Tenth exercise sheet "Class field theory" summer term 2025. The definition of the norm residue symbol $(x, y)_n$ used in the previous two sheets can be applied to arbitrary local fields K containing n different $\sqrt[n]{1}$, including the case $K = \mathbb{R}$ or $K = \mathbb{C}$. In the latter case the symbol vanishes identically as \mathbb{C} is algebraically close. If several local fields are considered, the norm residue symbol will be denoted $(\cdot, \cdot)_{K,n}$.

Problem 1 (1 point). If $K = \mathbb{R}$, show that

$$(x, 1-x)_2 = (-1)^{\frac{(1-\operatorname{sgn} x)(1-\operatorname{sgn} y)}{4}} = \begin{cases} -1 & x < 0 \text{ and } y < 0\\ 1 & otherwise! \end{cases}$$

Problem 2 (3 points). Let K be an ultrametric local field with residue field \mathfrak{k} . For $x \in \mathcal{O}_K$ let \overline{x} denote its image in \mathfrak{k} . If n is a positive integer which is not divisible by the characteristic of \mathfrak{k} , show that

$$\mathcal{O}_K^{\times n} = \left\{ x \in \mathcal{O}_K^{\times} \mid \overline{x} \in \mathfrak{k}^n \right\}$$

and that the $\mathbb{Z}/n\mathbb{Z}$ -module $K^{\times}/K^{\times n}$ is isomorphic to $(\mathbb{Z} / n\mathbb{Z}) \oplus (\mathfrak{k}^{\times} / \mathfrak{k}^{\times n})!$

Problem 3 (3 points). Let L/K be an at most tamely ramified finite Galois extension of ultrametric local fields. If $x \in \mathcal{O}_K^{\times}$ and $\overline{x} = 1$, show that $\psi_{L/K}(x) = 1$!

Definition 1. Let K be an ultrametric local field with residue field \mathfrak{k} and valuation exponent v. For $(x, y) \in K^{\times}$, the tame symbol is defined by

$$(x,y)_{\mathrm{T}} = \overline{(-1)^{v(x)v(y)}x^{v(y)}y^{-v(x)}}.$$

Obviously, this is a \mathbb{Z} -bilinear map $K^{\times} \times K^{\times} \to \mathfrak{k}^{\times}$.

Problem 4 (3 points). If $x \in K \setminus \{0, 1\}$, show that $(x, 1 - x)_T = 1$.

Remark 1. Let K be an arbitrary field. We define the $K_2(K)$ as the quotient of the abelian group $K^{\times} \otimes_{\mathbb{Z}} K^{\times}$ by the subgroup generated by the elements of the form $x \otimes (1 - x)$ with $x \in K \setminus \{0, 1\}$. Note that there is a definition of the algebraic K-groups $K_n(R)$ for arbitrary $n \in \mathbb{N}$ und arbitrary rings R, and the fact that for n = 2 and R = Kthe above elementary description holds is a theorem of Matsumoto, the other elementary low-degree algebraic K-groups of fields being $K_1(K) = K^{\times}$ and $K_0(K) = \mathbb{Z}$.

If K is a local field, then by a theorem of Moore [Mil72, Theorem A.14] $K_2(K)$ is the direct sum of the group $\mu(K)$ of roots of 1 in K and a divisible group. If K is non-isomorphic to \mathbb{C} then $\mu(K)$ can be shown to be finite. It follows from this that the norm residue symbol $(x, y)_{K,n}$ is uniquely determined by the property established in Problem 5 from the previous sheet and a single value $(x, y)_n$ where the image of $x \otimes y$ under

$$K^{\times} \otimes K^{\times} \to K_2(K) \to \mu(K)$$

qenerates the finite cyclic group $\mu(K)$.

Definition 2. If \mathfrak{k} is a finite field with q elements, the positive integer n divides q-1 and $x \in \mathfrak{k}^{\times}$, we define the Legendre symbol

$$\left(\frac{x}{\mathfrak{k},n}\right)_{\mathrm{L}} = x^{\frac{q-1}{n}}$$

Remark 2.

- **park 2.** Obviously, this is a $\sqrt[n]{1}$ in \mathfrak{k} . An alternative definition is $\left(\frac{x}{\mathfrak{k},n}\right)_{\mathrm{L}} = \frac{\mathrm{Frob}_q(\xi)}{\xi}$ where ξ is a $\sqrt[n]{x}$ in $\overline{\mathfrak{k}}$ and Frob_q is the topological generator of $\operatorname{Gal}(\overline{\mathfrak{k}}/\mathfrak{k})$ defined by $\operatorname{Frob}_{q}(y) = y^{q}$.
- In particular, $x \in \mathfrak{k}^n$ if and only if $\left(\frac{x}{\mathfrak{k},n}\right)_{\mathfrak{r}} = 1$.

Remark 3. If L/K is a finite unramified extension of ultrametric local fields then $\operatorname{Gal}(L/K)$ is generated by the Frobenius element $\operatorname{Frob}_{L/K}$ and $\psi_{L/K}(x) = \operatorname{Frob}_{L/K}^{v_K(x)}$ by Corollary 2.3.2 from the lecture.

If \mathfrak{k} is the residue field of the ultremetic K and n not a multiple of the characteristic of \mathfrak{k} then K contains all $\sqrt[n]{1}$ if and only if q-1 is a multiple of n. If $y \in \mathcal{O}_K^{\times}$ then $K(\sqrt[n]{y})$ is unramified over K and $(x, y)_n$ can be calculated usind the previous remark. Also, $\mu_n(K) \xrightarrow{x \to \overline{x}} \mu_n(\mathfrak{k})$ is an isomorphism in this case, hence $(x, y)_n$ is uniquely determined by the following result:

Problem 5 (4 points). Under the above assumptions,

$$\overline{(x,y)_n} = \left(\frac{(x,y)_{\mathrm{T}}}{\mathfrak{k},n}\right)_{\mathrm{L}}.$$

In the case where n = 2, the only remaining case for the calculation of $(x, y)_{K_v, n}$ when $K = \mathbb{Q}$ is the field $K_v = \mathbb{Q}_2$ of 2-adic numbers. Let $\mathbb{Z}_2 = \mathcal{O}_{\mathbb{Q}_2}.$

Problem 6 (3 points). If $x \in \mathbb{Z}_2$ is $\equiv 1 \mod 4$, show that $\mathbb{Q}_2(\sqrt{x})$ is unramified over \mathbb{Q}_2 ! If $x \equiv 1 \mod 8$, show that x is a square in \mathbb{Q}_2 !

It follows that the images of -1, -3 and 2 generate the \mathbb{F}_2 -vector space $\mathbb{Q}_2^{\times}/\mathbb{Q}_2^{\times 2}$ and in fact these elements turn out to form a base, by the following result:

Problem 7 (3 points). If $x, y \in \mathbb{Z}_2^{\times}$ then

$$(x,y)_2 = (-1)^{\frac{(x-1)(y-1)}{2}}$$

 $(2,x)_2 = (-1)^{\frac{x^2-1}{8}}.$

Moreover, $(2, 2)_2 = 1$.

If L/K is a finite abelian extension of global fields, v a place of Kand w a place of L lying above v, $\operatorname{Gal}(L_w/K_v)$ can be identified with the stabilizer of w in $\operatorname{Gal}(L/K)$. Using this identification, we define

$$\psi_{L/K,v}(x) = \psi_{L_w/K_v}(x)$$

for $x \in K_v$. The abelianness of L/K implies that this does not depend on the choice of w. The reciprocity law of global class field theory states that

$$\prod_{v} \psi_{L/K,v}(x) = \mathrm{Id}_L$$

for $x \in K^{\times}$, where the product over all places v of K has only finitely many factors $\neq \operatorname{Id}_L$.

Problem 8 (1 point). If K is a global field containing n different $\sqrt[n]{1}$, deduce that

$$\prod_{v} (x, y)_{K_v, n} = 1$$

for $x, y \in K^{\times}$.

Problem 9 (3 points). Derive the quadratic reciprocity law and its two supplementary laws by applying the previous result to $K = \mathbb{Q}$ and n = 2!

Four of the 24 points from this sheet are bonus points which do not count in the calculation of the 50%-limit for passing the exercises module.

Solutions should be submitted to the tutor by e-mail before Tuesday July 1 24:00.

References

[Mil72] JOHN MILNOR, Introduction to Algebraic K-Theory, Annals of Math. Studies 72 (Princeton University Press 1972)