta + i(l\omega - r\omega\beta).$$

Thus there exists an $i_0$ such that for all $i \geq i_0$ the object $O(-i)[1]$ lies in $P(0, a]$. Let us assume (after relabelling) that already $i_0 = 1$ is sufficient.

Consider a morphism of sheaves

$$O(-i)^{\oplus m} \to O(-j)^{\oplus n}.$$ 

Since $F_{\omega, \beta}$ is the free part of a torsion pair and hence closed under subobjects, the kernel $K = \text{Ker}(\alpha)$ lies in $F_{\omega, \beta}$. Similarly, $R = \text{Image}(\alpha)$ is a subsheaf of $O(-j)^{\oplus n}$ and lies in $F_{\omega, \beta}$. Therefore the distinguished triangle

$$K[1] \to O(-i)^{\oplus m}[1] \to R[1]$$
in $D^b(S)$ yields a short exact sequence in $\mathcal{P}[0,1]$. In particular, $K[1] \in \mathcal{P}[0,a]$.

Let $E \in D^b(S)$ be an object. Using the line bundles $O_E \cong E$ in the homotopy category $K(S) = K(\text{Coh}(S))$, where $O_E = (\ldots O_E^{-1} \to O_E \to \ldots)$ is a (possibly only bounded above) complex whose components are all direct sums of the line bundles $\mathcal{O}(i)$ for $i > 0$. Let $c$ be the smallest integer such that the cohomology $\mathcal{H}^c(E) \in \text{Coh}(S)$ is not isomorphic to zero. Define a new complex

$$F_E = (\ldots 0 \to \text{Ker}(\partial^{-1}) \to O_E^c \to O_E^{c+1} \to \ldots).$$

This is a subcomplex of $O_E$ which is bounded and the composition yields a quasi-isomorphism $F_E \cong E$.

From the above discussion we infer that $F_E[1]$ is a bounded complex whose all lie inside $\mathcal{P}[0,a]$. In particular, the complex $F_E[2]$ viewed inside $K^b(\mathcal{P}[1,1+a])$ is an element in $D^b(A)$. This shows that the realization functor

$$\Phi: D^b(A) \to D^b(\mathcal{P}[0,1]) \cong D^b(S)$$


**Corollary A.4.** For all $\sigma = (A, Z) \in U(S)$ we have $D^b(A) \cong D^b(S)$.

**Proof.** Any $\sigma \in U(S)$ is a $\widetilde{\text{GL}}^+(2,\mathbb{R})$-translate of a unique $\tau \in V(S)$. Thus we have $A = \mathcal{P}(a,a + 1)$ for some $a \in \mathbb{R}$, where $\mathcal{P}$ is the slicing corresponding to $\tau$. The assertion follows from Proposition [A.3] $\square$

**Proof of Proposition A.7** Corollary A.4 proves the assertion for abelian surfaces. Hence we can assume that $S$ is a K3 surface.

If $\Phi: D^b(S) \to D^b(S)$ is a derived auto-equivalence and $A \subset D^b(S)$ is a heart, then the restriction $\Phi|_A: A \to \Phi(A)$ induces an equivalence $D^b(A) \cong D^b(\Phi(A))$. Hence $D^b(A) \cong D^b(S)$ if and only if $D^b(\Phi(A)) \cong D^b(S)$. Moreover any auto-equivalence commutes with the $\widetilde{\text{GL}}^+(2,\mathbb{R})$-action. Since, as discussed earlier, any stability condition in $\text{Stab}^+(S)$ can be mapped by an auto-equivalence into the closure of $U(S)$, and we know the claim for elements in the interior of $U(S)$ by Corollary A.4 we may therefore assume that $\sigma$ lies on the boundary of $U(S)$.

As $\sigma$ is contained in $\overline{U(S)}$, all skyscraper sheaves $\mathbb{C}_x$ are semistable. After applying an element of $\widetilde{\text{GL}}^+(2,\mathbb{R})$ we may further assume that all skyscraper sheaves have phase 1 with respect to $\sigma$.

Following ideas of [6] we will consider a stability condition $\sigma' = (A', Z') \in U(S)$ such that skyscraper sheaves have slope 1 and approach $\sigma = (A, Z) \in \partial U(S)$ by first deforming only the real part of $Z'$ and afterwards the imaginary part of the central charge.

Concretely, consider the covering map $\pi: \text{Stab}^+(S) \to \mathcal{P}_0^+(S) \subset \text{NS}_\mathbb{Z} \otimes \mathbb{C}$ and choose an open ball $B \subset \mathcal{P}_0^+(S)$ of small radius containing $Z$. Choose a stability condition $\sigma' = (A', Z') \in U(S)$ such that skyscraper sheaves have slope 1 and such that the line from $Z'$ to $\mathbb{R}Z + \mathbb{Z}Z'$ and the line from $\mathbb{R}Z + \mathbb{Z}Z'$ to $Z$ viewed in the vector space $\text{NS}_\mathbb{Z} \otimes \mathbb{C}$ are contained inside $B$. Let $\tilde{Z}$ be the stability function $\mathbb{R}Z + \mathbb{Z}Z'$ and let $\tilde{\sigma} = (\tilde{A}, \tilde{Z})$ be the stability condition obtained from the covering property of $\pi$. By construction all skyscraper sheaves remain of phase 1 along this deformation from $\sigma$ to $\sigma'$.
The crucial observation now is that the stability condition $\tilde{\sigma}$ is still contained in the open subset $U(S)$. Indeed, recall that the set $U(S)$ can be characterized as the set of all stability conditions for which all skyscraper sheaves $C_x$ are stable of the same phase. Assume that a skyscraper sheaf $C_x$ becomes unstable along the line segment from $Z'$ to $\tilde{Z}$. Since semistability is a closed property, there would have to exist a $\tau$ on this line segment where $C_x$ becomes semistable. Since the imaginary part of the central charges stays constant along the path, $C_x$ is still contained in the abelian category $P(1)$, where $P$ is the slicing associated to $\tau$. As $C_x$ is semistable, there exists a stable object $F \in P(1)$ and a non-zero morphism $F \to C_x$ which is not an isomorphism. Since being stable is an open property [7, Prop. 2.10], the object $F$ was also stable for a stability condition on the line segment where $C_x$ is stable. However, a morphism between stable objects of the same phase is either an isomorphism or 0, yielding a contradiction. We conclude that $\tilde{\sigma} \in U(S)$.

Let $\tilde{P}$ be the the slicing associated to $\tilde{\sigma}$. Then as argued in [6, Lem. 5.2] the abelian category $\tilde{A} = \tilde{P}(1/2,3/2]$ is constant along a deformation that only changes the imaginary part of the stability condition. This yields $P(1/2,3/2] = \tilde{A}$, where $P$ is the slicing associated to $\sigma$.

Let $g \in \tilde{GL}^+(2,\mathbb{R})$ denote the rotation by $\pi/2$. Then $\tilde{A}$ is the heart of both $g\tilde{\sigma}$ and $g\sigma$. Since $\tilde{GL}^+(2,\mathbb{R})$ preserves $U(S)$, we have $g\tilde{\sigma} \in U(S)$ and therefore by Corollary A.4 we conclude $D^b(\tilde{A}) \cong D^b(S)$.

**Remark A.5.** Given an algebraic stability condition $\sigma = (A,Z) \in \text{Stab}^1(S)$, the proof above shows that in Proposition A.1 one can choose the element $g$ such that $g\sigma$ is algebraic as well. Indeed, this is immediate for stability conditions which are mapped by some auto-equivalence into $U(S)$. For $\sigma \in \partial U(S)$, we first applied an element from $\tilde{GL}^+(2,\mathbb{R})$ so that skyscraper sheaves get mapped to $-1$ and then applied the rotation by $\pi/2$. If $\sigma$ is algebraic, both steps can be achieved by multiplying $Z$ with elements from $\mathbb{Q} + i\mathbb{Q}$.

**References**


Universität Bonn, Mathematisches Institut
Email address: beckmann@math.uni-bonn.de

Universität Bonn, Mathematisches Institut
Email address: georgo@math.uni-bonn.de