Another source of examples for the fact that positivity is an unavoidable assumption in Schmüdgen's Positivstellensatz are the homogeneous versions of the examples given by Motzkin and Choi/Lam of polynomials whith positive values everywhere which are not sums of squares in the polynomial ring. This is because of the following fact:

Problem 1 (3 points). *Let K be any formally real field and $A = K[X_1, \dots, X_m]$. If $g \in A$ is a homogeneous polynomial which f belongs to the cone of A generated by $f = 1 - \sum_{i=1}^m X_i^2$ then it is a sum of squares in A.*

Problem 2 (5 points). *Let K be a real closed field, P one of the polynomials*

$$P(X,Y,T) = X^4T^2 + Y^2T^4 + X^2T^2 - 3X^2Y^2T^2$$
$$P(X,Y,T) = X^4Y^2 + Y^4X^2 + kT^6 - 3X^2Y^2T^2$$
$$P(X,Y,Z,T) = X^2Y^2 + X^2Z^2 + Y^2Z^2 + kT^4 - 4XYZT$$

where $k \in [1, \infty)_K$. If g is as before (i. e., $1 - g$ the sum of squares of the coordinates) then $P(x) \geq 0$ on all of K^3 (resp. K^4) but P does not belong to the cone generated by g in the polynomial ring.

Note that if $k > 1$ in the examples involving k then the values of P are actually positive safe for $P(0) = 0$.

The previous class of examples also leads to counterexamples which apply when all polynomial values are positive but some of them are infintesimally small. For this, let K be a non-Archimedean real closed field,

$$R = \bigcup_{n=1}^{\infty} [-n, n]_K$$

the convex hull of $\mathbb{Q}$ in K, $A = K[X_1, \dots, X_m]$ and $\mathcal{A} = R[X_1, \dots, X_m]$. It follows from polynomial interpolation that any $P \in A$ actually belongs to $\mathcal{A}$ if its values on

$$Q = \prod_{i=1}^{m} (a_i, b_i)_K, \tag{1}$$

where $a_i < b_i$ are rational numbers, belong to R.

Problem 3 (2 points). *Let $g \in \mathcal{A}$ be such that there is a set Q as in* (1) *such that for every $q \in Q$ there is a positive rational number ε such that $g(q) > \varepsilon$. If $P \in \mathcal{A}$ belongs to the cone of A generated by g, show that P belongs to the cone of $\mathcal{A}$ generated by g.*

Problem 4 (5 points)**.** *Let K be a real closed field containing $\iota > 0$ which is smaller than any positive rational number. Let P be one of the polynomials*

$$P(X,Y,T) = X^4T^2 + Y^2T^4 + X^2T^2 - 3X^2Y^2T^2 + \iota$$
$$P(X,Y,T) = X^4Y^2 + Y^4X^2 + kT^6 - 3X^2Y^2T^2 + \iota$$
$$P(X,Y,Z,T) = X^2Y^2 + X^2Z^2 + Y^2Z^2 + kT^4 - 4XYZT + \iota$$

where $k \in [1,\infty)_K$. If g is as in Problem 2 then $P(x) > 0$ for all $x \in K^3$ (resp. $x \in K^4$) but P does not belong to the cone generated by g in the polynomial ring.

Another subtility in any attempt to generalize the Schmüdgen result to non-Archimedean K is the assumption of boundedness on the set of Nichtnegativstellen of C. This must actually hold in Sper$\mathcal{A}$ (as opposed to Sper$A$) and this boundedness assertion cannot be checked looking only at Nichtnegativstellen in K^m.

Problem 5 (5 points)**.** *Let K be a real closed field containing $\iota > 0$ which is smaller than any positive rational number. Let*

$$g(X,Y,T) = (T^2 + \iota)(1 - X^2 - Y^2 - T^2)$$

and

$$P(X,Y,T) = X^4Y^2 + Y^4X^2 + 2 - 3X^2Y^2.$$

Then P does not belong to the cone generated by g in $K[X,Y,T]$ although its values on K^3 are ≥ 1, its coefficients are in R and the Nichtnegativstellen of g in K^3 all have Euclidean absolute value ≤ 1.

For one non-Archimedean version of the Schmüdgen result see Theorem 8.3.3 in Prestel/Delzell, Positive Polynomials. Another one will be given in the lecture.

All points from this sheet are bonus points which are disregarded in the determination of the $\geq$ 50%-threshold for passing the exercise module.

Solutions should be submitted before Thursday January 29.