Exercises to „Algebraic geometry I“, 12

DEFINITION 1. For locally ringed spaces X and T, let $X(T)$ denote the set of morphisms $T \to X$ in the category of locally ringed spaces.

REMARK 1. At least in the case of preschemes, this is often called the set of *T-valued points* of X. The same terminology and notation is often employed when only morphisms in the category of preschemes over a fixed base S are considered.

EXERCISE 1 (2 points). *Let X and T be locally ringed spaces. Show that*

$$U \to X(U)$$

for open subsets $U \subseteq T$, together with the restricion of morphisms of locally ringed spaces to open subsets $V \subseteq U$, is a sheaf of sets on the topological space underlying T.

EXERCISE 2 (4 points). *Let*

$$\begin{tikzcd} X_T \arrow[r, "\xi_T"] \arrow[d] & T \arrow[d] \\ X \arrow[r, "\xi"] & S \end{tikzcd} \tag{1}$$

be a commuative diagram in the category of locally ringed spaces. Assume that T may be covered by open subsets U for which

$$\begin{tikzcd} \xi_T^{-1}U \arrow[r, "\xi_T"] \arrow[d] & U \arrow[d] \\ X \arrow[r, "\xi"] & S \end{tikzcd}$$

is Cartesian. Show that the original diagram (1) *is Cartesian!*

On the previous exercise sheet we did not investigate the question of the functorial behavious of Proj in its ring argument. The situation is more complicated than for Spec. While the preimage of a homogenuous prime ideal of $S.$ under a morphism $R. \to S.$ of graded rings is a homgenuous prime ideal, the requirement of failing to contain the augmentation ideal may be lost.

DEFINITION 2. Let $\mathcal{C}$ be the category with $\mathbb{N}$-graded rings as objects and morphisms $R. \xrightarrow{\sigma} S.$ of graded rings such that $S.$ is generated as an S_0-algebra by the image of σ.

Exercise 3 (8 points). *Equip the construction* Proj *of the previous exercise sheet with the structure of a functor from* $\mathcal{C}^{\mathrm{op}}$ *to schemes.*

Exercise 4 (4 points). *Construct a natural transformation*

$$\mathrm{Proj}(R.) \to \mathrm{Spec}(R_0)$$

of functors from $\mathcal{C}^{\mathrm{op}}$ *to schemes for which the underlying continuous map sends* $\mathfrak{p} \in \mathrm{Proj}(R.)$ *to* $\mathfrak{p} \cap R_0 \in \mathrm{Spec}(R_0)$.

Exercise 5 (3 points). *Show that for an object* $R.$ *of* $\mathcal{C}$ *and an (ungraded)* R_0*-algebra* A*, one has a Cartesian diagram*

$$\begin{array}{ccc} \mathrm{Proj}\big(R. \otimes_{R_0} A\big) & \longrightarrow & \mathrm{Proj}(R.) \\ \downarrow & & \downarrow \\ \mathrm{Spec} A & \longrightarrow & \mathrm{Spec} R_0 \end{array}$$

Exercise 6 (without points). *Construct an example of a ring* R *such that* $\mathrm{Spec} R$ *is a connected topological space but not irreducible or empty and for which the localizations* $R_{\mathfrak{p}}$ *at arbitrary* $\mathfrak{p} \in \mathrm{Spec} R$ *are domains.*

Remark 2. The topological space $\mathrm{Spec} R$ will be non-Noetherian, by what has been shown about integral schemes in the lecture. While I do not assign points to this exercise, I am (in the unlikely case that such a student needs them) probably willing to give bonus points to anyone who comes up with an example of his own.

Of the 21 points for this exercise sheet, one is a bonus point. Solutions should be submitted Friday, January 26 in the lecture.