

Nineth exercise sheet “Class field theory” summer term 2025.

Let G be a finite group with n elements acting by K -linear automorphisms on a finite-dimensional K -vector space V . If n is prime to the characteristic of K and $W \subseteq V$ a G -invariant subspace then there is a projection $V \xrightarrow{p} W$ commuting with the G -action. For instance, one can take

$$p(v) = \frac{1}{n} \sum_{\gamma \in G} \gamma \pi(\gamma^{-1}v)$$

where $V \xrightarrow{\pi} W$ is any projection. It easily follows that V can be decomposed into a sum of non-zero G -invariant subspaces which are *irreducible* in the sense that they contain no G -invariant subspace besides $\{0\}$ and themselves.¹

Remark 1. *This is known as Maschke’s theorem and can be used in the solutions to the following problems. A counterexample in the case where the group order is a multiple of the characteristic is given by the group of matrices $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ with k running over \mathbb{F}_p acting on \mathbb{F}_p^2 .*

Let N be prime to the characteristic of the field K and assume that K contains all N -th roots of 1 in its algebraic closure. Thus the group μ_N of N -th roots of 1 in K is cyclic of order N . Let L/K be a Galois extension which is cyclic of order N and let σ be a generator of $G = \text{Gal}(L/K)$. For $\zeta \in \mu_N$ we put

$$L_\zeta = \{l \in L \mid \sigma(l) = \zeta l\}$$

Problem 1 (6 points). *Show that the subspaces L_ζ are of dimension 1 and are the only irreducible G -invariant subspaces of the K -vector space L !*

Problem 2 (2 points). *In the situation of the previous problem, show that every proper G -invariant subspace of L is contained in $\bigoplus_{\eta \in \mu_n \setminus \{\zeta\}} L_\eta$, for some $\zeta \in \mu_n$.*

It easily follows that L/K has a normal base because we have

Problem 3 (1 point). *Let L/K be a finite Galois extension. Show that $l \in L$ defines a normal base of L/K if and only if l is contained in no proper $\text{Gal}(L/K)$ -invariant subspace of the K -vector space L !*

In the following problem we do *not* assume that L has a non-trivial $\sqrt[p]{1}$.

¹We adopt the convention that $\{0\}$ is never irreducible.

Problem 4 (5 points). *Let L/K be a Galois extension whose degree p is a prime number different from the characteristic of K . Show that it has a normal base!*

Together with problem 3 from sheet 5 this finishes the proof of the existence of normal bases in the case of Galois extensions of prime degree. This is sufficient for our purposes. As explained at the end of sheet 6 we already have a proof of vanishing of \hat{H}^* for the additive group of a finite Galois extensions which does not use normal bases.

From now on, let K be a local field and n a positive integer such that $\mu_n = \{x \in K \mid x^n = 1\}$ has n elements. In particular, n is not a multiple of the characteristic of K . The norm residue symbol $(\cdot, \cdot)_n$ has been used from Problem 8 of the previous sheet onwards in the proof of much of the existence theorem of local class field theory. But the main reasons for the importance of that symbol is its usefulness for deriving the classical reciprocity laws (including the quadratic reciprocity law) from the reciprocity law of global class field theory.

Problem 5 (5 points). *For $x \in K \setminus \{0; 1\}$, show that $(x, 1 - x)_n = 1$!*

Remark 2. *It all boils down to show that $x \in N_{L/K}(L^\times)$ where $L = K(\sqrt[n]{x})$. But some care is needed when there is an integer $a > 1$ dividing n and such that x is an a -th power in K . Solutions which only work when n is prime will be given 3 points when they are otherwise correct.*

Problem 6 (2 points). *Let K be a field, G an abelian group written multiplicatively and $K^\times \times K^\times \xrightarrow{(\cdot, \cdot)} G$ a \mathbb{Z} -bilinear map such that $(x, 1 - x) = 1$ when $x \in K \setminus \{0; 1\}$. Show that $(x, -x) = 1$ when $x \neq 0$!*

Problem 7 (2 points). *In the previous situation, show that $(x, y)(y, x) = 1$!*

Three of the 23 points from this sheet are bonus points which do not count in the calculation of the 50%-limit for passing the exercises module.

Solutions should be submitted to the tutor by e-mail before Tuesday June 24 24:00.