Exercises to „Algebraic geometry I“, 7

For a field $\mathfrak{k}$, we equip $\operatorname{Spec}\mathfrak{k}$ with the structure sheaf

$$\mathcal{O}_{\operatorname{Spec}\mathfrak{k}}(U) = \begin{cases} \mathfrak{k} & U = \operatorname{Spec}\mathfrak{k} \\ \{0\} & U = \emptyset. \end{cases}$$

It is easy to see that there is a unique way of defining the restriction homomorphisms and that $\operatorname{Spec}\mathfrak{k}$ becomes a locally ringed space.

For a locally ringed space X, let $[X]$ be the underlying set.

Exercise 1 (10 points). *For a locally ringed space, give a description of* $\operatorname{Hom}_{\mathrm{LRS}}(\operatorname{Spec}\mathfrak{k}, X)$ *and use this description to prove the injectivity of*

$$\left[\operatorname{Ker}(X \underset{b}{\overset{a}{\rightrightarrows}} Y)\right] \to \operatorname{Ker}\left([X] \underset{[b]}{\overset{[a]}{\rightrightarrows}} Y\right)$$

for a pair $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ *of morphisms in* LRS *and the surjectivity of*

$$\left[X \underset{S}{\times} Y\right] \to [X] \underset{[S]}{\times} [Y]$$

for morphisms $X \xrightarrow{\xi} S \xleftarrow{v} Y$ *in* LRS *without using our explicit descriptions of equalizers or fibre products in that category (but using their existence and potentially other facts presented in the lecture).*

Exercise 2 (5 points). *Let $\mathcal{R}$ be a sheaf of rings on the topological space X. For each $x \in X$, let an ideal $\mathcal{I}_{[x]}$ in the stalk $\mathcal{R}_x$ be given. Show that the following conditions are equivalent:*

- *There exists a sheaf of ideals $\mathcal{I} \subseteq \mathcal{R}$ such that the canonical embedding $\mathcal{I}_x \to \mathcal{R}_x$ has image $\mathcal{I}_{[x]}$.*
- *For every $x \in X$ and every $i \in \mathcal{I}_{[x]}$ there are an open neigbourhood U of x in X and $\iota \in \mathcal{R}(U)$ such that $\iota_x = i$ and such that for every $y \in U$, $\iota_y \in \mathcal{I}_{[y]}$.*

Also, show that $\mathcal{I}$ is unique if it exists.

Definition 1. Let $\mathcal{C} \xrightarrow{F} \mathcal{A}$ be a functor. A *limit* of F is an object L of $\mathcal{A}$ together with morphisms $L \xrightarrow{\lambda_C} F(C)$ for every object C of $\mathcal{C}$ such that for every morphism $C \xrightarrow{\gamma} D$ in $\mathcal{C}$, we have $\lambda_D = F(\gamma)\lambda_C$ and such that universal property for such data holds: If T is any object of $\mathcal{A}$ together with morphisms $T \xrightarrow{\tau_C} F(C)$ for every object C of $\mathcal{C}$ such that for every morphism $C \xrightarrow{\gamma} D$ in $\mathcal{C}$, we have $\tau_D = F(\gamma)\tau_C$, then there exists a unique morphism $T \xrightarrow{t} L$ in $\mathcal{A}$ such that $\tau_C = \lambda_C t$ holds for every object C of $\mathcal{C}$.

Remark 1. A convenient way of formulating the sheaf axiom when the existence of the required products in the target category is uncertain is as follows: A presheaf $\mathcal{G}$ on a topology base $\mathfrak{B}$ satisfies the sheaf axiom if for every open covering $\mathcal{U}$ of $U \in \mathfrak{B}$ by elements of $\mathfrak{B}$, $\mathcal{G}(U)$ together with the obvious restriction maps is a limit of $\mathcal{G}(V)$ where V runs over all elements of $\mathfrak{B}$ contained in some element of $\mathcal{U}$.

Definition 2. We say that $\mathcal{A}$ *has finite limits* if in the situation of the previous definition, a limit exists whenever $\mathrm{Ob}(\mathcal{C})$ is a finite set and for arbitrary objects X and Y, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is finite. We say that $\mathcal{C}$ *has arbitrary limits* if the limit exists whenever $\mathcal{C}$ is a small category.

The following finishes much of the material about general categories used in this lecture:

Exercise 3 (5 points). *For a category $\mathcal{A}$, show that the following conditions are equivalent:*

- *$\mathcal{A}$ has finite products and equalizers of arbitrary morphism pairs $X \underset{b}{\overset{a}{\rightrightarrows}} Y$.*
- *$\mathcal{A}$ has a final object and the fibre product $X \times_S Y$ exists for every diagram $X \xrightarrow{\xi} S \xleftarrow{v} Y$ in $\mathcal{A}$.*
- *$\mathcal{A}$ has arbitrary finite limits.*

Remark 2. The result still holds, with essentially the same proof, when „finite“ is dropped in the first and third point while fibre products with infinitely many factors are allowed in the second.

Solutions should be submitted Friday, December 8 in the lecture.