Exercises to „Algebraic Geometry II“, 4

Exercise 1 (4 points). Give an example of an affine scheme X and a (necessarily non-qc) sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$ and an affine covering $\mathcal{U}$ of X such that $\check{H}^1(\mathcal{U}, \mathcal{I}) \neq 0$.

If $\mathcal{C}^*$ is a cochain complex of sheaves of abelian groups on a topological space X, it is possible to define *hypercohomology groups* $\mathbb{H}^q(X, \mathcal{C}^*)$ which, in the case where $\mathcal{C}^k$ vanishes when $k \neq p$, are canonically isomorphic to $H^{q-p}(X, \mathcal{C}^p)$. In the general case this is done using the machinery developed by Grothendieck. We are mostly interested in the case where the $\mathcal{C}^p$ are quasi-coherent sheaves of modules on a quasi-compact scheme and the cochain complex is bounded from below. This special case is accessible to techniques similar to the one used in the lecture. A few facts about *double complexes* must be introduced (or recalled) first.

Definition 1. A double complex (with anticommuting differentials) of objects of an abelian category is a double sequence $\left(\mathcal{C}^{p,q}\right)_{p,q=-\infty}^{\infty}$ of objects of that category, together with morphisms

$$C^{p,q} \xrightarrow{d'} C^{p+1,q}$$
$$C^{p,q} \xrightarrow{d''} C^{p,q+1}$$

satisfying

$$d'd' = 0$$
$$d''d'' = 0$$
$$d'd'' = -d''d'. \tag{1}$$

A *morphism of double complexes* is a sequence of morphisms $C^{p,q} \xrightarrow{\alpha_{p,q}} \tilde{C}^{p,q}$ satisfying $\tilde{d}'\alpha_{p,q} = \alpha_{p+1,q}d'$ and $\tilde{d}''\alpha_{p,q} = \alpha_{p+1,q}d''$.

Remark 1.
- One can show that the category of double complexes is again an abelian category. The same holds for the subcategories of double complexes bounded from above (in the sense that $C^{p,q} = 0$ whenever $p > B$, for some B) or from below (in the sense that $C^{p,q} = 0$ when $p < B$) in one direction, or of double complexes satisfying a combination of such conditions, like boundedness from above and below in both directions.
- For our purposes it is not necessary to develop this in full detail, as it is sufficient to define the required notions of kernel, cokernel or exactness componentwise. For instance, a sequence $\mathcal{C}^{*,*} \to \mathcal{D}^{*,*} \to \mathcal{E}^{*,*}$ of double complexes is exact iff

the sequence $\mathcal{C}^{p,q} \to \mathcal{D}^{p,q} \to \mathcal{E}^{p,q}$ is exact for any pair (p,q) of integers. Also, solutions to the exercises are not required to bother about the case of general abelian categories and shall be given the full number of points if they work for categories of modules over a ring.

- The above definition follows the usual convention. However double complexes with commuting differentials occur more frequently. In this case, (1) is replaced by $d'd'' = d''d'$.

The following exercise collects a number of facts about double complexes.

Exercise 2 (1 point). Let $\mathcal{C}^{*,*}$ be a double complex with commuting differentials. Show that a double complex with anti-commuting differentials is obtained by replacing $\mathcal{C}^{p,q} \xrightarrow{d''} \mathcal{C}^{p,q+1}$ by

$$\mathcal{C}^{p,q} \xrightarrow{(-1)^p d''} \mathcal{C}^{p,q+1}.$$

In the following exercise we only consider the case of double complexes $\mathcal{X}^{*,ast}$ with anti-commuting differentials which are bounded from below in both directions, where $\mathcal{X}$ is a placeholder for several letters.

Exercise 3 (5 points). With these assumptions, show the following:

- The sequence of objects
$$\mathrm{Tot}^r\big(\mathcal{C}^{*,*}\big) = \oplus_{p+q=r}\mathcal{C}^{p,q}.$$
becomes a cochain complex (called the *total complex* of the double complex) with differential
$$d\big(c_{p,q}\big)_{p+q=r} = \big(d'c_{p-1,q} + d''c_{p,q-1}\big)_{p+q=r+1}.$$
- For a short exact sequence of double complexes, one has a long exact sequence of cohomology objects of the total complexes.
- When $\mathcal{C}^{*,*} \to \mathcal{D}^{*,*}$ is a morphism and both double complexes are bounded in both directions and for arbitary integers p and q the morphism $\mathcal{C}^{*,q} \xrightarrow{\mathcal{D}} {}^{*,q}$ of cochain complexes induces an isomorphism on cohomology objects
$$H^p(\mathcal{C}^{*,q}, d') \xrightarrow{\cong} H^p(\mathcal{D}^{*,q}, d')$$
then the morphism induced on total complexes also induces an isomorphism on cohomology objects.
- Subject only to the assumption formulated at the beginning of the exercise, show that for any finite set S of integers there is a morphism $\mathcal{C}^{*,*} \to \tilde{\mathcal{C}}^{*,*}$ inducing an isomorphism on the p-th

cohomology of the total complex for all $p \in S$ and where $\tilde{C}^{*,*}$ is bounded from above and below in one or in both directions.
- Establish the third point under the weaker assumption that both complexes are bounded from below.

Remark 2. The boundedness assumption implies that for any given integer r, all but finitely many summands in

$$\prod_{p+q=r} \mathcal{C}^{p,q}$$

vanish. This shows that the product exists in an arbitrary abelian category (without needing AB3*) and is isomorphic to the coproduct. The same of course holds when the double complex is bounded from above in both directions, or from both sides in one direction. Without any assumption of this type, one must decide whether the total complex is to be formed using infinite coproducts or infinite products.

When $\mathcal{C}^*$ is a cochain complex of sheaves of abelian groups on the topological space X and $\mathcal{U}$ an open covering of X, applying the Čech construction componentwise and to the differentials of the complex considered as morphisms of sheaves of abelian groups, one obtains a double complex $\check{C}^*(\mathcal{U}, \mathcal{C}^*)$ with commuting differentials. There is a sub-double complex ${}^a\check{C}^*(\mathcal{U}, \mathcal{C}^*) \subset \check{C}^*(\mathcal{U}, \mathcal{C}^*)$ formed by the alternating Čech complexes ${}^a\check{C}^*(\mathcal{U}, \mathcal{C}^p)$. Using the second exercise one obtains double complexes with anti-commuting differentials. Assuming $\mathcal{C}^*$ to be bounded from below, the cohomology groups of the total complex $\mathrm{Tot}\check{C}^*(\mathcal{U}, \mathcal{C}^*)$ are called the *Čech hypercohomology groups* of $\mathcal{U}$ and denoted $\check{\mathbb{H}}^p(\mathcal{U}, \mathcal{C}^*)$. Their alternating counterparts are denoted ${}^a\check{\mathbb{H}}^p(\mathcal{U}, \mathcal{C}^*)$.

Exercise 4 (3 points). Let $\mathcal{V}$ be a refinement of $\mathcal{U}$. Construct a morphism $\check{\mathbb{H}}^*(\mathcal{U}, \mathcal{C}^*) \to \check{\mathbb{H}}^*(\mathcal{V}, \mathcal{C}^*)$ which is independent of the choice of a refinement map from the index set of $\mathcal{V}$ to that of $\mathcal{U}$.

Exercise 5 (3 points). Let in addition X be a quasi-compact scheme, the coverings be affine and the individual $\mathcal{C}^p$ be quasi-coherent $\mathcal{O}_X$-modules. Show that the previous morphism is an isomorphism. Also, show that ${}^a\check{\mathbb{H}}^*(\mathcal{U}, \mathcal{C}^*) \to \check{\mathbb{H}}^*(\mathcal{U}, \mathcal{C}^*)$ is an isomorphism.

This allows us to define the hypercohomology groups of any bounded from below complex of quasi-coherent sheaves of modules on any quasi-compact scheme by making a suitable choice of Čech coverings, e. g., by considering the covering by all affine open subsets. It is not necessary to assume the differentials of the complex to be $\mathcal{O}_X$-linear.

Remark 3. For instance, if X is a regular variety over a field of characteristic 0, $\Omega_{X/\mathfrak{k}}$ is a vector bundle which has exterior powers $\Lambda^p \Omega_{X/\mathfrak{k}}$ with the universal property for alternating p-linear morphisms from $\Omega_{X/\mathfrak{k}}$ to any sheaf of modules. On these exterior powers an $\mathfrak{k}$-linear (but not $\mathcal{O}_X$-linear, unless it is trivial) exterior derivative

$$\Lambda^p \Omega_{X/\mathfrak{k}} \xrightarrow{d} \Lambda^{p+1} \Omega_{X/\mathfrak{k}}$$

may be defined in a similar way as in Analysis III or differential geometry. One obtains a de Rham complex whose hypercohomology has interesting properties.

The case of positive characteristic may also be studied, but then the properties of de Rham cohomology become more difficult. The same holds for some related elementary considerations. For instance, when C is a smooth curve over a field of characteristic 0 and $f \neq 0$ a rational function, the rational section df of the line bundle $\Omega_{C/\mathfrak{k}}$ has a zero of precise order $k-1$ (resp. a pole of precise order $k+1$) at x when f has a zero (resp. a pole) of precise order $k > 0$ at x. In positive characteristic it is only possible to bound the zero order (resp. pole order) of df from below (resp. above). Whether this causes results valid in characteristic zero to fail in general depends on the specific situation.

Exercise 6 (2 points). For a ring R, identify $\Omega_{\mathbb{P}^1_R/R}$ with one of the line bundles $\mathcal{O}(k)$ investigated on the previous sheet.

Exercise 7 (4 points). Under the same assumption, calculate the hypercohomology of $X = \mathbb{P}^1_R$ with coefficients in the complex

$$\mathcal{O}_X \xrightarrow{d_{X/\mathrm{Spec} R}} \Omega_{X/\mathrm{Spec} R}.$$

Remark 4. There are 22 points for the exercises on this sheet. I will try to keep the sum of points from this and the next sheet close to 40, treating any excess above 40 as bonus points.

Solutions should be submitted Monday, May 28, in the lecture.