Exercises to „Algebra II“, 2

EXERCISE 1 (Principle of Noetherian induction, 3 points). *Let an assertion P about closed subsets of a Noetherian topological space X satisfy the following property:*

> *If $A \subseteq X$ is a closed subset such that $P(B)$ holds for all subsets B which are closed and strictly contained in A, then $P(A)$ holds.*

Then $P(A)$ holds for arbitrary closed subsets of X.

EXERCISE 2 (7 points). *Let X be a Noetherian topological space. Show that it may be decomposed as*

$$X = \bigcup_{i=1}^{n} X_i \tag{1}$$

where the X_i are irreducible closed subsets such that $X_i \not\subseteq X_j$ when $1 \leq \min(i,j) < \max(i,j) \leq n$. Moreover, show that for any such decomposition, $\{X_i \mid 1 \leq i \leq n\}$ coincides with the $\subseteq$-minimal elements of the set of closed subsets of X having an interior point.

REMARK 1. The decomposition (1) is called the decomposition of X into *irreducible components*. It follows from the second assertion that it is unique up to permutations of the X_i.

EXERCISE 3 (5 points). *Let X be any topological space and $U \subseteq X$ be open. Show that there is a bijection between the irreducible closed subsets A of U and the irreducible closed subsets B of X meeting U, sending B to $U \cap B$ and A to its closure in X.*

Let $\mathfrak{k}$ be an algebraically closed field.

EXERCISE 4 (5 points). *Let $I \subseteq R = \mathfrak{k}[X_1, \ldots, X_n]$ be an ideal satisfying $I = \sqrt{I}$. Show that it is prime if and only if $V(I)$ (the set of all zeroes of I in $\mathfrak{k}^n$) is irreducible.*

Solutions should be submitted Monday, October 30, in the lecture.