Exercises to „Algebraic Geometry II“, 9

Exercise 1 (3 points). Let $\mathcal{L}$ be a line bundle on a locally ringed space X and $\mathcal{M}$ any $\mathcal{O}_X$-module. Show that there is a morphism

$$\mathcal{L}^{-1} \otimes \mathcal{M} \to \underline{\mathrm{Hom}}(\mathcal{L}, \mathcal{M})$$

of $\mathcal{O}_X$-modules sending, for $l \in \mathcal{L}^{-1}(U) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{L}|_U, \mathcal{O}_U)$ and $m \in \mathcal{M}(U)$, $l \otimes m$ to the morphism $\mathcal{L}|_U \xrightarrow{\phi} \mathcal{M}|_U$ defined by

$$\phi(\ell) = l(\ell) \cdot m|_V ,$$

for open $V \subseteq U$ and $l \in \mathcal{L}(V)$! Moreover, show that this morphism is an isomorphism!

Let $\mathfrak{k}$ be any field and $X = \mathbb{P}^1_{\mathfrak{k}}$.

Exercise 2 (4 points). For a vector bundle $\mathcal{V}$ on X, show that the set $\mathfrak{L}_{\mathcal{V}}$ of integers l for which there is a non-zero homomorphism $\mathcal{O}(l) \to \mathcal{V}$ is bounded from above!

Definition 1. A *subbundle* of a vector bundle $\mathcal{V}$ on a locally ringed space X is a $\mathcal{O}_X$-submodule $\mathcal{W} \subseteq \mathcal{V}$ such that both $\mathcal{W}$ and $\mathcal{V} / \mathcal{W}$ are vector bundles.

Exercise 3 (5 Points). In the situation of the previous exercise, assume that -1 is the largest element of $\mathfrak{L}_V$. As it is an element of this set, there is a non-zero morphism $\mathcal{O}(-1) \to \mathcal{V}$ of $\mathcal{O}_X$-modules.

- Show that any such morphism is an isomorphism of $\mathcal{O}(-1)$ with a subbundle $\mathcal{L}$ of $\mathcal{V}$, and show that $0 \notin \mathfrak{L}_{\mathcal{V}/\mathcal{L}}$!
- Moreover, show that all elements of $\mathfrak{L}_{\mathcal{V}/\mathcal{L}}$ are ≤ -1.

It follows from the second exercise on the previous sheet and from the description of line bundles on X recalled in the lecture this Thursday that $\mathcal{V}$ always has a sub-line bundle and that any such line bundle is isomorphic to $\mathcal{O}(l)$ for a unique integer l. This may be used in the proof of the following exercise:

Exercise 4 (Dedekind-Weber 1892, Grothendieck 1956, 4 points). Show that any vector bundle $\mathcal{V}$ on X admits a decomposition

$$\mathcal{V} \cong \oplus_{i=1}^{d} \mathcal{O}(k_i).$$

Remark 1. The idea is, of course, to use induction on the dimension d of $\mathcal{V}$. One first confirms that the question is invariant under the 'twist' by the line bundle $\mathcal{O}(n)$ replacing $\mathcal{V}$ by $\mathcal{V}(n) := \mathcal{V} \otimes \mathcal{O}(n)$, then uses this to reduce to the situation of the previous exercise. One then has to use the induction assumption to show that the extension of $\mathcal{V}/\mathcal{L}$ by $\mathcal{L} \cong \mathcal{O}(-1)$ defined by $\mathcal{V}$ splits.

As

$$\dim H^0\big(X, \mathcal{V}(k)\big) = \sum_{i=1}^{d} \max(k+1+k_i, 0),$$

it is easy to see that the sequence of the k_i is unique up to reordering.

This result about vector bundles on X has an equivalent formulation as a classification of invertible matrices over the ring $\mathfrak{k}[T, T^{-1}]$ due to Dedekind and Weber, which is quite old. There is also a similar result of Plemelj about invertible holomorphic matrices on $\mathbb{C} \setminus \{0\}$ which is equivalent to the classification of holomorphic bundles on $\mathbb{P}^1_{\mathbb{C}}$.

When $n \geq 2$, the theory of vector bundles on $\mathbb{P}^n_{\mathfrak{k}}$ is much more complicated.

Exercise 5 (4 points). Deduce the following result of Dedekind and Weber from the result of the previous exercise:

$$\mathrm{GL}_d\big(\mathfrak{k}[T, T^{-1}]\big) = \bigcup_{\vec{k} \in \mathbb{Z}^d} \mathrm{GL}_d\big(\mathfrak{k}[T]\big) \cdot \begin{pmatrix} T^{k_1} & 0 & \dots & 0 \\ 0 & T^{k_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T^{k_d} \end{pmatrix} \mathrm{GL}_d\big(\mathfrak{k}[T^{-1}]\big)$$

Solutions should be submitted Thursday, June 28, in the exercises.