EXERCISES TO "ALGEBRA II", 12

In the following we consider varieties over a fixed algebraically closed field. Let the affine cone over a subset U of a projective space be denoted $\mathcal{C}(U)$.

Exercise 1 (12 points)**.** *Let us use $[S,T]$ as homogenuous coordinates on $\mathbb{P}^1$ and $[W,X,Y,Z]$ on $\mathbb{P}^3$. Consider*

$$C = V(X^3 - W^2Y, Y^3 - Z^2X, XY - WZ) \subseteq \mathbb{P}^3$$

and consider

$$\mathbb{P}^1 \xrightarrow{F} \mathbb{P}^3$$
$$F\big([S,T]\big) = [S^3, S^2T, ST^2, T^3].$$

Let $\mathbb{A}^2 \xrightarrow{f} \mathbb{A}^4$, $f(S,T) = (S^3, S^2T, ST^2, T^3)$ be the similar map between affine spaces.

- *Show that F maps $\mathbb{P}^1$ homeomorphically onto C. In particular, C is a one-dimensional irreducible subset of $\mathbb{P}^1$.*
- *Show that $\mathbb{P}^1 \xrightarrow{F} C$ is an isomorphism of algebraic varieties.*
- *More generally, show that for an integer l and an open subset $U \subseteq C$, a function $\mathcal{C}(U) \setminus \{0\} \xrightarrow{\lambda} \mathfrak{k}$ which is homogenuous degree l is an element of $\big(\mathcal{O}_C(l)\big)(U)$ if and only if*
$$(S,T) \to \lambda(f(S,T))$$
(a function on $\mathcal{C}(F^{-1}U) \setminus \{0\}$) is an element of
$$\big(\mathcal{O}_{\mathbb{P}^1}(4l)\big)(F^{-1}U).$$
- *Determine for which l the map*
$$\mathfrak{k}[W,X,Y,Z]_l \to \big(\mathcal{O}_C(l)\big)(C)$$
is surjective.
- *Calculate the degree and the Hilbert polynomial of $C \subseteq \mathbb{P}^3$.*

Remark 1. The assertions about F being an isomorphism are perhaps most easily established by proving them on affine open subsets of the projective spaces where an inverse map is given by straighforward formulas.

Remark 2. The assertion of the third point can be understood as identifying the line bundle $\mathcal{O}(4l)$ on $\mathbb{P}^1$ with the pull-back via F of $\mathcal{O}(1)$ on C.

The following exercises extend the results of the previous sheet about associated prime ideals.

Exercise 2 (1 point)**.** *Let R be a ring, $\mathfrak{p} \in \operatorname{Spec} R$. Show that*

$$\operatorname{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}.$$

Exercise 3 (3 points)**.** *Let $R = \mathfrak{k}[X, Y]$, $I \subseteq R$ the ideal generated by X^2 and XY. Calculate* $\operatorname{Ass}(R/I)$*.*

Exercise 4 (4 points)**.** *Let R be a Noetherian ring, M a finitely generated R-module.*

- *Show that* $\operatorname{Ass}(M)$ *is finite.*
- *Show that the set of $r \in R$ for which there exists $m \in M \setminus \{0\}$ equals*

$$\bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}$$

Solutions should be submitted Monday, January 22, in the lecture.