Exercises to „Algebraic Geometry II“, 7

Let X be a locally ringed space.

DEFINITION 1.
- Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{O}_X$-modules. Let $\mathcal{H} = \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B})$ be the $\mathcal{O}_X$-module defined by
$$\mathcal{H}(U) = \mathrm{Hom}_{\mathcal{O}_U}\bigl(\mathcal{A}|_U, \mathcal{B}|_U\bigr).$$
- An extension of $\mathcal{B}$ by $\mathcal{A}$ is a short exact sequence $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{B} \to 0$. A morphism of extensions of $\mathcal{B}$ by $\mathcal{A}$ from $(\mathcal{E}, \alpha, \beta)$ to $(\tilde{\mathcal{E}}, \tilde{\alpha}, \tilde{\beta})$ is a morphism $\mathcal{E} \xrightarrow{e} \tilde{\mathcal{E}}$ such that $e\alpha = \tilde{\alpha}$ and $\tilde{\beta}e = \beta$.

EXERCISE 1 (12 points).
- Show that all morphisms of extensions are isomorphisms.
- In the case where X is a quasi-compact scheme and $\mathcal{V}$ a vector bundle on X and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{A}$, show that $\underline{\mathrm{Hom}}(\mathcal{V}, \mathcal{A})$ is quasi-coherent and construct a bijection between $H^1\bigl(X, \underline{\mathrm{Hom}}(\mathcal{V}, \mathcal{A})\bigr)$ and the isomorphism classes of extensions of $\mathcal{V}$ by $\mathcal{A}$.

REMARK 1. Most of the points (e. g., 8–9) should be awarded for the construction of the bijection of the second point. This has been sketched in the lecture, but the construction of the functors between extensions and torseurs should be described reasonably carefully in both directions. The well-known facts (which have been used in the lecture) about sheafification of presheaves defined on a topology base may be used liberally.

EXERCISE 2 (4 points). Let X be a regular one-dimensional connected Noetherian scheme, η its general point, $K = \mathcal{O}_{X,\eta}$ the stalk of the structure sheaf at η (a regular zero-dimensional local ring, hence a field), $\mathcal{V}$ a vector bundle on X, V the K-vector space $\mathcal{V}_\eta$, $W \subseteq V$ a sub K-vector space and let

$$\mathcal{W}(U) = \bigl\{ v \in \mathcal{V}(U) \bigm| U = \emptyset \text{ or the image of } v \text{ under } \mathcal{V}(U) \to V \text{ is an element of } W. \bigr\}$$

Show that $\mathcal{W}$ and $\mathcal{V} \,/\, \mathcal{W}$ are vector bundles.

EXERCISE 3 (4 points). Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$, M a finitely generated R-module and $\vec{x} = (x_1, \ldots, x_n)$ a sequence of elements of $\mathfrak{m}$ such that the cohomology $H^p(\vec{x}, M)$ of the Koszul complex vanishes when $p \neq n$. Show that the sequence is $\mathfrak{m}$-regular.

Solutions should be submitted Thursday, June 21, in the exercises.