Exercises to "Algebra II", 11

The following exercise finishes the proof, started on the previous exercise sheet, that all fields are universally Japanese. This also finishes the proof of the claims made in the lecture about global sections of the line bundles $\mathcal{O}(k)$ on projective algebraic varieties.

Exercise 1 (9 points). *Let $\mathfrak{k}$ be a field of characteristic $p > 0$, $R = \mathfrak{k}[X_1, \ldots, X_n]$ and K the quotient field of R.*

- *Let $S \subseteq \mathfrak{k}$ be a finite subset, $q \in p^{\mathbb{N}}$, let $\mathfrak{l}$ be the finite field extension obtained from $\mathfrak{k}$ by adjoining q-th roots of the elements of S, and let M be the field extension of K obtained by adjoining q-th roots of the X_i and of the elements of S. Show that the integral closure of R in M is isomorphic to a polynomial ring in n variables over $\mathfrak{k}$ and finitely generated as an R-module.*
- *Let L/K be a finite field extension. Show that, when q and S in the previous step are chosen large enough, there is a finite field extension N of K containing both L and M as subextensions and such that N/M is separable.*
- *Use these observations to show that the integral closures of R in N and L are finitely generated R-modules, and finish the proof that $\mathfrak{k}$ is universally Japanese.*

Recall the definition of a general variety over an algebraically closed field $\mathfrak{k}$ given in the lecture: A pair $(X, \mathcal{O}_X)$ consisting of a Noetherian irreducible topological space X and a subsheaf $\mathcal{O}_X$ of the sheaf of $\mathfrak{k}$-valued functions on X such that for every point x of X there is an open neighbourhood U such that $(U, \mathcal{O}_X|_U)$ is isomorphic to an affine algebraic variety $Y \subseteq \mathfrak{k}^n$. Open subsets with that property are called affine. Morphisms are continuous maps $X \xrightarrow{f} Y$ such that the pull-back of a section of the structure sheaf on Y is a section of the structure sheaf on X.

Exercise 2 (8 points). *For a pair of morphisms $X \overset{f}{\underset{g}{\rightrightarrows}}$ of general varieties over $\mathfrak{k}$, consider the question whether*

$$K = \{x \in X \mid f(x) = g(x)\}$$

is closed in X.

- *Show that K is closed if Y is affine.*
- *Show that K is closed if for two arbitrary points of Y, there exists an affine open subset of Y containing both of them.*

- *Show that Y has the property of the previous point if it is quasi-affine or quasi-projective. In particular, K is closed in this case.*

Remark 1. It may be used that sections of the structure sheaf are continuous functions. It may also be used that (as was shown in Algebra 1) for X affine and $f \in \mathcal{O}(X) \setminus \{0\}$, $X \setminus V(f)$ is isomorphic to an affine algebraic variety.

Definition 1. Let R be a ring and M an R-module. Let $\operatorname{Ass}(M) \subseteq \operatorname{Spec} R$ denote the set of all prime ideal $\mathfrak{p}$ of R such that there is $m \in M$ such that

$$\mathfrak{p} = \operatorname{Ann}_R(m) := \{r \in R \mid r \cdot m = 0\}.$$

Exercise 3 (3 points).

- *If $N \subseteq M$, show that*

$$\operatorname{Ass}(M) \subseteq \operatorname{Ass}(N) \cup \operatorname{Ass}(M/N).$$

- *If R is Noetherian and M not zero, show that* $\operatorname{Ass}(M)$ *is not empty.*

Remark 2. Obviously, we have $\operatorname{Ass}(N) \subseteq \operatorname{Ass}(M)$ in the situation of the first point. One does not normally have $\operatorname{Ass}(M/N) \subseteq \operatorname{Ass}(M)$ as is shown by $M = R\mathbb{Z}$ and $N = 2\mathbb{Z}$.

Solutions should be submitted Monday, January 15, in the lecture.