Exercises to „Algebraic Geometry II“, 10

We finish the proof that higher direct images of coherent sheaves of modules under a proper morphism between locally Notherian schemes are coherent.

We will always let R denote a Noetherian ring and X and Y locally Noetherian preschemes.

Exercise 1 (5 points). Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. Show that the *support*

$$\mathrm{supp}\mathcal{M} = \{x \in X \mid \mathcal{M}_x \neq 0\}$$

of $\mathcal{M}$ is a closed subset of X. Moreover, show that there is a smallest closed subprescheme $S \xrightarrow{i} X$ for which there is a coherent $\mathcal{O}_S$-module $\mathcal{N}$ such that $\mathcal{M}$ is isomorphic to $i_*\mathcal{N}$, and that the underlying set of points of S equals $\mathrm{supp}\mathcal{M}$.

Exercise 2 (15 points). Let $X \xrightarrow{f} \mathrm{Spec}R$ be a proper morphism. Show that for any coherent $\mathcal{O}_X$-module $\mathcal{M}$ and any $p \in \mathbb{N}$, $H^p(X, \mathcal{M})$ is a finitely generated R-module.

Remark 1. The proof is usually done by Noetherian induction on the scheme theoretic support of $\mathcal{M}$. Chow's lemma from the previous sheet may be applied when that support is an integral scheme. The other cases of the induction argument are easier to carry out.

The part of the proof involving Chow's lemma is typically formulated using the Leray spectral sequence but direct hypercohomology as explained on sheet 6 may also be used.

Remark 2. The following assertion (to be formulated as a theorem in the upcoming lecture this Thursday, July 5) is local with respect to Y, hence follows from the previous exercise: If $X \xrightarrow{f} Y$ is a proper morphism between locally noetherian preschemes and $\mathcal{M}$ a coherent $\mathcal{O}_X$-module, then all $R^p f_*\mathcal{M}$ are coherent $\mathcal{O}_Y$-modules.

Solutions should be submitted Thursday, July 12, in the exercises.