Exercises to „Algebraic Geometry II“, 6

This exercise sheet finishes our lengthy discussion of hypercohomology and spectral sequences, safe for its application to the finite generation of the cohomology of coherent sheaves of modules on proper A-schemes. This will use the following:

Exercise 1 (8 points). Let A be a Noetherian ring, $X \xrightarrow{\xi} \mathrm{Spec} A$ a quasi-compact A-scheme and $\mathcal{C}^*$ a bounded from below complex of quasi-coherent $\mathcal{O}_X$-modules. Let $\mathcal{H}^q(\mathcal{C}^*)$ be its q-th cohomology sheaf.

- If all the A-modules $H^p\big(X, \mathcal{H}^q(\mathcal{C}^*)\big)$ are finitely generated, show that all the A-modules $\mathbb{H}^r(X, \mathcal{C}^*)$ are finitely generated.
- Conversely, assume that all $\mathbb{H}^r(X, \mathcal{C}^*)$ are finitely generated and that $H^p\big(X, \mathcal{H}^q(\mathcal{C}^*)\big)$ is finitely generated for $q \neq 0$ and all p. Show that $H^p\big(X, \mathcal{H}^0(\mathcal{C}^*)\big)$ is finitely generated for all p.

Remark 1. In particular, the differentials of the cochain complex are assumed to be $\mathcal{O}_X$-linear. The result holds under the weaker assumption of A-linearity but then the filtration of the complex by the subcomplexes

$$\Big(\tau_{\leq p}\mathcal{C}^*\Big)^q = \begin{cases} \mathcal{C}^q & q < p \\ \mathcal{Z}^p = \mathrm{Ker}\big(\mathcal{C}^p \xrightarrow{d_{\mathcal{C}^*}} \mathcal{C}^{p+1}\big) & p = q \\ 0 & p < q \end{cases}$$

or by the subcomplexes

$$\Big(\tilde{\tau}_{\leq p}\mathcal{C}^*\Big)^q = \begin{cases} \mathcal{C}^q & q \leq p \\ \mathcal{B}^q & q = p+1 \\ 0 & q > p+1 \end{cases}$$

which are quasi-isomorphic to the previous ones becomes harder to exploit within the limitations of the lectures approach as the $\mathcal{Z}^p$ may then fail to be $\mathcal{O}_X$-modules. The subcomplexes described here are called the *soft truncations*. Soft truncations $\tau_{>p}\mathcal{C}^*$ are defined as the quotients by these subcomplexes.

Remark 2. If $\mathcal{U}$ is an affine covering of X then a short exact sequence $0 \to \mathcal{A}^* \to \mathcal{C}^* \to \mathcal{B}^* \to 0$ of cochain complexes of quasi-coherent $\mathcal{O}_X$-modules induces a short exact sequence of Čech double complexes hence a long exact sequence of hypercohomology groups. This may be used without further comment in the exercises.

Remark 3. The second point holds under the stated assumptions but solutions may assume that $\mathcal{C}^*$ vanishes in negative degrees, which simplifies things slightly.

We now resume our treatment of spectral sequences, retaining the notations of the previous exercise sheet. For cohomological spectral sequences, the typical situation is that the D-component of the exact couple has a bigrading

$$D = \coprod_{p,q\in\mathbb{Z}} D^{p,q}$$

and α has double degree $(-1, 1)$. We are interested in the direct limit L^k of the sequence

$$\dots \xrightarrow{\alpha} D^{p+1,k-p-1} \xrightarrow{\alpha} D^{p,k-p} \xrightarrow{\alpha} D^{p-1,k+1-p} \xrightarrow{\alpha} \dots \tag{1}$$

and its filtration by subobjects F^pL^k, the image of $D^{p,k-p}$ in L^k. Moreover, the E-component has a similar double grading $E^{p,q}$ and β is of double degree $(0, 0)$ while γ has degree $(r, 1-r)$. It follows that the differential d of the E-term has degree $(r, 1-r)$. We are mostly interested in the case where the grading of the E-term is bounded from below in both directions. In this case the sequence (1) stabilizes with α being an isomorphism when p is sufficiently small, and L^k could also be defined as $D^{p,k-p}$ for sufficiently small p and F^qL^k as the image of $D^{q,k-q}$ in it. If the D-component of the derived exact couple is given the double grading where $D^{p,q}_{r+1}$ is the image of $D^{p,q}_r \xrightarrow{\alpha_r} D^{p-1,q+1}_r$ or, alternatively, the cokernel of $E^{p-r,q+r-1}_r \xrightarrow{\gamma_r} D^{p,q}_r$, then the limit and its filtration do not change on passing to the derived couple.

Exercise 2 (4 points). Let $(D_r, E_r, \alpha_r, \beta_r, \gamma_r)$ be an exact couple with double grading as above and let its derived couple

$$(D_{r+1}, E_{r+1}, \alpha_{r+1}, \beta_{r+1}, \gamma_{r+1})$$

be given the double grading described above for D and where $E^{p,q}_{r+1}$ is the cokernel of $E^{p-r,q+r-1}_r \xrightarrow{d_r} \operatorname{Ker}\big(E^{p,q}_r \xrightarrow{d_r} E^{p+r,q+1-r}_r\big)$.

- Show that the degrees of α_{r+1}, β_{r+1} and γ_{r+1} are $(-1, 1)$, $(0, 0)$ and $(r + 1, -r)$.
- In the case where the initial term $E^{p,q}_r$ is bounded from below in both the p and the q-direction, show that for fixed (p, q) and sufficiently large k, we have $D^{p,q}_k \cong F^pL^{p+q}$ and $\gamma^{p,q}_k = 0$, giving an identification of $E^{p,q}_k$ with the quotient of the limit filtration:

$$0 \to F^{p+1}L^{p+q} \to F^pL^{p+q} \to E^{p,q}_k \to 0.$$

Remark 4. Most applications to algebra may be obtained as consequences of the spectral sequence of a filtered complex. To derive it from the machinery of exact couples, let C^* be a cochain complex equipped with a descending filtration by subcomplexes F^pC^* and put

$$E_1^{p,q} = H^{p+q}\big(F^pC^* \;/\; F^{p+1}C^*\big)$$

and

$$D_1^{p,q} = H^{p+q}\big(F^pC^*\big).$$

Moreover, let α be induced by the the inclusion $F^{p+1}C^* \to F^pC^*$ on cohomology, β by the projection $F^pC^* \to F^pC^* \;/\; F^{p+1}C^*$, and let γ be given by the connecting morphism of the short exact cohomology sequence for

$$0 \to F^{p+1}C^* \to F^pC^* \to F^pC^* \;/\; F^{p+1}C^* \to 0.$$

Then we are in the situation of the first point of the previous exercise. When the cochain complex is bounded from below and $F^pC^* = C^*$ for sufficiently small p, then the second point may also be applied, $L^k = H^k(C^*)$ and F^pL^k is the image of $H^k(F^pC^*)$ in it.

When $C^{*,*}$ is a double complex bounded from below in both directions, the total complex may be equipped with the filtration by the subcomplexes

$$F^p\mathrm{Tot}^k(C^{*,*}) = \oplus_{l=p}^{\infty} C^{l,k-l}$$

(the total complex of the *hard truncation* or *brutal truncation* in the second direction) and the E_1-term becomes $E_1^{p,q} = {}''H^q(C^{p,*})$, the double prime indicating that the cohomology is taken with respect to the differential d'' of degree $(0,1)$.

Exercise 3 (4 points).

- In the previous situation, identify the differential of the E_1-term with the morphism induced by $C^{p,*} \xrightarrow{d'} C^{p+1,*}$ on cohomology, giving
$$E_2^{p,q} = {}'H^p\big({}''H^q(C^{*,*})\big).$$
- Let X be a quasi-compact scheme and $\mathcal{C}^*$ a bounded from below cochain complex of quasi-coherent $\mathcal{O}_X$-modules. Let $\mathcal{U}$ be an affine open covering of X and $C^{p,q} = \check{C}^p(\mathcal{U}, \mathcal{C}^q)$. Identify the E_2-term with
$$H^p\big(X, \mathcal{H}^q(\mathcal{C}^*)\big).$$

Remark 5. In particular, if $Y \xrightarrow{f} X$ is a quasi-compact and separated morphism, $\mathcal{M}$ a quasi-coherent $\mathcal{O}_Y$-module and $\mathcal{C}^* = \boldsymbol{R}f_*\mathcal{M}$, $\mathcal{H}^q(\mathcal{C}^*) = R^qf_*\mathcal{M}$ and we obtain the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^qf_*\mathcal{M}) \Rightarrow H^{p+q}(Y, \mathcal{M})$$

explained without proof in the lecture.

Remark 6. The first exercise was placed first to make clear that it can be solved without using the hypercohomology spectral sequence. However, both the solution sketched in the first Remark and a (carefully worked out) application of the hypercohomology spectral sequence will be accepted. Such solutions may also use the slightly simplifying assumption that the complex of sheaves is concentrated in non-negative degrees.

Exercise 4 (4 points). Let X be a quasi-compact scheme and $\mathcal{C}^* \xrightarrow{f} \mathcal{D}^*$ a morphism of bounded from below cochain complexes of quasi-coherent $\mathcal{O}_X$-modules which induces isomorphisms of all cohomology sheaves $\mathcal{H}^p(\mathcal{C}^*) \cong \mathcal{H}^p(\mathcal{D}^*)$. Show that f induces an isomorphism $\mathbb{H}^q(\mathcal{C}^*) \cong \mathbb{H}^q(\mathcal{D}^*)$ of hypercohomology groups!

Remark 7. In particular ($\mathcal{D}^* = 0$), when $\mathcal{C}^*$ satisfies the assumptions and its cohomology sheaves vanish, then all hypercohomology groups $\mathbb{H}^q(X, \mathcal{C}^*)$ vanish.

Solutions should be submitted Thursday, June 14, in the exercises.