Exercises to "Algebra II", 7

Definition 1. Let R and S be rings which are not necessarily commutative. An (R, S)-bimodule is an abelian group N with multiplication maps $R \times N \to N$ defining the structure of a left R-module and $N \times S \to N$ defining the structure of a right S-module on N, and such that we have $(r \cdot n) \cdot s = r \cdot (n \cdot s)$ for arbitrary $n \in N$, $r \in R$ and $s \in S$.

Exercise 1 (3 points)**.** *In the above situation, let M be a right R-module and N an (R, S)-bimodule. Show that there is a unique structure of right S-module on the abelian group $M \otimes_R N$ such that we have $(m \otimes n) \cdot s = m \otimes (n \cdot s)$ for arbitrary $m \in M$, $n \in N$ and $s \in S$.*

Remark 1. When Q is another ring and M has the additional structure of an (Q, R)-bimodule then one has a unique structure of a (Q, S)-bimodule on $M \otimes N$ such that $q \cdot (m \otimes n) = (q \cdot m) \otimes n$. When N is only a left R-module, this still defines the structure of a left Q-module on $M \otimes N$.

In the commuative case, every left R-module is also a right R-module and an (R, R)-bimodule by $m \cdot r := r \cdot m$. This gives the tensor product the structure of an R-module:

Exercise 2 (1 point)**.** *Let R be a commutative ring, M and N R-modules. Show that the (R, R)-bimodule structure on $M \otimes_R N$ defined by the previous exercise and the subsequent remarks satisfies $r \cdot (m \otimes n) = (m \otimes n) \cdot R$.*

Remark 2. In other words, it comes from the structure of an ordinary R-module in the aforementioned way.

Exercise 3 (2 points)**.** *In the situation of the previous exercise, show that $M \times N \to M \otimes N$, $(m, n) \to m \otimes n$ has the universal property for R-bilinear maps $M \times N \to T$ stated in the lecture.*

It is OK to do the following two exercises in the commutative case only (as they are formulated) but the results still hold in the non-commutative case, with essentially the same proofs. All tensor products are over R and we drop the subscript R, which is often done when there is no ambiguity about the ground ring.

Exercise 4 (4 points)**.** *Let $M' \subseteq M$ be a submodule of the R-module M, let N be any R-module and let $K \subseteq M \otimes N$ be the image of the R-linear map $M' \otimes N \to M \otimes N$ sending the element $m \otimes n$ of $M' \otimes N$ to the element $m \otimes n$ of $M \otimes N$. Show that the R-bilinear map $(M/M') \times N \to Q = (M \otimes N)/K$ sending $(m \bmod M', n)$ to the image of $m \otimes n$ in Q is well-defined and has the universal property characterizing $(M/M') \otimes N$.*

Remark 3. Note that the surjective map $M' \otimes N \to K$ may fail to be injective.

Exercise 5 (2 points)**.** *Let* $M' \xrightarrow{\mu} M$ *be an* R*-linear map and* $I \subseteq M$ *its image. Show that for any* R*-module* N*, the maps* $M' \otimes N \xrightarrow{\mu \otimes \mathrm{Id}_N} M \otimes N$ *(sending* $m \otimes n$ *to* $\mu(m) \otimes n$*) and* $I \otimes N \to M \otimes N$ *have the same image.*

Exercise 6 (2 points)**.** *Let* a *and* b *be natural numbers and let* $c = \gcd(a, b)$*. Show that* $\mathbb{Z}/c\mathbb{Z}$ *with*

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\begin{aligned}
(\mathbb{Z}/a\mathbb{Z}) \times (\mathbb{Z}/b\mathbb{Z}) &\to \mathbb{Z} \,/\, \gcd(a, b)\mathbb{Z} \\
(k \bmod a) \times (l \bmod b) &\to (kl) \bmod c\mathbb{Z}
\end{aligned}
$$

is a tensor product of $\mathbb{Z}/a\mathbb{Z}$ *and* $\mathbb{Z}/b\mathbb{Z}$ *over* $\mathbb{Z}$*.*

Exercise 7 (2 points)**.** *Calculate the tensor product of the embedding* $(22\mathbb{Z}/88\mathbb{Z}) \to (\mathbb{Z}/88\mathbb{Z})$ *over* $\mathbb{Z}$ *with* $\mathbb{Z}/6\mathbb{Z}$*.*

Exercise 8 (4 points)**.** *Let* K *be a perfect field,* L/K *be a finitely generated field extension and* $(l_1, \ldots, l_n)$ *an* n*-Tupel of Elements of* L *such that* L *is algebraic and separable over* $K(x_1, \ldots, x_n)$ *and such that no* $(n-1)$*-Tuple with that property exists. Show that* $(x_1, \ldots, x_n)$ *is a transcendence base of* L/K*.*

Remark 4.

- It follows that the dimension of the L-vector space $\Omega_{L/K}$ equals the transcendence degree of the field extension, completing the proof of the claim made in the lecture that if X is an algebraic variety over an algebraically closed field $\mathfrak{k}$ and $\Omega_{\mathcal{O}(X)/\mathfrak{k}}$ is a locally free $\mathcal{O}(X)$-module, then its rank must equal $\dim X$.
- When the characteristic is zero there are no inseparable algebraic field extensions and the assertion is a consequence of the basic theory of algebraic independence and transcendence degrees. For this reason, solutions may silently assume the the fields are of characteristic $p > 0$.
- Note that a field is called perfect if it has no inseparable algebraic extension or, equivalently, if its characteristic p is either zero or every element of K is a p-th power.

Solutions should be submitted Monday, December 4, in the lecture.