Eighth exercise sheet "Class field theory" summer term 2025. In the case of Archimedean local fields, the only non-trivial case in the construction of the isomorphism (1) from sheet 7 is that of \mathbb{C}/\mathbb{R} , which is easy to do.

Problem 1 (4 points). Calculate $H^2(\mathbb{C}/\mathbb{R}, \mathbb{C}^{\times})!$

Most other results from local class field theory are trivial in the Archimedean case. In the solution to the following problems, it is therefore allowed ¹ to discuss the non-Archimedean case only, although the claims will hold as stated. In the non-Archimedan case, we put $K^o = \{x \in K \mid |x| \leq 1\}$. In addition to the results of local class field theory stated at the end of the previous sheet, we will also need the fact that

$$\psi_{L/K}(k) = \rho(\psi_{M/K}(k))$$

for finite Galois extensions $L \subseteq M$ of the local field K and $k \in K^{\times}$, where $\operatorname{Gal}(M/K)_{ab} \xrightarrow{\rho} \operatorname{Gal}(L/K)_{ab}$ is the group homomorphisms obtained from $\operatorname{Gal}(M/K) \to \operatorname{Gal}(L/K)$ (the restriction to L).

Note that it follows from the isomorphism (3) from the last sheet that $[K^{\times} : N_{L/K}L^{\times}] < \infty$ in the case of finite Galois extensions of local fields.

Problem 2 (1 point). Show that $[K^{\times} : N_{L/K}L^{\times}] < \infty$ for finite separable extensions of local fields!

Using Problem 4 of sheet 2 and the compactness of the group of units $L^{o\times}$ one gets

Problem 3 (5 points). For finite separable extensions L/K of local fields, show that $N_{L/K}L^{\times}$ is an open subgroup of K^{\times} !

Remark 1. Note that when [L:K] is prime to the characteristic then openness can also be shown using $K^{\times[L:K]} \subseteq N_{L/K}L^{\times}$ and showing the openness of the first group by using Hensel's Lemma. But this does not work in the case of local fields of function fields of characteristic p as $K^{\times p}$ fails to be open in K^{\times} in that case.

Remark 2. If the characteristic p is positive and $q \in p^{\mathbb{N}}$ then K has precisely one purely inseparable extension of degree q, namely $L = K^{1/q}$. We have $L^q = K$ and $N_{L/K}(x) = x^q$, hence $N_{L/K}L^{\times} = K^{\times}$. Using this it is easy to show that the separability assumption in the previous two problems can be dropped.

 $^{^{1}\}mathrm{Of}$ course with the exception of Problem 5 which only makes an assertion about the Archimedian case

Problem 4 (5 points). Let L/K be a finite Galois extension of local fields with Galois group G, U a subgroup of K^{\times} containing $N_{L/K}L^{\times}$, $H \subseteq G$ the inverse image under $G \to G_{ab}$ of $\psi_{L/K}(U) \subseteq G_{ab}$ and $E = L^H$ the fixed field of H. Show that $N_{E/K}E^{\times} = U!$

Remark 3. Recall that the kernel [G, G] of $G \to G_{ab}$ is the subgroup generated by elements of the form $g\gamma g^{-1}\gamma^{-1}$ with g and γ running over G. If $U = N_{L/K}L^{\times}$ then H = [G, G] and $E = L^H$ is the maximal subextension of L/K which is Galois over K abelian Galois group. The result then is $N_{E/K}E^{\times} = N_{L/K}L^{\times}$, which is sometimes called the limitation theorem of local class field theory.

As a result of the previous discussion (using Remark 2 for the inseparable case), for a subgroup U of the multiplicative group K^{\times} of a local field, the following conditions are equivalent:

- There is a finite abelian extension L/K such that $U = N_{L/K}L^{\times}$.
- There is a finite extension L/K such that $N_{L/K}L^{\times} \subseteq U$.

Such U are called *norm subgroups* and the previous discussion shows that they are open and of finite index in K^{\times} . The *existence theorem* of local class field theory is the assertion that every open subgroup of finite index in K^{\times} is a norm subgroup.

Problem 5 (1 point). Show that the existence theorem holds when $K = \mathbb{R}$.

Problem 6 (1 point). Let U be a subgroup of K^{\times} whose preimage under $L^{\times} \xrightarrow{N_{L/K}} K^{\times}$ is a norm subgroup of L^{\times} . Show that U is a norm subgroup of K^{\times} !

Let K be a local field of characteristic prime to n and such that K contains the group μ_n of solutions ζ to $\zeta^n = 1$ in the algebraic closure of K.

Problem 7 (5 points). Show that $K^{\times n}$ is an open subgroup of finite index in $K^{\times !}$

If $x \in K^{\times}$ then $L = K(\sqrt[n]{x})$ is a Galois extension of K with Galois group G isomorphic to a subgroup of μ_n by the homomorphism

$$G \xrightarrow{\iota_{x,n}} \mu_n$$

sending $\sigma \in G$ to the unique $\zeta \in \mu_n$ such that $\sigma(\xi) = \zeta \xi$ when $\xi^n = x$. We put

$$(x,y)_n = \iota_{x,n}(\psi_{L/K}(y))$$

It is clear that this *norm residue symbol* is multiplicative in y.

Problem 8 (3 points). Show that $(x, y)_n$ is multiplicative in x! In other words, show that the equation

$$(x,y)_n(\xi,y)_n = (x\xi,y)_n$$

holds!

With an appropriate definition of non-degenerateness, the following also holds for composite numbers n. But for the sake of brevity we restrict our considerations to the case where n = p is a prime number different from the characteristic of the local field K.

Problem 9 (3 points). Show that $(\cdot, \cdot)_p$ is a non-degenerate μ_p -valued bilinear form on the \mathbb{F}_p -vector space $K^{\times}/K^{\times p}$!

Problem 10 (3 points). Let L be the extension of K obtained by adjoining a $\sqrt[p]{x}$ for all $x \in K$. Show that $N_{L/K}(L^{\times}) \subseteq K^p$!

Actually $N_{L/K}L^{\times} = K^{\times p}$ but the stated assertion is sufficient to see that $K^{\times p}$ is a norm subgroup.

Problem 11 (2 points). Show that the intersection of two norm subgroups is a norm subgroup!

Problem 12 (6 points). Let K be local field and U an open subgroup of K^{\times} such that $[K^{\times} : U]$ is finite and prime to the characteristic of K. Show that U is a norm subgroup!

In particular, the existence theorem of local class field theory holds in the number field case. Of course this proof depends on the results about the Brauer group $H^2(L/K, L^{\times})$ stated at the end of the last exercise sheet.

From the 38 points from this sheet, 18 are bonus points which do not count in the calculation of the 50%-limit for passing the exercises module. Solutions should be submitted to the tutor by e-mail before Tuesday June 17 24:00.