Remark 1. *Recall that the following conditions to a topological space X are equivalent:*

- *There is a topology base $\mathfrak{B}$ closed under finite intersections in X and such that all elements of $\mathfrak{B}$ are quasicompact.*
- *The set $\mathfrak{Qc}(X)$ of quasicompact open subsets of X is such a topology base.*

The following two problems provide the proof of the Sura-Bura theorem omitted from the lecture:

Problem 1 (4 points). *Let X be a quasicompact topological space with the following property:*

> *If Q is a quasi-component of X and $Q = A \cup B$ where A and B are disjoint closed subsets of X then there are disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.*

Show that every quasi-component of X is connected.

The existence of U and V is obvious when X is T_4. By a well-known result of general topology, a quasicompact space is T_4 if and only if it is T_2. Thus, for compact spaces the spaces of connected components and quasicomponents are canonically homeomorphic and compact (in particular, Hausdorff). That the same holds for spectral spaces follows from

Problem 2 (4 points). *Let X be a topological space satisfying the equivalent conditions of Remark 1. Show that Problem 1 can be applied to X.*

Problem 3 (7 points). *From the lecture, recall the bijection beween closed subsets $Z \subseteq \mathrm{Spec} R$ and ideals I such that $I = \sqrt{I}$, sending I to $V(I)$ and Z to $\bigcap_{\mathfrak{p} \in Z} \mathfrak{p}$. Show that an ideal $\mathfrak{p}$ of R is a prime ideal if and only if $Z = V(\mathfrak{p})$ is irreducible. In this case, show that $X = \overline{\{\mathfrak{p}\}}$!*

Definition 1. *Let K be a field. A* valuation ring *of K is a subring $R \subseteq A$ such that for all $k \in K^{\times}$, $R \cap \{k; k^{-1}\}$ is not empty.*

Problem 4 (8 points). *For a valuation ring R of a field K, show the following:*

- *For two ideals I and J of R, $I \subseteq J$ or $J \subseteq I$.*
- *R is a local ring with maximal ideal*

$$\mathfrak{m} = \big\{x \in K \;\big|\; x = 0 \text{ or } x^{-1} \notin R\big\}.$$

- *An ideal $I \subseteq R$ is a prime ideal if and only if it is proper and $I = \sqrt{I}$.*

- *R is integrally closed in K.*
- *R is Noetherian if and only if $R = K$ or R is a principal ideal domain with precisely one prime element.*[1]

Three of the 23 points from this sheet are bonus points with are disregarded in the determination of the $\geq 50\%$-threshold for passing the exercise module.

Solutions should be submitted before Thursday December 11. Good luck!

[1] In other words, a discrete valuation ring