Exercises to „Algebraic geometry I", 9

EXERCISE 1 (5 points). *Let* $\mathcal{R} \underset{b}{\overset{a}{\rightrightarrows}} \mathcal{S}$ *be a pair of morphisms of sheaves of rings on a topological space* X*. Show that there exists a sheaf of ideals* $\mathcal{I}$ *in* $\mathcal{S}$ *such that* $\mathcal{I}_x$ *is the ideal generated by*

$$\{a(r) - b(r) \mid r \in \mathcal{R}_x\}$$

in $\mathcal{S}_x$*. Moreover, show that* $\mathcal{S} \,/\, \mathcal{I}$ *is a coequalizer of the given pair of morphisms in the category of sheaves of rings.*

EXERCISE 2 (10 points). *Let the image* $\mathcal{I}$ *of a morphism* $\mathcal{M} \xrightarrow{f} \mathcal{N}$ *of sheaves of modules over a sheaf of rings* $\mathcal{R}$ *on a topological space* X *be*

$$\begin{aligned}\mathcal{I}(U) &= \{n \in \mathcal{N}(U) \mid \text{every } x \in U \text{ has an open neighbourhood } V \text{ in } \\ &\qquad U \text{ such that } n|_V \text{ in the image of } \mathcal{M}(U) \xrightarrow{f} \mathcal{N}(U)\} \\ &= \{n \in \mathcal{N}(U) \mid \text{for every } x \in U, \text{ the image of } n \text{ in } \mathcal{N}_x \text{ is} \\ &\qquad \text{in the image of } \mathcal{M}_x \xrightarrow{f} \mathcal{N}_x\}.\end{aligned}$$

Moreover, let $\operatorname{coker}(f) = \mathcal{N} \,/\, \mathcal{I}$ *and let* $\mathcal{N} \xrightarrow{q} \operatorname{coker}(f)$ *denote the projection.*

Show that:

- *The two characterizations of sections of* $\mathcal{I}$ *are equivalent.*
- $\mathcal{I}$ *is a sub-*$\mathcal{R}$*-module of* $\mathcal{N}$ *over which* f *factors, and any sub-*$\mathcal{R}$*-module of* $\mathcal{N}$ *over which* f *factors contains* $\mathcal{I}$*.*
- *We have the universal property that*

$$\begin{aligned}(1)\quad \operatorname{Hom}_{\mathcal{R}}(\operatorname{coker}(f), \mathcal{T}) &\to \operatorname{Ker}\big(\operatorname{Hom}_{\mathcal{R}}(\mathcal{N}, \mathcal{T}) \xrightarrow{\cdot \circ f} \operatorname{Hom}_{\mathcal{R}}(\mathcal{N}, \mathcal{T})\big) \\ \tau &\to \tau q\end{aligned}$$

  is bijective.
- *Show that forming images and cokernels of morphisms of* $\mathcal{R}$*-modules commutes with taking stalks.*

REMARK 1. The cokernel of a morphism $M \xrightarrow{f} N$ is, by definition, the quotient of N by the image of f. It is easy to verify that it enjoys a universal property similar to (1).

EXERCISE 3 (5 points). *Show that the functor* $M \to \tilde{M}$ *from* R*-modules to* $\mathcal{O}_{\operatorname{Spec} R}$*-modules commutes with taking kernels, cokernels and images.*

Remark 2. Such covariant functors between *abelian categories* (a class of categories suitable for doing homological algebra, and containing all categories of R-modules or $\mathcal{R}$-modules) are called *exact.* By earlier results on this and the previous sheet, the stalk functors are other examples of exact functors. Contravariant exact functors are studied as well (the opposite of an abelian category being abelian). Naturally, they exchange kernels and cokernels.

Solutions should be submitted Friday, December 22 in the lecture.