Exercises to „Algebraic Geometry II“, 1

Exercise 1 (3 points). Prove one of the following assertions:

- Let

$$\begin{array}{ccccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C & \xrightarrow{c} & D \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\delta}\downarrow \\ \tilde{A} & \xrightarrow{\tilde{a}} & \tilde{B} & \xrightarrow{\tilde{b}} & \tilde{C} & \xrightarrow{\tilde{c}} & \tilde{D} \end{array}$$

  be a commutative diagram of modules over a ring, where the rows are exact sequences and β and δ are injective maps while α is surjective. Show that γ is injective.
- Let

$$\begin{array}{ccccccc} B & \xrightarrow{b} & C & \xrightarrow{c} & D & \xrightarrow{d} & E \\ {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\delta}\downarrow & & {\scriptstyle\varepsilon}\downarrow \\ \tilde{B} & \xrightarrow{\tilde{b}} & \tilde{C} & \xrightarrow{\tilde{c}} & \tilde{D} & \xrightarrow{\tilde{d}} & \tilde{E} \end{array}$$

  be a commutative diagram of modules over a ring, where the rows are exact sequences and β and δ are surjective maps while ε is injective. Show that γ is surjective.

Remark 1.
- As most participants seem familiar with diagram chase, I would like to limit the burden on Mr. Hamann who has to correct the solutions. Thus, only one of the two assertions should be proved. For solutions containing proofs of both assertions whithout making clear which of the two proofs the authors would like to get points for, Mr. Hamann is free to make the choice he considers most reasonable.
- Combining the two points, one gets the classical 5-lemma about isomorphisms: In the case of a ten node ladder diagram with α epi, ε mono and β and δ iso, γ is an isomorphism. The assertion holds as stated for any Abelian category, and can be proved by diagram chase in the case of the category of R-modules.

Exercise 2 (3 points). Let $(M_i)_{i\in I}$ be a (possibly infinite) family of modules over a ring R, and $P = \prod_{i\in I} M_i$ the set of all families $(m_i)_{i\in I}$ where $m_i \in M_i$. Let the R-module operations on P be defined componentwise, and let $P \xrightarrow{\pi_j} M_j$ be the map sending $(m_i)_{i\in I}$ to m_j. Show that that the object P and the family $(\pi_i)_{i\in I}$ of morphisms in the category of R-modules satisfy the universal property characterizing products: For any R-module T and any family $\left(T \xrightarrow{t_i} M_i\right)_{i\in I}$

of morphisms of R-modules, there is a unique morphism $T \xrightarrow{\tau} P$ of R-modules such that $t_i = \pi_i \tau$ holds for all $i \in I$.

EXERCISE 3 (4 points). Let $(M_i)_{i\in I}$ and P be as before, let $C \subseteq P$ denote the subset of all $(m_i)_{i\in I} \in P$ for which $\{i \in I | m_i \neq 0\}$ is finite, and let $M_j \xrightarrow{\iota_j} C$ be defined by

$$\iota_j(m)_i = \begin{cases} m & j = i \\ 0 & \text{otherwise.} \end{cases}$$

Show that that the object C and the family $(\iota_i)_{i\in I}$ of morphisms in the category of R-modules satisfy the universal property characterizing coproducts: For any R-module T and any family $\left(M_i \xrightarrow{t_i} T\right)_{i\in I}$ of morphisms of R-modules, there is a unique morphism $C \xrightarrow{\tau} T$ of R-modules such that $t_i = \tau \iota_i$ holds for all $i \in I$.

EXERCISE 4 (4 points). Let $(\mathcal{M}_i)_{i\in I}$ be a (possibly infinite) family of sheaves of modules over the sheaf of rings $\mathcal{R}$ on the topological space X, and let $\mathcal{P}$ be defined by

$$\mathcal{P}(U) = \prod_{i\in I} \mathcal{M}_i(U)$$

for open subsets $U \subseteq X$, where the product on the right hand side is the product in the category of $\mathcal{R}(U)$-modules as in the second exercise. Let the restriction maps to $V \subseteq U$ be defined componentwise. Specify morphisms $\mathcal{P} \xrightarrow{\pi_i} \mathcal{M}_i$ and show that the universal property of a product in the category of $\mathcal{R}$-modules is satisfied.

EXERCISE 5 (6 points). Let $(\mathcal{M}_i)_{i\in I}$ and $\mathcal{P}$ be as in the previous exercise and let $\mathcal{C} \subseteq \mathcal{P}$ be the subsheaf defined by

$$\mathcal{C}(U) = \Big\{(m_i)_{i\in I} \;\Big|\; \text{For every } x \in U \text{, there exists an open neighbourhood } V \subseteq U \text{ such that } m_i|_V \text{ vanishes for all but finitely many } i \in I\Big\}$$

Specify morphisms $\mathcal{M}_i \xrightarrow{\iota_i} \mathcal{C}$ and show that the universal property of a coproduct in the category of $\mathcal{R}$-modules is satisfied.

Solutions should be submitted Thursday, April 26, in the exercises.