Exercises to „Algebraic Geometry II“, 1

Exercise 1 (3 points). Let $\mathcal{R}$ be a sheaf of rings on the topological space X.

- Let

$$\begin{array}{ccccccc}
\mathcal{A} & \xrightarrow{a} & \mathcal{B} & \xrightarrow{b} & \mathcal{C} & \xrightarrow{c} & \mathcal{D}\\
{\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow & & {\scriptstyle\delta}\big\downarrow\\
\tilde{\mathcal{A}} & \xrightarrow{\tilde{a}} & \tilde{\mathcal{B}} & \xrightarrow{\tilde{b}} & \tilde{\mathcal{C}} & \xrightarrow{\tilde{c}} & \tilde{\mathcal{D}}
\end{array}$$

  be a commutative diagram of $\mathcal{R}$-modules, where the rows are exact sequences and β and δ are monomorphisms while α is an epimorphism. Show that γ is a monomorphism.
- Let

$$\begin{array}{ccccccc}
\mathcal{B} & \xrightarrow{b} & \mathcal{C} & \xrightarrow{c} & \mathcal{D} & \xrightarrow{d} & \mathcal{E}\\
{\scriptstyle\beta}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow & & {\scriptstyle\delta}\big\downarrow & & {\scriptstyle\varepsilon}\big\downarrow\\
\tilde{\mathcal{B}} & \xrightarrow{\tilde{b}} & \tilde{\mathcal{C}} & \xrightarrow{\tilde{c}} & \tilde{\mathcal{D}} & \xrightarrow{\tilde{d}} & \tilde{\mathcal{E}}
\end{array}$$

  be a commutative diagram of $\mathcal{R}$-modules, where the rows are exact sequences and β and δ are epimorphisms while ε is an monomorphism. Show that γ is an epimorphism.

In the following three exercises, the properties of abelian categories from the definition given in the lecture may all be used. In particular, for morphisms $A \xrightarrow{i} B \xrightarrow{p} C$ the following conditions are equivalent:

- i is a monomorphism and p a cokernel of i.
- p is an epimorphism and i a kernel of i.

This is the case if and only if

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0 \tag{1}$$

is exact. The five lemma, proved for categories of modules on the previous sheet and for sheaves of modules in the previous exercise, may also be used for arbitrary abelian categories and applied to exact sequences of the form

$$0 \to \mathrm{Ker}(f) \to X \xrightarrow{f} Y \to \mathrm{Coker}(f) \to 0.$$

Otherwise, only the methods available from the lecture (eg, universal properties of kernels and cokernels, cancellation of identities by a monomorphism on the left) may be used.

Exercise 2 (3 points). Let $X \xrightarrow{f} Y$ be a morphism in an abelian category and $I \xrightarrow{i} X$ a kernel of $Y \xrightarrow{\mathrm{Coker}} (f)$ (called the *image* of

f, as in the lecture). Show the existence and uniqueness of a morphism $X \xrightarrow{\tilde{f}} I$ such that $f = i\tilde{f}$ and show that $\tilde{f}$ is an epimorphism.

EXERCISE 3 (6 points). For a short exact sequence (1) in an abelian category and a morphism $B \xrightarrow{f} X$, let $K \xrightarrow{\kappa} B$ be a kernel of f. In the case where fi vanishes, the universal property of p as a cokernel of i shows the existence and uniqueness of a morphism $C \xrightarrow{g} X$ such that $f = gp$. Let $L \xrightarrow{\lambda} C$ be a kernel of g.

- Show the existence and uniqueness of a morphism $K \xrightarrow{q} L$ such that $\lambda q = p\kappa$.
- Show that q is an epimorphism.
- Show the existence and uniqueness of a morphism $A \xrightarrow{j} K$ such that $\kappa j = i$.
- Show that q is a cokernel of j and j a kernel of q.

REMARK 1. In the case where κ is an epimorphism, the second point can be shown using the five lemma, and the general case can be reduced to this by the previous exercise.

EXERCISE 4 (8 points). Let $0 \to C_o^* \xrightarrow{a} C^* \xrightarrow{b} \hat{C}^* \to 0$ be a short exact sequence of cochain complexes in an abelian category, and let $X^i \xrightarrow{\kappa} C^i$ be a kernel of $C^i \xrightarrow{\hat{d}^i b} \hat{C}^{i+1}$. Let $Z_o^i \xrightarrow{\zeta_o^i} C_o^i$, $Z^i \xrightarrow{\zeta} C^i$ and $\hat{Z}^i \xrightarrow{\hat{\zeta}^i} \hat{C}^i$ be kernels of $C_o^i \xrightarrow{d_o^i} C_o^{i+1}$, $C^i \xrightarrow{d^i} C^{i+1}$ and $\hat{C}^i \xrightarrow{\hat{d}^i} \hat{C}^{i+1}$. By $d^2 = 0$ and the universal property of the kernel, $C^{i-1} \xrightarrow{d} C^i$ factors uniquely as $C^{i-1} \to Z^i \xrightarrow{\zeta^i} C^i$. For the sake of simplicity, we overload the notation d^{i-1} denote the first of these morphisms by $C^{i-1} \xrightarrow{d^{i-1}} Z^i$. The cokernel $Z^i \xrightarrow{\eta^i} H^i$ of this morphism is the i-th cohomology object H^i of C^*. Similarly for $Z_o^i \xrightarrow{\eta_o^i} H_o^i$ and $\hat{C}^i \xrightarrow{\hat{\eta}^i} \hat{H}^i$.

- Show the existence and uniqueness of a morphism $C_o^i \xrightarrow{a_1} X^i$ such that $\kappa a_1 = a$ and construct a morphism $X^i \xrightarrow{b_1} \hat{Z}^i$ which is a cokernel of a_1.
- Show the existence and uniqueness of a morphism $X^i \xrightarrow{\delta} Z_o^{i+1}$ such that $a\zeta_o^{i+1}\delta = d^{i+1}\kappa$.
- Show that $\delta a_1 = d_o^i$.
- Show the existence and uniqueness of a morphism $\hat{Z}^i \xrightarrow{\delta_1} H_o^{i+1}$ such that $\delta_1 b_1 = \eta_o^{i+1}\delta$.
- Show that the composition

$$\hat{C}^{i-1} \xrightarrow{\hat{d}^{i-1}} \hat{Z}^i \xrightarrow{\delta_1} H_o^{i+1}$$

vanishes.

REMARK 2. By the universal property of the cokernel $\hat{Z}^i \xrightarrow{\hat{\eta}^i} \hat{H}^i$ of $\hat{C}^{i-1} \xrightarrow{\hat{d}^{i-1}} \hat{Z}^i$, we obtain a morphism

$$\hat{H}^i \xrightarrow{d} H_o^{i+1}$$

generalizing the well known diagram chase construction of the connecting morphism of the long exact cohomology sequence for a short exact sequence of cochain complexes of abelian groups or modules over a ring. Participants not yet familiar with this construction should look it up in the literature on cohomology and should familiarize themselves with the diagram chase proof of the exactness of the long exact cohomology sequence.

Solutions should be submitted Thursday, May 3, in the exercises.