Exercises to „Algebraic geometry I“, 11

EXERCISE 1 (3 points). *Let $R.$ be a $\mathbb{Z}$-graded ring which has a homogenuous element of degree $\neq 0$ which is a unit in R. Show that we have a bijection between* $\mathrm{Spec}R_0$ *and the set of homogenuous prime ideals in $R.$ sending $\mathfrak{p} \in \mathrm{Spec}R_0$ to $\mathfrak{q} = \sqrt{\mathfrak{p} \cdot R.}$ and the homogenuous prime ideal $\mathfrak{q} \subseteq R.$ to $\mathfrak{p} = \mathfrak{q} \cap R_0$.*

EXERCISE 2 (6 points). *Let $R.$ be an $\mathbb{N}$-graded ring. Let* $\mathrm{Proj}(R.)$ *denote the set of homogenuous prime ideals of $R.$ not containing R_+. For a homogenuous ideal $I \subseteq R.$, let $V(I)$ denote the set of all elements of* $\mathrm{Proj}(R.)$ *containing I. For a homogenuous element f of $R.$, let $V(f) = V(fR.)$.*

- *Show that there is a topology (called the Zariski topology) on* $\mathrm{Proj}(R.)$ *whose closed subsets are precisely the sets $V(I)$, for homogenuous prime ideals $\mathfrak{p}$.*
- *For $f \in R_d$ with positive d, construct a homeomorphism between* $\mathrm{Proj}(R) \setminus V(f)$ *and* $\mathrm{Spec}\big((R_f)_0\big)$.
- *Show that the open subsets of the form* $\mathrm{Proj}(R.) \setminus V(f)$, *with f as in the previous point, form a topology base of* $\mathrm{Proj}(R.)$ *and that $V(f) \supseteq V(g)$ if and only if some power of f is divisible by g.*

REMARK 1.
- The fact that prime ideals containing R_+ are excluded corresponds to the fact that in classical projective algebraic geometry we have $V\big(\mathfrak{k}[X_0, \ldots, X_n]_+\big) = \emptyset$. It is easy to see that the prime ideals containing R_+ are automatically homogenuous and that they are in canonical bijection with $\mathrm{Spec}(R_0)$.
- Note that the fact that f has positive degree is essential for the last claim, as (e. g.) $\mathrm{Proj}(R.)$ is empty when $R_+ = 0$, such that $V(g)$ may be empty for non-units g. However, g is allowed to be of degree 0.

It follows from the last point that the localization $(M.)_f$ of a graded $R.$-module $M.$ up to canonical isomorphism only depends on f. Let $\widetilde{M.}$ be the sheafification of the presheaf

$$\big(\mathrm{Proj}(R.) \setminus V(f)\big) \Rightarrow \big((M.)_f\big)_0$$

on the topology base of the last point of the previous exercise. In the case where $M. = R.$ this has the structure of a sheaf of rings, as $\big((R.)_f\big)_0$ is a ring. We denote this sheaf of rings by $\mathcal{O}_{\mathrm{Proj}R.}$.

EXERCISE 3 (6 points).
- *Show that under the homeomorphism of the second point, the restriction of $\widetilde{M.}$ to* $\mathrm{Proj}(R.) \setminus$

$V(f)$ is isomorphic to the presheaf $\widetilde{\big((M.)_f\big)_0}$ on $\operatorname{Spec}\big((R.)_f\big)_0$, where in the case $M. = R.$ this isomorphism is an isomorphism of sheaves of rings. If follows that $\operatorname{Proj}(R.)$ is a prescheme and $\widetilde{M}.$ a quasi-coherent sheaf of modules on it.

- *Show that $\operatorname{Proj}(R.)$ is a scheme.*
- *Decide whether $\operatorname{Proj}(R.)$ is always quasi-compact.*

Remark 2.

- One easily derives that

$$\big(\mathcal{O}_{\operatorname{Proj} R.}\big)_{\mathfrak{p}} \cong \big((R.)_{\mathfrak{p}}\big)_0$$
$$\big(\widetilde{M}.\big)_{\mathfrak{p}} \cong \big((M.)_f\big)_0$$

where by convention in the graded case the localization $M_{\mathfrak{p}}$ at a homogenuous prime ideal inverts the *homogenuous* elements of $R \setminus \mathfrak{p}$.

- In the case $M. = R.[d]$, $\widetilde{M}.$ is the sheafification of

$$\big(\operatorname{Proj}(R.) \setminus V(f)\big) \Rightarrow \big((R.)_f\big)_d$$

and is denoted $\mathcal{O}(d)$. This is a line bundle when R_+ is generated by R_1, since in this case the open subsets $\operatorname{Proj}(R.) \setminus V(f)$ with $f \in R_1$, on which f^d is a free generator, cover $\operatorname{Proj}(R.)$. However, they are not line bundles in general. The assumption of R_+ being generated by R_1 is typically but not always satisfied. For instance, it is perfectly reasonable (and sometimes useful) to study weighted projective spaces $\operatorname{Proj}(\mathfrak{k}[X_0, \dots, X_n])$ where the X_i have differing weights.

Exercise 4 (5 points). *Let*

$$\begin{array}{ccc} X_T & \longrightarrow & T \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \tau} \\ X & \longrightarrow & S \end{array}$$

be a Cartesian square of preschemes. For the sheaves of Kähler differentials and for the pull-back functors of quasicoherent sheaves of modules constructe on the previous exercise sheet, construct an isomorphism $\xi^*\Omega_{X/S} \cong \Omega_{X_T/T}$.

Solutions should be submitted Friday, January 19 in the lecture.