Exercises to "Algebra II", 3

Exercise 1 (1 point). *Let R be a ring such that, for any $f \in R \setminus \{0\}$, the ring R/fR is Noetherian. Show that R is Noetherian!*

Exercise 2 (8 points). *Let $\mathfrak{k}$ be a field and*

$$R = \mathfrak{k}\big[X_i \bigm| i \text{ is a positive integer}\big]$$

be the polynomial ring in countably infinitely many variables over $\mathfrak{k}$. For $k \in \mathbb{N}$ let $\mathfrak{p}_k \subseteq R$ the ideal generated by the X_i such that $k^2 < i \leq (k+1)^2$. Finally, let $S = R \setminus \bigcup_{k=0}^{\infty} \mathfrak{p}_k$. Show that S is a multplicative subset of R, that R_S is Noetherian and that $\big\{\mathrm{ht}(\mathfrak{q}) \bigm| \mathfrak{q} \in \mathrm{Spec} R_S\big\}$ is unbounded, such that R_S has infinite Krull dimension.

For the following two exercises, Theorems 11 and 12 of the lecture and their corollaries (e. g., the finite dimensionality of Noetherian local rings) may be used. The fact a Notherian domain is factorial if and only if the irreducible elements are prime elements may also be used.

Exercise 3 (3 points). *Let R be any Noetherian ring and $I \subseteq R$ any ideal different from R. Show that there is $\mathfrak{p} \in \mathrm{Spec} R$ which is $\subseteq$-minimal among all prime ideals containing I.*

Exercise 4 (3 points). *Let R be a Noetherian domain. Show that R is factorial if and only if every prime ideal of height one in R is principal.*

Exercise 5 (3 points). *Let B be any domain and $A \subseteq B$ any subring such that B is integral over A. Show that B is a field if and only if A is a field.*

Exercise 6 (2 points). *Let B be any ring and $A \subseteq B$ a subring over which B is integral. Show that $\mathfrak{p} \in \mathrm{Spec} B$ is a maximal ideal if and only if $\mathfrak{p} \cap A$ is maximal.*

The part of the Krull-Cohen-Seidenberg Theorem (Theorem 7 as formulated in the lecture) which was used in the proof of corollary 1.1.1 (to the principal ideal Theorem Theorem 11) can easily be reduced to the previous observation by making $\mathfrak{p}$ maximal by passing to an appropriate localization and using the relation between prime ideals in the localization and in the orignal ring.

Exercise 7 (3 points). *Let B be any ring and $A \subseteq B$ a subring over which B is integral, and let $\mathfrak{p}$ in $\mathrm{Spec} A$. Show that there are no proper inclusions among the $\mathfrak{q} \in \mathrm{Spec} B$ satisfying $\mathfrak{q} \cap A = \mathfrak{p}$.*

Solutions should be submitted Monday, November 6, in the lecture.