Exercises to “Algebra II”, 5

Exercise 1 (5 points). *As in exercise 2 on sheet 3, let $\mathfrak{k}$ be a field and*

$$R = \mathfrak{k}\big[X_i \bigm| i \text{ is a positive integer}\big]$$

be the polynomial ring in countably infinitely many variables over $\mathfrak{k}$, let $\mathfrak{p}_k \subseteq R$ the ideal generated by the X_i such that $k^2 < i \leq (k+1)^2$, let $S = R \setminus \bigcup_{k=0}^{\infty} \mathfrak{p}_k$, and let $A = R_S$ be the example of an infinite-dimensional Noetherian ring considered in the aforementioned exercise.

- *Show that any prime ideal $\mathfrak{r}$ of A has the form $\mathfrak{q} \cdot R_S$ where $\mathfrak{q} \in \operatorname{Spec} R$ is contained in one of the $\mathfrak{p}_k$.*
- *Show that $A_{\mathfrak{r}}$ is a regular local ring.*

Remark 1. For the first point, it is known from the relation between prime ideals in R and R_S it is known that $\mathfrak{r}$ has the stated form where $\mathfrak{q}$ is disjoint from S. To conclude that it must be contained in one of the $\mathfrak{p}_i$ the following prime avoidance lemma, known from Algebra 1, is useful:

> Let A be any ring, I and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ ideals of A. If $I \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i$ and at most two of the $(\mathfrak{p}_i)_{i=1}^{n}$ fail to be prime, then I is contained in one of the $\mathfrak{p}_i$.

Eisenbud, Commutative Algebra with a View toward Algebraic Geometry, devotes subsection 3.2 to assertions of this type. The fact that we have inifinitely many $\mathfrak{p}_i$ must be worked around somehow.

Exercise 2 (5 points). *Let A be a ring, M an A-module, $d \in \operatorname{Der}(A, M)$, $a = (a_i)_{i=1}^{n} \in A^n$. Moreover, for*

$$P = \sum_{\alpha \in \mathbb{N}^n} p_\alpha X^\alpha \in A[X_1, \ldots, X_n]$$

we put

$$d(P) = \sum_{\alpha \in \mathbb{N}^n} d(p_\alpha) X^\alpha \in A[X_1, \ldots, X_n].$$

Show that

$$d\big(P(a)\big) = \big(d(P)\big)(a) + \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(a) \cdot d(a_i).$$

Remark 2. Applying this when $A = R[X_1, \ldots, X_n]$, $a_i = X_i$ where $P \in A$ and d is R-linear we have $d(P) = 0$. Denoting $m_i = d(a_i)$ we obtain

$$d(P) = \sum_{i=1}^{n} \frac{\partial P}{\partial X_i} \cdot m_i$$

as was claimed in the lecture when we calculated $\Omega_{A/R}$.

The module of Kähler differentials may be interesting even in the case of field extensions.

Exercise 3 (4 points)**.** *Let $\mathfrak{l}/\mathfrak{k}$ be a separable algebraic field extension. Show that $\Omega_{\mathfrak{l}/\mathfrak{k}} = 0$. More generally, show that any $\mathfrak{k}$-valued derivation of $\mathfrak{k}$ has a unique extension to an $\mathfrak{l}$-valued derivation of $\mathfrak{l}$.*

Remark 3. It is straightforward to obtain the following generalization of the last assertion: Let V be a $\mathfrak{k}$-vector space and $W = \mathfrak{l} \otimes_{\mathfrak{k}} V$ be its field extension to $\mathfrak{l}$. Then any V-valued derivation of $\mathfrak{k}$ has a unique extension to W-valued derivation of $\mathfrak{l}$. This is so because any base of V over $\mathfrak{k}$ defines a base of W over $\mathfrak{l}$ and because every vector space has a base.

In positive characteristic it is easier for the derivative of a polynomial to vanish, which makes it harder to obtain vanishing or uniqueness results for derivations in the way needed for the above exercise. But the fact that more polynomials have vanishing derivatives can also make it easier to show that a derivation must vanish.

Exercise 4 (1 point)**.** *Let $\mathfrak{l}/\mathfrak{k}$ be a field extension of positive characteristic, where $\mathfrak{l}$ is algebraically closed. Show that $\Omega_{\mathfrak{l}/\mathfrak{k}}$ vanishes.*

Remark 4. In fact, it is sufficient to assume that $\mathfrak{l}$ is perfect in the sense that it has no inseparable algebraic extension.

Exercise 5 (5 points)**.** *Let R be a ring and*

$$A \to B \to C \to 0 \qquad (+)$$

be a sequence of R-modules. Show that (+) is exact if and only if the sequence

$$0 \to \mathrm{Hom}_R(C, T) \to \mathrm{Hom}_R(B, T) \to \mathrm{Hom}_R(A, T)$$

is exact for any R-module T.

Solutions should be submitted Monday, November 20, in the lecture.