EXERCISES TO "ALGEBRA II", 6

Exercise 1 (2 points). *For the field of rational functions*

$$K = \mathfrak{k}(X_1, \ldots, X_n)$$

over an arbitrary field $\mathfrak{k}$*, show that* $\big(dX_i\big)_{i=1}^n$ *is a base of the* K*-vector space* $\Omega_{K/\mathfrak{k}}$.

Remark 1. As in the case of the polynomial ring, the result generalizes to an infinite number of variables.

Exercise 2 (6 points). *Let* $K/\mathfrak{k}$ *be a finitely generated field extension. Show that* $\Omega_{K/\mathfrak{k}}$ *vanishes if and only if* $K/\mathfrak{k}$ *is algebraic and separable.*

Exercise 3 (5 points). *Let* $\mathfrak{k}$ *be a field,* $\bar{\mathfrak{k}}$ *an algebraic closure of* $\mathfrak{k}$*,* $x = (x_i)_{i=1}^n \in \bar{\mathfrak{k}}^n$*, and* $\mathfrak{k}(x) \subseteq \bar{\mathfrak{k}}$ *the subfield generated by* $\mathfrak{k}$ *and the* x_i*. Moreover, let* $\mathfrak{m} \subset \mathfrak{k}[X_1, \ldots, X_n]$ *be the (maximal) ideal of polynomials vanishing at* x*. Show the equivalence of the following conditions:*

- *The field extension* $\mathfrak{k}(x)/\mathfrak{k}$ *is separable.*
- *The map*

$$\begin{aligned} \mathfrak{m} \big/ \mathfrak{m}^2 &\to \mathfrak{k}(x)^n \\ P \mod \mathfrak{m}^2 &\to \mathrm{grad} P(x) \end{aligned}$$

  is bijective.

Remark 2. As $\mathfrak{k}(x)$ (being a field) is a regular ring, the result shows that the Jacobian criterion does no longer hold as stated in the lecture when the ground field is not perfect. The condition of smoothness becomes relevant, and $\mathfrak{k}(x)/\mathfrak{k}$ is smooth if and only it is regular.

In the following exercise the ring R is assumed to be associative and with 1 but not necessarily commutative.

Exercise 4 (7 points). *Let* M *be a right and* N *a left* R*-module. Show that there exist an abelian group* $M \otimes_R N$ *and a map*

$$\begin{aligned} M \times N &\to M \otimes_R N \\ (m, n) &\to m \otimes n \end{aligned}$$

with the following properties:

- *The map* $\otimes$ *is* $\mathbb{Z}$*-bilinear: The identities* $(m + m') \otimes n = (m \otimes n) + (m' \otimes n)$ *and* $m \otimes (n + n') = (m \otimes n) + (m \otimes n')$ *hold for arbitrary* $m, m' \in M$ *and* $n, n' \in N$.
- *For* $r \in R$*,* $m \in M$*,* $n \in N$ *we have* $(mr) \otimes n = m \otimes (rn)$.

- *The map* $\otimes$ *satisfies the universal property for maps with the previous two properties: If* T *is any abelian group and* $M \times N \xrightarrow{t} T$ *any* $\mathbb{Z}$*-bilinear map with the property* $t(mr, n) = t(m, rn)$ *(for* r*,* m *and* n *as above), then there is a unique homomorphism* $M \otimes_R N \xrightarrow{\tau} T$ *such that* $t(m, n) = \tau(m \otimes n)$*.*

Remark 3. It is not hard to see that the universal property characterizes $M \otimes N$ uniquely up to unique isomorphism.

Solutions should be submitted Monday, November 27, in the lecture.