Exercises to „Algebraic Geometry II“, 5

Let X be a quasi-compact scheme. We resume the considerations from the previous sheet in which hypercohomology was introduced.

Exercise 1 (4 points). Let $\mathcal{C}^*$ be a bounded from below complex of sheaves of abelian groups on X such that the individual $\mathcal{C}^q$ are quasi-coherent $\mathcal{O}_X$-modules. Construct a homomorphism of abelain groups

$$H^p\big(\mathcal{C}^*(X)\big) \to \mathbb{H}^p(X, \mathcal{C}^*)$$

(from the cohomology of the cochain complex of global sections of the individual $\mathcal{C}^q$ to the hypercohomology of X with coefficients in $\mathcal{C}^*$) which is functorial in $\mathcal{C}^*$ and which is an isomorphism when all groups $H^p(X, \mathcal{C}^q)$ with $p \in \mathbb{Z}$ and $q > 0$ vanish.

Let $X \xrightarrow{f} Y$ be a continuous map between topological spaces, $\mathcal{U}$ an open covering of X, $\mathcal{M}$ a sheaf of abelian groups on X and $\check{\mathcal{C}}^p(f, \mathcal{U}, \mathcal{M})$ the sheaf of abelian groups

$$V \to \check{C}^p\big(\mathcal{U} \cap f^{-1}V, \mathcal{M}\big). \tag{1}$$

The Čech differential of the individual complexes $\check{C}^*\big(\mathcal{U} \cap f^{-1}V, \mathcal{M}\big)$ is compatible with restriction to smaller V, defining a cochain complex $\check{\mathcal{C}}^*(f, \mathcal{U}, \mathcal{M})$ of sheaves of abelian groups on Y.

Exercise 2 (3 points). Let $X \xrightarrow{f} Y$ be a quasi-compact and separated morphism, where Y is a quasi-compact scheme, and let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume that $\mathcal{U}$ is an affine covering. Construct an isomorphism from $R^p f_* \mathcal{M}$ to the p-th cohomology sheaf of $\check{\mathcal{C}}^*(f, \mathcal{U}, \mathcal{M})$.

Remark 1. Assuming that the constructions of $R^p f_*$ of the lecture is used, the easiest way to get a morphism in the indicated direction is to use the universal property of sheafification. To show that it is an isomorphism, it is sufficient to show that it is an isomorphism on sections on affine open subsets V, as these V form a topology base. For this it is useful to know that, Y being a scheme, the morphism $V \to Y$ is affine. The base change $f^{-1}V \to X$ of this morphism is also affine, as the class of affine morphisms is stable under base-change. This fact may be used without further discussion in the solutions to the above exercise.

Exercise 3 (4 points). We keep the assumptions of the previous exercise. Under the addition assumption that for any open subset U

of the covering $\mathcal{U}$, $f(U)$ is contained in some affine open subset of Y, construct an isomorphism

$$H^*(X, \mathcal{M}) \cong \mathbb{H}^*\big(Y, \check{\mathcal{C}}^*(f, \mathcal{U}, \mathcal{M})\big)$$

between the cohomology of X with coefficients in $\mathcal{M}$ and the hypercohomology of Y with coefficients in $\check{\mathcal{C}}^*(f, \mathcal{U}, \mathcal{M})$.

REMARK 2. The additional assumption in the exercise may be dropped, as we will possibly see on one of the remaining exercise sheets.

REMARK 3. In view of the previous two exercises, it is natural to ask whether the $\check{\mathcal{C}}^*(f, \mathcal{U}, \mathcal{M})$ for individual $\mathcal{U}$ are just one particular construction of an object $\boldsymbol{R}f_*\mathcal{M}$, just as the Čech cohomology groups $\check{H}^*(\mathcal{U}, \mathcal{M})$ for affine $\mathcal{U}$ are just one particular way of constructing $H^*(X, \mathcal{M})$. It is in fact possible to take this point of view, despite the fact that the isomorphism class of the cochain complex $\check{\mathcal{C}}^*(f, \mathcal{U}, \mathcal{M})$ will usually change when the covering $\mathcal{U}$ is changed. It is thus necessary to replace the category of cochain complexes of sheaves on Y by a different target category. To sketch its definition, we call a morphism $\mathcal{C}^* \to \mathcal{H}^*$ of cochain complexes a *quasi-isomorphism* if it induces an isomorphism on cohomology sheaves. It follows from exercise 2 that for a refinement $\mathcal{V}$ of $\mathcal{U}$ and any choice of a refinement map between the underlying index sets, one has a quasi-isomorphism $\check{\mathcal{C}}^*(f, \mathcal{U}, \mathcal{M}) \to \check{\mathcal{C}}^*(f, \mathcal{V}, \mathcal{M})$. This motivates the choice of the target category as the category $\mathcal{D}^+\big(\underline{\mathrm{Qc}}(Y)\big)$ obtained from the category of bounded from below cochain complexes of quasi-coherent $\mathcal{O}_Y$-modules by inverting the quasi-isomorphisms. This procedure of selectively inverting certain morphisms in a category is conceptionally similar to localizing a ring, but is more involved. In addition, there are set-theoretic difficulties when the category to be "localized" is not small. Nevertheless, it can be shown that the *derived category* $\mathcal{D}^+\big(\underline{\mathrm{Qc}}(Y)\big)$ introduced above exists, that the functor of hypercohomology

$$\mathcal{C}^* \to \mathbb{H}^p(X, \mathcal{C}^*)$$

factors over it and that $\check{\mathcal{C}}^*(\mathcal{U}, \mathcal{M})$ is, up to canonical isomorphism in the derived category, independent of $\mathcal{U}$, defining a functor

$$\underline{\mathrm{Qc}}(X) \xrightarrow{\boldsymbol{R}f_*} \mathcal{D}^+\big(\underline{\mathrm{Qc}}(Y)\big)$$

such that

$$H^p(X, \mathcal{M}) \cong \mathbb{H}^p(Y, \boldsymbol{R}f_*\mathcal{M}).$$

There is also, more generally, a version of the construction (1) of $\check{\mathcal{C}}^*(\mathcal{U}, \mathcal{M})$ giving a double complex $\check{\mathcal{C}}^*(\mathcal{U}, \mathcal{M}^*)$ from a complex $\mathcal{M}^*$ of sheaves of abelian groups on X. In the case where X is a prescheme, f

quasi-compact and separated, and $\mathcal{M}^*$ a bounded from below complex of quasi-coherent $\mathcal{O}_X$-modules, this defines a functor

$$\mathcal{D}^+\big(\underline{\mathrm{Qc}}(X)\big) \xrightarrow{\boldsymbol{R}f} \mathcal{D}^+\big(\underline{\mathrm{Qc}}(Y)\big)$$

sending $\mathcal{M}^*$ to the total complex of $\check{\mathcal{C}}^*(f, \mathcal{U}, \mathcal{M}^*)$. There are, for suitable g, canonical isomorphisms

$$\boldsymbol{R}g_*\boldsymbol{R}f_*\mathcal{M}^* \cong \boldsymbol{R}(gf)_*\mathcal{M}$$

between these derived functors. Also

$$\mathbb{H}^*(X, \mathcal{M}^*) \cong \mathbb{H}^*(Y, \boldsymbol{R}f_*\mathcal{M}^*).$$

We will have to avoid this modern framework of derived categories in the lecture and (safe for a few remarks like the current one) in the exercises since introducing the technical foundations for this advanced technique would require a lecture of its own. A good source for derived categories is the Gel'fand/Manin textbook on homological algebra.

An easier technique are spectral sequences, which are still indispensible today. In general, the relation between the hypercohomology of a complex $\mathcal{C}^*$ of sheaves on X and the cohomology groups of X with coefficients in the cohomology sheaves of $\mathcal{C}^*$ is given by a spectral sequence. A brief description of spectral sequences will be given in the lecture on Thursday (May 17), or may be looked up in the literature, including wikipedia. In the special case of the hypercohomology of the complexes $\boldsymbol{R}f_*\mathcal{M}$ considered in the previous two exercises, this hypercohomology spectral sequences gives the Leray spectral sequence which will be explained in the lecture next Thursday.

The construction of spectral sequences preferred by topologists is Massey's construction by *exact couples*, as it is the most general. I also consider it simpler than the other alternatives.

Definition 1. An *exact couple* in an abelian category $\mathcal{A}$ is a quintuple $(D, E, \alpha, \beta, \gamma)$ consisting of objects E and F and morphisms α, β and γ forming an exact sequence

$$D \xrightarrow{\alpha} D \xrightarrow{\beta} E \xrightarrow{\gamma} D \xrightarrow{\alpha} D.$$

Remark 4. This sequence is often arranged as a triangle with vertices D, D and E.

Let $(D_0, E_0, \alpha_0, \beta_0, \gamma_0) = (D, E, \alpha, \beta, \gamma)$ be an exact couple. It is easy to see that $E \xrightarrow{d = \beta\gamma} E$ satisfies $d^2 = 0$. In fact, both

$$d\beta \text{ and } \gamma d \tag{2}$$

vanish. Let E_1 denote the cohomology object $\mathrm{Coker}\big(E \xrightarrow{d} \mathrm{Ker}(d)\big)$.

It is also easy to see that the restriction of α to the image

$$\mathrm{Coker}(\gamma) \cong \mathrm{Image}(\alpha) = \mathrm{Ker}(\beta) = D_1 \subseteq D_0$$

of α factors uniquely over $D_1 \to D$, defining $D_1 \xrightarrow{\alpha_1} D_1$. By the vanishing of (2), the morphism $D \xrightarrow{\beta} E$ factors as

$$D \xrightarrow{\tilde{\beta}} \mathrm{Ker}(d) \to E.$$

Let $D \xrightarrow{\hat{\beta}} E_1$ be the composition $D \xrightarrow{\tilde{\beta}} \mathrm{Ker}(d) \to E_1$. Its composition with γ equals the composition $E \xrightarrow{d} \mathrm{Ker}(d) \to E_1$ which is zero. It follows that $D \xrightarrow{\hat{\beta}} E_1$ factors over $D_1 \cong \mathrm{Coker}(\gamma) \xrightarrow{\beta_1} E_1$.

Finally, the vanishing of (2) implies that the composition

$$\mathrm{Ker}(d) \to E \xrightarrow{\gamma} D \tag{3}$$

with $E \xrightarrow{d} \mathrm{Ker}(d)$ vanishes. Since $\beta\gamma = d$, (3) also factors uniquely as the composition of a morphism

$$\mathrm{Ker}(d) \to \mathrm{Ker}(\beta) = D_1 \tag{4}$$

with $D_1 \to D$. As the last morphism is a monomorphism, the composition of (4) with $E \xrightarrow{d} \mathrm{Ker}(d)$ also vanishes. It follows that (4) factors over a morphism

$$E_1 = \mathrm{Coker}\big(E \xrightarrow{d} \mathrm{Ker}(d)\big) \xrightarrow{\gamma_1} D_1.$$

Exercise 4 (7 points). Show that $(D_1, E_1, \alpha_1, \beta_1, \gamma_1)$ is an exact couple.

Remark 5. It is OK for solutions to do this by diagram chase in the case of the category of modules over a ring. By checking exactness on stalks, this also covers the case of the categories of sheaves of abelian groups or of modules, and thus all abelian categories relevant to the lecture.

Solutions should be submitted Thursday, June 7, in the exercises.