Exercises to "Algebra II", 10

Let $\mathfrak{k}$ be an algebraically closed field. In the following exercise we differentiate between the structure sheaves $\mathcal{O}_{\mathrm{aff}}$ in the affine and $\mathcal{O}_{\mathrm{proj}}$ in the projective case. Recall that for an element f of $\mathcal{O}_{\mathrm{aff}}(U)$ and $x \in U$ there must be an open neighbourhood V of $x \in \mathbb{A}^n$ and polynomials p, q in n variables such that we have $q(y) \neq 0$ for $y \in V$ and $f(y) = \frac{p(y)}{q(y)}$ when $y \in U \cap V$. We identify $\mathfrak{k}^n$ with $\mathbb{P}^n \setminus V(X_0)$ identifying $(x_1, \ldots, x_n)$ with $[1, x_1, \ldots, x_n]$.

Exercise 1 (4 points)**.** *Let $U \subseteq \mathbb{P}^n \setminus V(X_0)$ be an open subset of a Zariski-closed subset of $\mathbb{P}^n$. Show that a $\mathfrak{k}$-valued function f on U is an element of $\mathcal{O}_{\mathrm{aff}}(U)$ if and only if it is an element of $\mathcal{O}_{\mathrm{proj}}(U)$.*

The following result was proved in my lecture "Algebra I", but for those who have not attended this lecture it may be useful to work out a proof, which can be obtained by simplifying the proof given in the projective case, replacing the projective Nullstellensatz by the affine Version.

Exercise 2 (5 points)**.** *Let $X \subseteq \mathfrak{k}^n$ be Zariski-closed. Show that every element of $\mathcal{O}_{\mathrm{aff}}(X)$ can be represented as $f = p|_X$ where p is a polynomial in n variables with coefficients in $\mathfrak{k}$.*

Recall that an element x of a field L is called *integral* over a subring R if it satisfies an algebraic equation $x^n = \sum_{i=0}^{n-1} r_i x^i$ with $r_i \in R$. The integral closure of R in L are the elements integral over R. It has been shown in "Algebra I" (and is sometimes shown in even earlier lectures) that this is a subring of L.

Exercise 3 (3 points)**.** *Let R be a domain with quotient field K, $I \subseteq R$ a finitely generated ideal and $x \in K$ such that $x \cdot I \subseteq I$. Show that x is integral over R.*

Remark 1. The full number of points should be given for solutions which assume R to be Noetherian (which is the only case needed below), but the result holds as stated and can be proved in this form using only knowledge required for this lecture.

Exercise 4 (5 points)**.** *Let L/K be a finite separable field extension, $R \subseteq K$ a Noetherian domain which is integrally closed in its field of quotients K and S the integral closure of R in L.*

- *Show that $(x, y) \to \mathrm{Tr}_{L/K}(xy)$ is a non-degenerate K-bilinear form on L.*

- *When $M \subseteq L$ is a finitely generated free R-module, show that its dual $M^* = \{x \in L \mid \mathrm{Tr}_{L/K}(mx) \in R \text{ for all } x \in M\}$ is a finitely generated R-module.*
- *Show that S is a finitely generated R-module.*

Remark 2. The results about traces and norms for finite field extensions and their values on elements of an integral ring extension from the (approximately) third lecture, where they were formulated in preparation for our proof of a special case of the principal ideal theorem, may all be used. In particular, it may be used in the above situation $\mathrm{Tr}_{L/K}$ is not identically 0.

A Noetherian ring R is called *universally Japanese* if for every domain A which is an R-algebra of finite type and any finite field extension L of the quotient field of A, the integral closure of A in L is a finitely generated A-module. By a result of Nagata (cf. Matsumura, Commutative Algebra, Second Edition, (31.H)/Theorem 72) which may have motivated Grothendieck to coin the above notion, it is sufficient to prove this when $A = R/\mathfrak{p}$ with $\mathfrak{p} \in \mathrm{Spec}R$. It follows that any field is universally Japanese. The proof of this result of Nagata is lengthy and uses methods of commutative algebra not yet developed in this lecture. In the case where R is a field, an easier proof can be given using the Noether normalization theorem. This becomes especially easy when it is clear that the previous exercise can be applied.

Exercise 5 (4 points)**.** *Show that any field of characteristic zero is universally Japanese.*

Remark 3. Of course the harder of the two aforementioned criteria for "universally Japanese" must be established and the result of Nagata must not be used.

As an application, consider a projective variety $X \subseteq \mathbb{P}^n(\mathfrak{k})$. Let $C = C(X)$ be the affine cone over X, and $R = \mathcal{O}(C)$. Let $l \in \mathbb{N}$, and $f \in \big(\mathcal{O}(l)\big)(X)$. By definition, f defines an element of $\mathcal{O}(C \setminus \{0\})$ and hence of the quotient field of K, and it was was mentioned in the lecture (and follows from the second exercise) that f is given by a homogenuous polynomial of degree l in $X_0, \ldots, X_n$ if and only if it is actually an element of R.

Exercise 6 (6 points)**.**
- *In the above situation, show that f is integral over R.*
- *Show that $\big(\mathcal{O}(l)\big)(X)$ is a finite-dimensional $\mathfrak{k}$-vector space for all $l \in \mathbb{N}$.*

- *Deduce that there exists a natural number k such that for $l \geq k$, any $f \in \big(\mathcal{O}(l)\big)(X)$ is given by an element of $\mathfrak{k}[X_0, \ldots, X_n]_l$.*

Remark 4.

- The fact that $\mathfrak{k}$ is universally Japanese may be used, even when $\mathfrak{k}$ has positive characteristic.
- The assertions of Theorem 16 of the lecture which have been given a complete proof in the lecture may also be used.
- As was mentioned in the lecture, the conceptionally clearest proof uses the finite dimensionality for all l and the vanishing for sufficiently large l of the cohomology group $H^1(\mathbb{P}^n, \mathcal{J} \otimes \mathcal{O}(l))$ where $\mathcal{J}$ is the subsheaf of all elements of $\mathcal{O}_{\mathbb{P}^n}$ vanishing on X. But this is out of reach of the methods developed in this lecture, and the above proof and its ingredients are probably also worthwhile to know.

Of the 27 points of this exercise sheet, seven are bonus points which do not enter the calculation of the 50%-limit for passing the exercises.

Solutions should be submitted Monday, January 8, in the lecture.