EXERCISES TO "ALGEBRA II", 8

Exercise 1 (3 points). *Prove or disprove: Any ideal I in a domain R is a locally free R-module.*

Exercise 2 (4 points). *Let R be a ring, A and S R-algebras, $B = A \otimes_R S$ and M an A-module. Show that there is an isomorphism $M \otimes_R S \cong M \otimes_A B$ sending $m \otimes s$ to the tensor product of m with the image of s in B with an inverse sending $m \otimes (a \otimes s)$ to $(ma) \otimes s$.*

Exercise 3 (4 points). *In the situation of the previous exercise, show that there is a canonical isomorphism $\Omega_{B/S} \cong \Omega_{A/R} \otimes_R S$ sending $d_{B/S}(a \otimes s)$ to $d_{A/R}(a) \otimes s$ with an inverse sending $d_{A/R}(a) \otimes s$ to $d_{B/S}(a \otimes s)$.*

In the case of a finite inseparable field extension $\mathfrak{l}/\mathfrak{k}$, $\Omega_{\mathfrak{l}/\mathfrak{k}} \neq 0$ (shown in the previous exercises) implies that $\mathfrak{l}/\mathfrak{k}$ is not smooth in the sense defined in the lecture. In this case it can be shown that $\mathfrak{l} \otimes_{\mathfrak{k}} \overline{\mathfrak{k}}$ has nilpotent elements and fails to be regular. These properties are inherited by the polynomial ring $A = \mathfrak{l}[X_1, \ldots, X_n]$ for which $\Omega_{A/\mathfrak{k}}$ is a free module of rank bigger than n and for which regularity is also destroyed by a base change from $\mathfrak{k}$ to $\overline{\mathfrak{k}}$ which turns out to produce nilpotent elements in the ring. These are the paradigmatic examples of regular $\mathfrak{k}$-algebras of finite type which are not smooth over $\mathfrak{k}$. However, there are more complicated examples in which the failure of smoothness for a regular $\mathfrak{k}$-algebra A occurrs only at a proper closed subset of Spec$A$.

Exercise 4 (5 points). *Let $\mathfrak{k}$ be a field of characteristic $p > 0$ and τ an element of $\mathfrak{k}$. Let $A = \mathfrak{k}[X, Y] \,/\, (X^p - XY^p - \tau)$.*

- *Show that A is a regular ring if and only if τ is not a p-th power in $\mathfrak{k}$.*
- *Calculate $\Omega_{A/\mathfrak{k}}$ and describe the set of elements of* Spec$A$ *where it fails to be locally free.*

Remark 1.
- It may be used without any comment in the soltuation that the dimension of A is one, als this follows from the principal ideal theorem (in its easy special case of factorial domains) and basic dimension theory.
- As we do not assume $\mathfrak{k}$ to be perfect, the Jacobian criterion for regularity may fail, but it can fail only in one direction. In the above example where we have a single equation P, it is easy to see (and should be pointed out briefly in solutions) that there still is a one-direction implication between $P \in \mathfrak{m}_x^2$ (for $x \in \overline{\mathfrak{k}}^2$) and the vanishing of P and grad$P$ at x. It is then sufficient to

investigate regularity at the $\mathfrak{m}_x$ where it cannot be decided in this way.

- When $\mathfrak{l}/\mathfrak{k}$ is a field extension containing a $\sqrt[p]{\tau}$,
$$B = \mathfrak{l} \otimes_{\mathfrak{k}} A \cong \mathfrak{l}[X,Y] \,\big/\, (X^p - XY^p - \tau)$$
will (by the above results) become non-regular.
- In view of the base-change compatibility of Kähler differentials proved before, smoothness over $\mathfrak{k}$ is preserved by base-change to field extensions $\mathfrak{l}/\mathfrak{k}$. That this does not hold for regularity in the case of inseparable extensions has been pointed out in the previous remark and before the previous exercise. In fact, any regular $\mathfrak{k}$-algebra of finite type which is not smooth will become non-regular after base-change to an appropriate inseparable extension of $\mathfrak{k}$.

The following remark assumes some familiarity with the theory of extensions of Dedekind domains (introduced in Algebraic Number Theory textbooks, lectures or seminars) and the notion of the length of a module and may be skipped by those unfamiliar with it.

Remark 2. If $B \subseteq A$ denotes $\mathfrak{k}[Y^2]$ then $\Omega_{B/A}$ is a non-cyclic A-module and its annihilator is larger than the Dedekind different $\mathfrak{D} = \mathfrak{D}_{A/B}$, in contrast with the situation for separable residue field extensions (e. g., Neukirch, Algebraic Number Theory, Proposition III.2.7). The length of this module is however still equal to the valuation exponent of $\mathfrak{D}$ as this relation between the Dedekind different (obtained by a *dualizing* construction) and Kähler differentials can be derived for general extensions of Dedekind domains (with a separable extension of fields of quotients) from a suitable version of Grothendieck-Serre *duality* in which a suitable determinant bundle of Kähler differentials (or a relative cotangent complex related to them) plays a role.

Exercise 5 (4 points). *Let $\mathfrak{k}$ be an algebraically closed field and $I \subseteq R = \mathfrak{k}[X_0, \ldots, X_n]$ a homogenuous ideal such that $I = \sqrt{I}$. Show that the projective set of zeroes $V(I)$ is irreducible if and only if I is a prime ideal different from R_+.*

Solutions should be submitted Monday, December 11, in the lecture.