Exercises to “Algebra II”, 4

Our task for this week is to generalize some results of the lecture to ground fields $\mathfrak{k}$ which are not necessarily algebraically closed. Let $\overline{\mathfrak{k}}$ be an algebraic closure of $\mathfrak{k}$ and let $G = \mathrm{Aut}(\overline{\mathfrak{k}}/\mathfrak{k})$ be the automorphism group of that field extension. Let $R = \mathfrak{k}[X_1, \ldots, X_n]$.

Exercise 1 (2 points). *Let $\mathfrak{m} \subset R$ be a maximal ideal and $x = (x_1, \ldots, x_n) \in \overline{\mathfrak{k}}^n$ a zero of $\mathfrak{m}$. Show that there is a unique $\mathfrak{k}$-linear isomorphism between the residue field $\mathfrak{k}(\mathfrak{m}) = R/\mathfrak{m}$ of $\mathfrak{m}$ and the subfield of $\overline{\mathfrak{k}}$ generated by $\mathfrak{k}$ and the x_i, sending $X_i \bmod \mathfrak{m}$ to x_i.*

Exercise 2 (6 points). *Show that there is a bijection between the set $\mathrm{mSpec}R$ of maximal ideals $\mathfrak{m}$ of R and the set of G-orbits A in $\overline{\mathfrak{k}}^n$, sending A to the ideal $\mathfrak{m}$ of polynomials vanishing in one (and hence any) element of A and $\mathfrak{m}$ to the set A of zeroes of $\mathfrak{m}$ in $\overline{\mathfrak{k}}^n$.*

Remark 1. Recall that a G-orbit in a set X on which G acts is a non-empty subset $O \subseteq X$ which is invariant under the G-action and on which G acts transitively.

Exercise 3 (2 points). *Let $\mathfrak{m}$ be a maximal ideal of R and $\mathfrak{k}(\mathfrak{m}) = R/\mathfrak{m}$ its residue field, and let $0 \le i \le n$. Show that the image $\mathfrak{k}_i$ of $\mathfrak{k}[X_1, \ldots, X_i] \subseteq R$ in $\mathfrak{k}(\mathfrak{m})$ is a subfield.*

Exercise 4 (7 points). *Show that R is regular at all of its maximal ideals.*

Remark 2. It may be used without proof that $\mathrm{ht}\mathfrak{m} = n$ for any $\mathfrak{m} \in \mathrm{mSpec}(R)$. This follows from the fact (Proposition 2.4.1 in Algebra I or the proof of theorem 5.6 of Matsumura’s Commutatitve Ring Theory) that $\mathrm{degtr}(\mathfrak{k}(\mathfrak{p})/\mathfrak{k}) > \mathrm{degtr}(\mathfrak{k}(\mathfrak{q})/\mathfrak{k})$ holds whenever $\mathfrak{p} \subset \mathfrak{q}$ is a proper inclusion between prime ideals of R for the inequality $\le$. In the other direction one may use the surjetivity of $\mathrm{Spec}S \to \mathrm{Spec}R$ for the integral ring extension $S = \overline{\mathfrak{k}}[X_1, \ldots, X_n]/R$ together with the last exercise and the fact that the result is clear in the algebraically closed case where one can write down a strictly increasing chain of length n of affine subspaces. The constructions used in the most natural approach to the solution may also be used to directly write down a chain of prime ideals of R ending in $\mathfrak{m}$ and having length n.

The task then is to generate, for arbitrary $x \in \mathfrak{k}^n$, the ideal of all $f \in R$ vanishing at x by n elements. For $n = 1$ the minimal polynomial is used, and for the general case induction on n is a suitable approach.

Exercise 5 (4 points). *Show that R is regular at all of its prime ideals.*

Remark 3. Of course this should not be done by applying the theorem of Serre (if R is a regular local ring in the sense of our definition and if $\mathfrak{p} \in \mathrm{Spec} R$, then $R_{\mathfrak{p}}$ is regular) mentioned in the lecture. A more elementary way to do it is to reduce to the situation where $\mathfrak{p}$ is a maximal ideal utilizing a technique used to achieve a similar reduction in the proof of the aforementioned Proposition 2.4.1 in Algebra I. Note that this reduction sacrifices the algebraic closedness of the ground field, even if the result is only to be proved for algebraically closed $\mathfrak{k}$.

One of the 21 points which can be gained with solutions to these exercises is a bonus point which is not used in the calculation of the 50%-limit for successfully passing the exercises.

Solutions should be submitted Monday, November 13, in the lecture.