Exercises to „Algebraic Geometry II“, 9

The reduction to the case of projective morphisms of the proof of coherence of higher direct images of coherent sheaves under proper morphisms of locally Noetherian preschemes is by Chow's lemma.

Let R be a Noetherian ring and $X \xrightarrow{f} Y = \mathrm{Spec}R$ a proper morphism, where X is an integral (reduced and irreducible) scheme. Let $X = \bigcup_{i=1}^{m} U_i$ where $U_i \subseteq X$ is an affine open subset. Let $U_i \xrightarrow{\iota_i} X$ denote its open embedding to X.

Exercise 1 (1 point). Show that there is a closed embedding $U_i \xrightarrow{k_i} \mathbb{A}^{n_i}_R$.

Let $U = \bigcup_{i=1}^{n} U_i$ and $U \xrightarrow{\iota} X$ its open immersion to X. We view $\mathbb{A}^{n_i}_R$ as a closed subprescheme of $\mathbb{P}^{n_i}_R$ and consider the morphism

$$(1)\quad U \xrightarrow{k=(\iota,k_1,\ldots,k_n)} P = X \mathop{\times}_{Y} \mathbb{P}^{n_1}_Y \mathop{\times}_{Y} \ldots \mathop{\times}_{Y} \mathbb{P}^{n_m}_Y = \\ = \mathbb{P}^{n_1}_X \mathop{\times}_{X} \ldots \mathop{\times}_{X} \mathbb{P}^{n_m}_X$$

Let $\tilde{X}$ be the closure of its image, viewed as a reduced closed subprescheme of P. Let $\tilde{X} \xrightarrow{\pi} X$ be the composition of the closed embedding $\tilde{X} \to P$ with the projection $P \to X$.

Exercise 2 (4 points). Show that π is a projective morphism which maps $\pi^{-1}U$ isomorphically to U.

Let

$$P = X \mathop{\times}_{Y} \mathbb{P}^{n_1}_Y \mathop{\times}_{Y} \ldots \mathop{\times}_{Y} \mathbb{P}^{n_m}_Y \xrightarrow{q} Q = \mathbb{P}^{n_1}_Y \mathop{\times}_{Y} \ldots \mathop{\times}_{Y} \mathbb{P}^{n_m}_Y.$$

be the projection obtained by omitting the first factor of the fibre product and $\tilde{q}$ its restriction to $\tilde{X}$. We want to show that $\tilde{q}$ is a closed embedding, showing that $\tilde{X}$ is projective over Y.

Let

$$V_i = \mathbb{P}^{n_1}_Y \mathop{\times}_{Y} \ldots \mathop{\times}_{Y} \mathbb{A}^{n_i}_Y \mathop{\times}_{Y} \ldots \mathop{\times}_{Y} \mathbb{P}^{n_m}_Y \subset Q$$

when $1 \leq i \leq n$ and let $V_0 = Q \setminus \tilde{q}(\tilde{X})$.

Exercise 3 (3 points). Show that $j = (\iota_1, k_1, \mathrm{Id}_{\mathbb{P}^{n_2}_R}, \ldots, \mathrm{Id}_{\mathbb{P}^{n_m}_R})$ is a closed embedding

$$U_1 \mathop{\times}_{Y} \mathbb{P}^{n_2}_Y \mathop{\times}_{Y} \ldots \mathop{\times}_{Y} \mathbb{P}^{n_m}_Y \xrightarrow{j} X \mathop{\times}_{Y} \mathbb{A}^{n_1}_Y \mathop{\times}_{Y} \mathbb{P}^{n_2}_Y \mathop{\times}_{Y} \ldots \mathop{\times}_{Y} \mathbb{P}^{n_m}_Y = q^{-1}V_1$$

over which (1) factors.

Remark 1. A suitable approach is to apply the following.

Exercise 4 (3 points). Let $A \xrightarrow{j} B \xrightarrow{q} C$ be morphisms of preschemes where q is separated and qj a closed embedding. Show that j is a closed embedding.

Exercise 5 (5 points). Show that the V_i, $0 \leq i \leq n$, are open subsets of Q covering it.

Remark 2. For the fact that they cover Q, use the fact that the $\pi^{-1}U_i$ cover $\tilde{X}$.

Exercise 6 (4 points). Show that $\tilde{q}^{-1}V_i \to V_i$ is a closed embedding and conclude that $\tilde{X} \to Y$ is a projective morphism.

Remark 3. We have obtained a projective morphism $\tilde{X} \xrightarrow{\pi} X$ which is an isomorphism above some open dense subset of X and such that $\tilde{X} \xrightarrow{f\pi} Y$ is projective (and not only proper).

Remark 4. The fact (explained in the lecture) that there is a closed embedding from $\mathbb{P}^a_Y \times_Y \mathbb{P}^b_Y$ to $\mathbb{P}^{a+b+ab}_Y$ may be used here.

Remark 5. Solutions awarded the full number of points should explain briefly where the fact that f is proper is needed and whether the proof still works in the following situations:

- f is of finite type and universally closed.
- f is separated, of finite type and closed.

Solutions should be submitted Thursday, July 5, in the exercises.